

Complex Analysis

International Dance Day

Complex variableHere $x+iy$ is a complex variable& it is denoted by z .

i.e. $z = x+iy$ where $i = \sqrt{-1}$

Function of Complex variable→ Denoted by $f(z)$ on w .

i.e. $w = f(z)$

$\Rightarrow w = u+iv$

where u & v are functions of (x,y) ,

i.e. $w = u(x,y) + i v(x,y)$,

 $\downarrow$
Real Part $\downarrow$
Im. PartLimit

A function $f(z)$ is said to have a limit l , of $f(z)$ as $z \rightarrow z_0$ if
 $\forall \epsilon > 0 \exists \delta > 0 \forall z \neq z_0$ in the disk

$$|z - z_0| < \delta$$

$$\Rightarrow |f(z) - 1| < \epsilon.$$

i.e. for every $z \neq z_0$ in the δ -disk,
the value of f lies in the disk.

$$\text{i.e. } \lim_{z \rightarrow z_0} f(z) = 1.$$

Continuity

$f(z)$ is said to be continuous

at $z = z_0$ if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

Differentiability

Let $f(z)$ be a complex function

$$\text{then } f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

Ex of $f(z) = 4x + y + i(-x + 4y)$

and discuss $\frac{df}{dz}$.

Solⁿ $f(z) = 4x + y + i(-x + 4y)$
 $= u + iv.$

$$u = 4x + y$$

$$v = -x + 4y.$$

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \rightarrow 0} \frac{f(x + iy + \delta x + i\delta y) - f(x + iy)}{\delta x + i\delta y}$$

$$= \lim_{\delta z \rightarrow 0} \frac{f(x + \delta x + i(y + \delta y)) - f(x + iy)}{\delta x + i\delta y}$$

$$= \lim_{\delta z \rightarrow 0} \frac{4(x + \delta x) + (y + \delta y) + i(-(x + \delta x) + 4(y + \delta y)) - (4x + y + i(-x + 4y))}{\delta x + i\delta y}$$

$$f(x + \delta x + i(y + \delta y))$$

$$= y(x+\delta x) + y + \delta y - i(x+\delta x) + i(y+\delta y)$$

$$= 4x + 4\delta x + y + \delta y - i x - i \delta x + 4i y + 4i \delta y$$

$$f(x+iy) = u(x,y) + i(-x+xy)$$

Now

$$f(x+\delta x + i(y+\delta y)) - f(x+iy)$$

$$= 4x + 4\delta x + y + \delta y - i\hbar - i\delta\hbar +$$

$$4iz + 4i8y - 4x - 8 + 2x - 4iy$$

$$= 4\delta x + 8y - 2\delta x + 4i\delta y$$

Now, $f'(z) = \lim_{\substack{\delta x \rightarrow 0 \\ \delta y \rightarrow 0}} \frac{4\delta x + \delta y - i\delta x + 2i\delta y}{\delta x + 2i\delta y}$

① Along Real Axis:

$$\delta y = 0$$

~~$\delta Z = \delta M$~~

$$g'(z) = \lim_{\delta x \rightarrow 0} \frac{4\delta x - i\delta x}{\delta x}$$

$$= \lim_{x \rightarrow 0} \frac{4-x^2}{1}$$

$$24 - 2$$

② Along Imaginary axis ↓

$$\delta x = 0$$

$$\delta Z = \tau \delta y$$

$$\frac{\partial f}{\partial z} = \lim_{\delta y \rightarrow 0} \frac{\delta y + 4i\delta y}{i\delta y}$$

$$= \frac{1+4i}{2}$$

$$= 4 - 2$$

© Along the line $y=x$

$$Z = u + i'v$$

if $y = x$

$$\Rightarrow Z = x + iy = (1+i)x$$

$$\Rightarrow \delta z = (1+i)\delta x$$

$$\delta y = \delta x$$

$$\frac{\partial f}{\partial z} = \lim_{\delta x \rightarrow 0} \frac{4\delta x + \delta x - i\delta x + 4i\delta x}{\delta x + i\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{4+1-i+4i}{1+i}$$

$$= \lim_{\delta x \rightarrow 0} \frac{5+3i}{1+i}$$

$$= \frac{(5+3i)(1-i)}{1+1}$$

$$= \frac{8-2i}{2}$$

$$= 4-i$$

Analytic Function

A function $f(z)$ is said to be analytic at $z=z_0$ if it is

differentiable at $z=z_0$.

N.B

* The point at which the function is not differentiable is called a Singular Point.

Necessary Condition for $f(z)$ to be Analytic

The necessary conditions for a function $f(z)$ to be analytic at all the points in a region R are

$$\text{if } f(z) = u + iv \rightarrow \begin{matrix} \downarrow \\ \text{Real} \\ \text{Part} \end{matrix} \quad \begin{matrix} \rightarrow \\ \text{Imaginary} \\ \text{part} \end{matrix}$$

$$\text{then } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\text{or } u_x = v_y \quad , \quad u_y = -v_x$$

Provided, u_x, u_y, v_x, v_y are exist.

These equations are called
C-R equations.

ie Cauchy-Riemann Equation.

Sufficient Condition for $f(z)$ to be Analytic

The sufficient condition for a function $f(z) = u + iv$ to be analytic at
on the points z in a region R are

$$(i) \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous
functions of x and y in region R .

Ex 1 Show that $f(z) = |z|^2$
is differentiable only at the origin.

Solⁿ $f(z) = |z|^2$ where $z = x + iy$
 $\Rightarrow f(z) = x^2 + y^2$

Hence $u = x^2 + y^2, v = 0$

If $f(z)$ is differentiable
 $\Rightarrow f(z)$ satisfies C-R equation.

i.e. $u_x = v_y$

$\Rightarrow 2x = 0 \Rightarrow x = 0$

and $u_y = -v_x$

$\Rightarrow 2y = 0 \Rightarrow y = 0$

C-R equations are satisfied only
when $x = y = 0$.

(Proved)

Sol Let $w = \sin z$
 $= \sin(x + iy)$

$$\Rightarrow w = \sin x \cos y + \cos x \sin y$$

$$\Rightarrow w = \sin x \cosh y + i \cos x \sinh y$$

Here $u = \sinh x \cosh y$, $v = \cosh x \sinh y$

$$\frac{\partial u}{\partial n} = \cos n \cosh y, \quad \frac{\partial v}{\partial n} = -\sin n \sinh y$$

$$\frac{\partial u}{\partial y} = 82h \sin hy, \quad \frac{\partial v}{\partial y} = \cos \sinh y$$

Thus, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

and $\frac{\partial y}{\partial y} = - \frac{\partial v}{\partial u}$

$\Rightarrow$ w satisfies the G-R equation

$\Rightarrow w = \sinh z$ is an analytic function.

$$\text{Now } \frac{d}{dz} (\sin z) = \frac{d}{dz} [\sin x \cosh y + i \cos x \sinh y]$$

$$= \frac{\partial}{\partial x} [\sin x \cosh y + i \cos x \sinh y]$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$= \cos \alpha \cos 2\gamma - \sin \alpha \sin 2\gamma$$

$$\begin{pmatrix} -z' \cos hy = \cos 2y & \& \\ \sin 2y = z' \sin hy \end{pmatrix}$$

$$= \cos(x + iy)$$

$$= \cos z$$

$$\Rightarrow \frac{d}{dz}(\sinh z) = \cosh z$$

Polar Form of C-R equation is

$$\frac{\partial u}{\partial \eta} = \frac{1}{\eta} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

PROOF

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Let $x = r \cos \theta$
 $y = r \sin \theta$
 $r^2 = x^2 + y^2$
 $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

& u is a function of (x, y)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

Since $r^2 = x^2 + y^2$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{2x}{2r} = \frac{r \cos \theta}{r}$$

$$\Rightarrow \boxed{\frac{\partial r}{\partial x} = \cos \theta}$$

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$$\& \frac{\partial r}{\partial y} = \frac{2y}{2r} = \frac{r \sin \theta}{r}$$

$$\Rightarrow \boxed{\frac{\partial r}{\partial y} = \sin \theta}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = -\frac{r \sin \theta}{r^2} = -\frac{\sin \theta}{r}$$

MAY

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$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2} = \frac{r \cos \theta}{r^2}$$

$$\Rightarrow \boxed{\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}}$$

$$\text{So } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \cos \theta + \frac{\partial u}{\partial \theta} \cdot \left(-\frac{\sin \theta}{r}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cos \theta - \frac{\partial u}{\partial \theta} \cdot \frac{\sin \theta}{r} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \sin \theta + \frac{\partial u}{\partial \theta} \cdot \frac{\cos \theta}{r} \quad \text{--- (2)}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$= \frac{\partial v}{\partial r} \cdot \cos \theta + \frac{\partial v}{\partial \theta} \cdot \left(-\frac{\sin \theta}{r}\right)$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \cos \theta - \frac{\partial v}{\partial \theta} \cdot \frac{\sin \theta}{r} \quad \text{--- (3)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$= \frac{\partial v}{\partial r} \cdot \sin \theta + \frac{\partial v}{\partial \theta} \cdot \frac{\cos \theta}{r}$$

By C-R equation

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial r} \cos \theta - \frac{\partial u}{\partial \theta} \cdot \frac{\sin \theta}{r} =$$

$$\frac{\partial v}{\partial r} \sin \theta + \frac{\partial v}{\partial \theta} \cdot \frac{\cos \theta}{r}$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial r} \sin \theta + \frac{\partial u}{\partial \theta} \cdot \frac{\cos \theta}{r} = -\frac{\partial v}{\partial r} \cos \theta + \frac{\partial v}{\partial \theta} \cdot \frac{\sin \theta}{r}$$

multiplying $\cos \theta$ & $\sin \theta$ to eqⁿ ⑤ & ⑥ respectively and adding ~~⑤ & ⑥~~ we have

$$\frac{\partial u}{\partial r} \cos^2 \theta + \frac{\partial u}{\partial \theta} \sin^2 \theta =$$

$$\frac{\partial v}{\partial \theta} \frac{\cos^2 \theta}{r} + \frac{\partial v}{\partial r} \frac{\sin^2 \theta}{r}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}}$$

and subtracting ~~⑤ & ⑥~~ by multiplying $\cos \theta$ to eqⁿ ⑤ & $\sin \theta$ to eqⁿ ⑥ we have

$$-\frac{\partial u}{\partial \theta} \frac{\sin^2 \theta}{r} - \frac{\partial u}{\partial r} \frac{\cos^2 \theta}{r} =$$

$$\frac{\partial v}{\partial r} \sin^2 \theta + \frac{\partial v}{\partial \theta} \cos^2 \theta$$

$$\Rightarrow -\frac{1}{r} \frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}}$$

(Proved)

Laplace Equation's

If $f(z) = u + iv$ be a complex function.

$$\text{and } \frac{\partial u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{--- (2)}$$

(1) & (2) are called Laplace Equation,

Harmonic Function's

The function which satisfies Laplace's equation is called Harmonic Function.

i.e. If $f(z) = u + iv$ is an analytic function and u & v are satisfy Laplace equation

$$\text{i.e. } u_{xx} + u_{yy} = 0$$

$$\& \ v_{xx} + v_{yy} = 0$$

Then u & v are called Conjugate Harmonic Function.

Q. ~~Let~~ Construct the analytic function $f(z)$ of which real part is $e^x \cos y$.

$$\text{Sol} \quad u = e^x \cos y$$

$$u_x = e^x \cos y$$

$$u_y = -e^x \sin y$$

Since $f(z)$ is analytic

$$u_x = v_y \quad \& \quad u_y = -v_x$$

$$\text{Now } dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\Rightarrow dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$\Rightarrow dr = e^x \sin y \, dx + e^x \cos y \, dy$$

Integrating both sides

$$\int dr = \int e^x \sin y \, dx + \int e^x \cos y \, dy$$

(y as const.) (terms indep. of x)

$$\Rightarrow r = e^x \sin y.$$

$$\begin{aligned} \text{Now } f(z) &= u + iv \\ &= e^x \cos y + i e^x \sin y \end{aligned}$$

$$\begin{aligned} \Rightarrow f(z) &= e^x (\cos y + i \sin y) \\ &= e^x \cdot e^{iy} \end{aligned}$$

$$\Rightarrow f(z) = e^{x+iy} = e^z$$

$$\Rightarrow f(z) = e^z.$$

Trigonometric Function / Circular Functions

Maclaurine Series expansion of Trigonometric Functions

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

and series expansion of e^z is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

Now

$$\cos z + i \sin z$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right) + i \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$

$$= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} - \dots$$

$$= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \dots$$

$$= e^{iz}$$

$$\Rightarrow \boxed{\cos z + i \sin z = e^{iz}} \quad \text{--- (1)}$$

Similarly, $\boxed{\cos z - i \sin z = e^{-iz}} \quad (2)$

Solving (1) & (2)

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Q Separate the Real and Imaginary part of $\cos(x+iy)$.

Soln

$$\begin{aligned}\cos(x+iy) &= \cos x \cos iy - \sin x \sin iy \\ &= \cos x \cosh y - i \sin x \sinh y\end{aligned}$$

$$\text{Real part} = \cos x \cosh y$$

$$\text{Imaginary part} = -\sin x \sinh y.$$

MAY

Expand $\cos 7\theta$ & $\sin 7\theta$ in terms of $\cos \theta$ & $\sin \theta$.

$$\cos 7\theta + i \sin 7\theta$$

$$= (\cos \theta + i \sin \theta)^7 \quad (\text{By De Moivre's Theorem})$$

By using Binomial Theorem,

$$\begin{aligned} & \cos^7 \theta + {}^7C_1 \cos^6 \theta (i \sin \theta) + \\ & {}^7C_2 \cos^5 \theta (i \sin \theta)^2 + {}^7C_3 \cos^4 \theta (i \sin \theta)^3 \\ & + {}^7C_4 \cos^3 \theta (i \sin \theta)^4 + {}^7C_5 \cos^2 \theta (i \sin \theta)^5 \\ & + {}^7C_6 \cos \theta (i \sin \theta)^6 + {}^7C_7 (i \sin \theta)^7 \end{aligned}$$

$$\begin{aligned} = & \cos^7 \theta + i {}^7C_1 \cos^6 \theta \sin \theta - 21 \cos^5 \theta \sin^2 \theta \\ & - i {}^7C_2 \cos^4 \theta \sin^3 \theta + 35 \cos^3 \theta \sin^4 \theta \\ & + i {}^7C_3 \cos^2 \theta \sin^5 \theta - 7 \cos \theta \sin^6 \theta \\ & - i \sin^7 \theta \end{aligned}$$

MAY

Now equating Real and Imaginary parts on both sides we have

$$\cos 7\theta = \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta$$

$$\begin{aligned} \text{and } \sin 7\theta = & {}^7C_1 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta \\ & + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta \end{aligned}$$

Expansion of $\cos^n \theta$ & $\sin^n \theta$ is

$$\text{Let } x = \cos \theta + i \sin \theta$$

$$\Rightarrow \frac{1}{x} = \frac{1}{\cos \theta + i \sin \theta} = \frac{\cos \theta - i \sin \theta}{1}$$

$$\Rightarrow \frac{1}{x} = \cos \theta - i \sin \theta$$

$$x + \frac{1}{x} = 2 \cos \theta, \quad x - \frac{1}{x} = 2i \sin \theta$$

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MAY

By De Moivre's Theorem,

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$$x^n = \cos n\theta + i \sin n\theta$$

$$\frac{1}{x^n} = \cos n\theta - i \sin n\theta$$

$$x^n + \frac{1}{x^n} = 2 \cos n\theta$$

$$x^n - \frac{1}{x^n} = 2i \sin n\theta$$

To expand $\cos^n \theta$,

$$(2 \cos \theta)^n = \left(x + \frac{1}{x}\right)^n$$

2024
SUNDAYTo expand $\sin^n \theta$,

$$(2i \sin \theta)^n = \left(x - \frac{1}{x}\right)^n$$

Ques Express $\sin^5 \theta$ in terms of sines of multiples of θ .

$$\sin^5 \theta = (2i \sin \theta)^5 = \left(x - \frac{1}{x}\right)^5$$

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By Binomial Theorem,

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$$\Rightarrow 2i \cdot 32 \sin^5 \theta = x^5 + {}^5C_1 x^4 \left(-\frac{1}{x}\right)$$

$$+ {}^5C_2 x^3 \left(-\frac{1}{x}\right)^2 + {}^5C_3 x^2 \left(-\frac{1}{x}\right)^3$$

$$+ {}^5C_4 x \left(-\frac{1}{x}\right)^4 + {}^5C_5 \left(-\frac{1}{x}\right)^5$$

$$\Rightarrow 2 \cdot 32 \sin^5 \theta = x^5 - 5x^4 \cdot \frac{1}{x} + 10x^3 \cdot \frac{1}{x^2}$$

$$- 10x^2 \cdot \frac{1}{x^3} + 5x \cdot \frac{1}{x^4} - \frac{1}{x^5}$$

$$\Rightarrow 2 \cdot 32 \sin^5 \theta = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

$$\Rightarrow 2 \cdot 32 \sin^5 \theta = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$\text{Since } x^n - \frac{1}{x^n} = 2i \sin n\theta$$

$$\Rightarrow 2 \cdot 32 \sin^5 \theta = 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta$$

$$\Rightarrow 16 \sin^5 \theta = \sin 5\theta - 5 \sin 3\theta + 5 \sin \theta$$

So Su Mo Tu We Th Fr Sa Su Mo

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$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)$$

Hence $4+3i = 5e^{i \tan^{-1}(\frac{3}{4})}$

$$(\because z = re^{i\theta})$$

Logarithmic Functions

If $z = x+iy$ be

Logarithmic function is represented by $\ln z$.

The General value of $\log z$ is

$$\boxed{\log z = \log z + 2n\pi i}$$

$$\boxed{\log z = \log z + i \operatorname{Arg} z}$$

Q. Find the value of $\log z$ at $z = -5$.

Solⁿ

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = 0 \quad \operatorname{Arg} z = \pi - \theta = \pi$$

$$\log z = \log |z| + i \operatorname{Arg} z$$

$$\Rightarrow \log(-5) = \log 5 + \pi i$$

In general

$$\log(-5) = \log 5 + (2n+1)\pi i$$

Q. Find the general value of $\log i$

Solⁿ

Let $z = i$

$$r = \sqrt{1^2 + 0^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$$

$$\operatorname{Arg} z =$$

$$\log i = \log 1 + i(2n+1)\frac{\pi}{2}$$

$$\Rightarrow \log i = (2n+1)\frac{\pi i}{2}$$

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Q. Find the general value of $\log(1+i) + \log(1-i)$.

soln

$$1+i =$$

$$\text{then } r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$1+i = \sqrt{2} e^{i\frac{\pi}{4}}$$

$$\log(1+i) = \log(\sqrt{2} e^{i\frac{\pi}{4}})$$

$$= \log \sqrt{2} + \log e^{i\frac{\pi}{4}}$$

$$= \log \sqrt{2} + i \frac{\pi}{4}$$

$$\log(1+i) = \log \sqrt{2} + i \frac{\pi}{4} + 2n\pi i$$

$$= \log \sqrt{2} + (2n\pi + \frac{\pi}{4})i$$

Similarly,

$$\log(1-i) = \log \sqrt{2} + (2n\pi - \frac{\pi}{4})i$$

$$\text{Now, } \log(1+i) + \log(1-i)$$

$$= [\log \sqrt{2} + (2n\pi + \frac{\pi}{4})i] +$$

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$$[\log \sqrt{2} + (2n\pi - \frac{\pi}{4})i]$$

$$= 2 \log \sqrt{2} + 4n\pi i$$

Ans

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Let C be a piecewise smooth path and $f(z)$ be a continuous function on C .

of C : $z = z(t)$,
 & $a \leq t \leq b$.

Then $\int_c^b f(z) dz$

$$= \int_a^b f[z(t)] \cdot \frac{dz}{dt} \cdot dt$$

Ex 8
Consider $f(z) = \frac{1}{z}$.

and C_1 counter clockwise unit circle about O .

$$z \text{ and } 0 \leq t \leq 2\pi.$$

Find $\int_C f(z) dz$

800 $c: e^{it}$ (Parametric form of unit circle about 0)
 $\forall Z(t) = e^{it}$

Now $\int_C f(z) dz$

$$= \int_0^{2\pi} \frac{1}{z} \cdot z e^{it} dt$$

$$= i \int_0^{2\pi} \frac{1}{e^{it}} e^{it} dt \quad (\because z = e^{it})$$

$$= \frac{1}{2} \int_0^{2\pi} dt$$

$$= \int_0^{2\pi} \tau + j_0$$

$$= 2\pi\tau$$

$$\oint_C f(z) dz = \int_C (u+iv)(dx+idy)$$

where $f(z) = u + iv$
 $dz = dx + i dy$

Page 1

eg Find the value of

$$\int_0^{1+i} (x^2 + iy) dz$$

① đường $y = x$ ② đường $y = x^2$

JUNE

② along $y = x^2$
 $\Rightarrow dy = 2x dx$

$$\int_{(0,0)}^{(1,1)} (x^2 + 2y) (dx + i dy)$$

$$= \int_{x=0}^1 (x^2 + i x^2) (dx + 2i x dx)$$

$$= (1+i) \int_0^1 x^2 dx (1+2ix)$$

$$= (1+i) \int_0^1 (x^2 + 2ix^3) dx$$

$$= (1+i) \left[\frac{x^3}{3} + 2i \frac{x^4}{4} \right]_0^1$$

$$= (1+i) \left(\frac{1}{3} + \frac{i}{2} \right)$$

$$= \frac{1}{3} + \frac{i}{2} + \frac{i}{3} - \frac{1}{2}$$

$$= -\frac{1}{6} + \frac{5i}{6}$$

TUESDAY

JUNE

World Day Against Child Labour

Ex 1 Find the value of the integral

$$\int_C (x+y) dx + x y dy$$

① along $y = x^2$ having $(0,0)$, $(3,9)$ end pt

② along $y = 3x$ between the same pts

Ans
~~Sol:~~ ① Along $y = x^2$

$$\Rightarrow dy = 2x dx$$

Now $\int_{(0,0)}^{(3,9)} (x+y) dx + x y dy$

$$= \int_{x=0}^3 (x + x^2) dx + x^4 \cdot 2x dx$$

$$= \int_0^3 (x + x^2 + 2x^5) dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{2x^6}{6} \right]_0^3$$

$$= \frac{9}{2} + \frac{27}{3} + \frac{129}{3} = \frac{513}{2}$$

(b) along $y = 3x$

$$\Rightarrow dy = 3 dx$$

$$\int_0^3 (x + 3x) dx + x^2 \cdot 3x \cdot 3 dx$$

$$= \int_0^3 (4x + 9x^3) dx$$

$$= 2x^2 + \frac{9}{4}x^4 \Big|_0^3$$

$$= 18 + \frac{729}{4}$$

$$= \frac{801}{4}$$

* $\int_C \operatorname{Re} z \, dz$, C : shortest path
from $1+i$ to $3+2i$
i.e. $(1,1)$ to $(3,2)$

Eqn of line is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} = t \text{ (say)}$$

$$\Rightarrow \frac{y-1}{2-1} = \frac{x-1}{3-1} = t$$

$$\Rightarrow \frac{y-1}{1} = \frac{x-1}{2} = t$$

$$\Rightarrow y = t+1, \quad x = 2t+1$$

$$\Rightarrow dy = dt \quad \Rightarrow dx = 2dt$$

$$\text{when } x=1 \quad t=0$$

$$y=1, t=0$$

$$x=3 \quad t=1$$

$$y=2, t=1$$

$$\text{Thus, } \int_0^1 (2t+1) (2dt + i dt)$$

$$= \int_0^1 (2t+1) dt (2+i)$$

$$= (2+i) \int_0^1 (2t+1) dt$$

$$= (2+i) \left[t^2 + t \right]_0^1$$

$$= (2+i) \cdot 2$$

$$= 4 + 2i$$

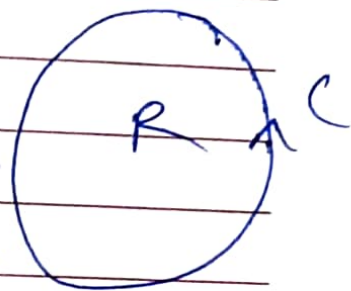
Cauchy's Integral Theorem's

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C , then

$$\int_C f(z) dz = 0.$$

Proof

Let R be the region enclosed by the curve C .



$$\text{Let } f(z) = u + iv$$

$$z = x + iy$$

$$\Rightarrow dz = dx + i dy$$

$$\text{Now } \int_C f(z) dz$$

$$= \int_C (u + iv)(dx + i dy)$$

$$= \int_C (u dx - v dy) + i \int_C (v dx + u dy)$$

By Green's Theorem,

$$\iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

$$+ i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

Since $f(z)$ is analytic,

By C-R equation,

$$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{ and}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\text{so } \int_C f(z) dz =$$

$$\iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx dy +$$

$$i \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y} \right) dx dy$$

$$= 0 + i \cdot 0$$

$$\Rightarrow \int_C f(z) dz = 0$$

(Proved)

Evaluate

$$\oint_C \frac{1}{z^2 - 4} dz, \quad C: |z| = 1$$

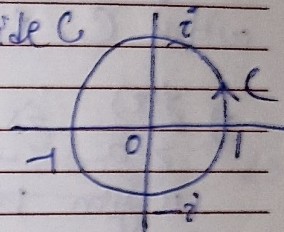
Solⁿ For singularity $z^2 - 4 = 0$
 $\Rightarrow z = \pm 2$

Here $z = 2$ lies outside C

So by Cauchy's

Integral Theorem

$$\oint_C \frac{1}{z^2 - 4} dz = 0$$



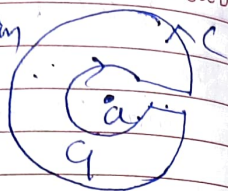
Cauchy's Integral Formula

If $f(z)$ is analytic within and on a closed curve C and if a is any point within C , then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$\text{i.e. } \int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Proof Consider the function $\frac{f(z)}{z-a}$ which is analytic at all points within C , except $z=a$.



$\Rightarrow \frac{f(z)}{z-a}$ is analytic in the region between C and G .

By Cauchy's Integral Theorem for multiply connected region

$$\int_C f(z) dz = \int_G f(z) dz$$

$$\text{Hence } \int_C \frac{f(z)}{z-a} dz = \int_G \frac{f(z)}{z-a} dz$$

$$= \int_G \frac{f(z) - f(a) + f(a)}{z-a} dz$$

$$= \int_G \frac{f(z) - f(a)}{z-a} dz + f(a) \int_G \frac{dz}{z-a}$$

$$\text{Now, } \int_G \frac{f(z) - f(a)}{z-a} dz$$

$$= \int_0^{2\pi} \frac{f(a + re^{i\theta}) - f(a)}{re^{i\theta}} \cdot ire^{i\theta} d\theta$$

$$\begin{aligned} \text{Since } z-a &= re^{i\theta} \\ \Rightarrow z &= a + re^{i\theta} \\ \Rightarrow dz &= ire^{i\theta} d\theta \\ \text{where } r &\rightarrow 0 \end{aligned}$$

$$= i \int_0^{2\pi} [f(a + re^{i\theta}) - f(a)] d\theta$$

$$\text{Since } r \rightarrow 0$$

$$= i \int_0^{2\pi} [f(a) - f(a)] d\theta$$

$$= 0$$

$$\text{And, } \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = i \int_0^{2\pi} d\theta$$

$$\Rightarrow \int_C \frac{dz}{z-a} = 2\pi i$$

Thus,

$$\int_C \frac{f(z)}{z-a} dz = 0 + f(a) \cdot 2\pi i$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

(Proved)

Generalized Form of Cauchy's Integral Formula

2024
SUNDAY

By Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz \quad \text{--- (1)}$$

Differentiating (1) w.r.t a

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$

$$f''(a) = \frac{1!}{2\pi i} \int_C \frac{2f(z)(z-a)}{(z-a)^4} dz$$

$$\Rightarrow f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz$$

$$f'''(a) = \frac{3!}{2\pi i} \int_C \frac{f(z)(z-a)^2}{(z-a)^6} dz$$

$$\Rightarrow f'''(a) = \frac{4!}{2\pi i} \int_C \frac{f(z)}{(z-a)^4} dz$$

$$\Rightarrow f'''(a) = \frac{5!}{2\pi i} \int_C \frac{f(z)}{(z-a)^5} dz$$

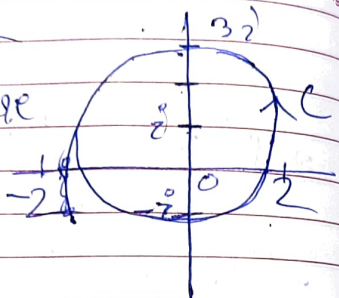
Similarly,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Ex Expression for Derivatives of Analytic function
Ex Evaluate the integral

$$\int_C \frac{z-1}{(z+1)^2(z-2)} dz \quad \text{where } C: |z-i|=2$$

Centre of the cone
= z
Radius = 2



For Singularity,

$$(z+1)^2(z-2)=0$$

$$\Rightarrow Z = -1, -1/2$$

$g(z)$ has two poles at $z = -1$ of order two.

and one simple pole at $z=2$.

Am and are inside C. except Z2

for $z = -1$,

$$\int_C \frac{(z-1)}{(z+1)^2(z-2)} dz = \int_C \frac{\frac{z-1}{z-2}}{(z+1)^2} dz$$

By ⁿ Cauchy's Integral Formula,

$$\int_C \frac{z-1}{(z-2)(z+1)^2} dz$$

$$= \left(\frac{2\pi i}{1!} \cdot \frac{1}{2} \left[\frac{z-1}{z-2} \right] \right)_{z=-1}$$

$$= \left[2\pi i \frac{(z-2) - (z-1)}{(z-2)^2} \right]_{z=1}$$

$$= \left[2\pi i \cdot \frac{-1}{(z-i)^2} \right]_{z=-1}$$

$$\geq 2\pi r \cdot \frac{1}{a}$$

$$= -\frac{2\pi^2}{9}$$

Since $z=2$ lies outside C ,
by Cauchy's Indegner theorem

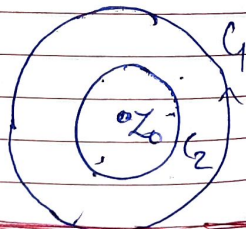
$$\int_C \frac{f(z)}{dz} = 0.$$

Hence $\int_C \frac{z-1}{(z+1)^2(z-2)} dz = -\frac{2\pi i}{9} + 0$
 $= -\frac{2\pi i}{9}$

Laurent's Series

If $f(z)$ is to be expanded about a point where $f(z)$ is not analytic, then it is expanded by Laurent's Series.

* If $f(z)$ is analytic on two concentric circles C_1 and C_2 with centre z_0 and in the annulus between them, then $f(z)$ can be expanded by the Laurent Series.



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

↓
Principal Part

$$\text{ex } f(z) = a_0 + a_1 (z-z_0) + a_2 (z-z_0)^2 + \dots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots$$

THURSDAY

JUNE

FRIDAY 28

$$\text{then } a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw$$

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{-(n+1)}} dw$$

→ w lies in C_1 .

Q. Expand $f(z) = \frac{1}{(z-1)(z-2)}$ for $1 < |z| < 2$.

Solⁿ

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$$= \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= -(z-1)^{-1} + (z-2)^{-1}$$

$$\Rightarrow f(z) = -(z)^{-1} \left(1 - \frac{1}{z}\right) + (z)^{-1} \left(1 - \frac{z}{2}\right)$$

$$\text{since } \frac{1}{|z|} < 1$$

$$f(z) = (z-2)^{-1} \left(1 - \frac{z}{2}\right) - (z)^{-1} \left(1 - \frac{1}{z}\right)$$

JUNE _____ By Binomial Theorem

2024
SATURDAY

$$-\frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \right)$$

$$\rightarrow f(z) = \frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \frac{z^3}{16} \dots$$

$$-\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4} \dots$$

Q. Find the Laurent Series of $z^{-5} \sinh z$ with centre 0.

2024
SUNDAY

So we know

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\text{Now } z^{-5} \sinh z = z^{-5} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} (-1)^n$$

$$\Rightarrow z^{-5} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4}$$

$$\Rightarrow z^5 \sin z = \frac{1}{z^4} - \frac{1}{2z^2} + \frac{1}{24} - \frac{1}{5040} z^4 + \dots$$

2024
MONDAY

National Doctor's Day (India)

Q. Develop $\frac{1}{1-z}$.

(d) in nonnegative powers of z

⑥ in negative powers of z

Soln (a) we know $\frac{1}{1-\epsilon} \geq \sum_{n=0}^{\infty} \epsilon^n$

$$\text{ex } \frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \dots$$

$$\textcircled{b} \quad \frac{1}{Fz} = \frac{1}{-z(1-\frac{1}{z})}$$

$$= -\frac{1}{2} \left(1 - \frac{1}{2}\right)^{-1}$$

$$= \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \right)$$

$$= -\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4} - \dots$$

Q. Write an possible Laurent series for the function .

$$f(z) = \frac{1}{z(z+2)^3}$$

about the pole $z = -2$.

$$\text{Soln } f(z) = \frac{1}{z(z+2)^3}$$

$$\text{Let } z+2 = t \\ \Rightarrow z = t-2$$

$$f(z) = \frac{1}{z(z+2)^3}$$

$$= \frac{1}{(t-2)t^3}$$

$$= \frac{1}{t^3} \cdot \frac{1}{(t-2)(1-\frac{t}{2})}$$

$$= -\frac{1}{2t^3} \left(1 - \frac{t}{2}\right)^{-1}$$

$$= -\frac{1}{2t^3} \left(1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \dots\right)$$

$$= -\frac{1}{2t^3} - \frac{1}{4t^2} - \frac{1}{8t} - \frac{1}{16} - \frac{t}{32} \dots$$

$$\Rightarrow f(z) = -\frac{1}{2(z+2)^3} - \frac{1}{4(z+2)^2} - \frac{1}{8(z+2)} - \frac{1}{16} \\ - \frac{(z+2)}{32} - \frac{(z+2)^2}{64} - \dots$$

Singularity

The point at which the function $f(z)$ is not analytic is known as a Singular Point or Singularity of the function.

$$\text{eg } f(z) = \frac{z^2}{z(z-2)}$$

$$\text{For singularity } z(z-2) = 0 \\ \Rightarrow z = 0, 2$$

So $z = 0, 2$ are singular points of $f(z)$.

Isolated Singularity

If the function $f(z)$ has a singularity at $z=a$ and there is

no other singularity within a small curve centre at $z=a$.

Then $z=a$ is an isolated Singularity of $f(z)$.

$$\text{eg } f(z) = \frac{1}{z(z-2)}$$

Here $z=0$ & $z=2$ are isolated singularities.

Non-Isolated Singularity

The singularity which is not isolated is called non-isolated singularity.

eg: $f(z) = \frac{1}{\sin(\frac{z}{2})}$

sin singularity $\sin(\frac{z}{2}) = 0$

$$\Rightarrow \frac{z}{2} = n\pi \Rightarrow z = \frac{1}{n}$$

for $n \in \mathbb{N}$, $z = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

~~$z = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, \dots$~~

are the points of singularity.

Here $z=0$ is an non-isolated singularity since in the

neighbourhood of $z=0$ there is

infinite no. of other singularity
 $z = \frac{1}{n}$

Pole of order m

if $f(z)$ has an isolated singular point $z=a$, $f(z)$ can be expanded

in Laurent's series around $z=a$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \dots + \frac{b_m}{(z-a)^m} \quad (b_m \neq 0)$$

* Then $z=a$ is said to be a pole if the Principal Part of Laurent

series ~~is~~ has only finitely many terms.

m is called order of pole.

* If $m=1$, the pole is called a Simple pole.

* If the Principal Part (i.e. terms of negative powers in expansion) is infinite ~~then~~ then $z=a$ is called an essential singularity of $f(z)$.

Method of Finding Residues

① Residue at Simple Pole,

if $f(z)$ has a simple pole at $z=a$

$$\text{then } \underset{z=a}{\text{Res } f(z)} = \lim_{z \rightarrow a} (z-a) f(z).$$

ex Find Residue of
 $f(z) = \frac{z^2}{z-4}$

Solⁿ For singularity $z-4=0$
 $\Rightarrow z=4$

$z=4$ is a simple pole.

$$\begin{aligned} \text{so } \text{Res}(z=4) &= \lim_{z \rightarrow 4} (z-4) f(z) \\ &= \lim_{z \rightarrow 4} (z-4) \frac{z^2}{(z-4)} \\ &= 16 \end{aligned}$$

② Residue at a pole of order n if

if $f(z)$ has a pole of order n at $z=a$, then

$$\text{Res}(at z=a) = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z) \right]_{z=a}$$

ex Find Residue of

$$f(z) = \frac{z^2+3}{z(z+3)^2}$$

Solⁿ For singularity

$$z(z+3)^2 = 0$$

$$\Rightarrow z=0, -3, -3$$

Here $z=0$ is a simple pole

& $z=-3$ is a pole of order 2

$$\begin{aligned} \text{Now } \text{Res}(z=0) &= \lim_{z \rightarrow 0} (z-0) \frac{z^2+3}{(z+3)^2} \\ &= \lim_{z \rightarrow 0} \frac{z^2+3}{(z+3)^2} \\ &= \frac{1}{3} \end{aligned}$$

$$\text{Res}_{(z=-3)}^{\text{at}}$$

$$= \frac{1}{(2-1)!} \left[\frac{d}{dz} (z+3)^2 f(z) \right]_{z=-3}$$

$$= \left[\frac{d}{dz} (z+3)^2 \frac{z^2+3}{z(z+3)^2} \right]_{z=-3}$$

$$= \left[\frac{d}{dz} \left(\frac{z^2+3}{z} \right) \right]_{z=-3}$$

$$= \left[\frac{z \cdot 2z - z^2 - 3}{z^2} \right]_{z=-3}$$

$$= \frac{z^2-3}{z^2} \Big|_{z=-3}$$

$$= \left(1 - \frac{3}{z^2} \right)_{z=-3}$$

$$= \left(1 - \frac{1}{9} \right)$$

$$= \frac{8}{9}$$

Residue Theorem

If $f(z)$ is analytic in a closed curve C , except at a finite no. of poles within C , then

$$\int_C f(z) dz = 2\pi i \sum_{j=1}^k \text{Res}_{z=z_j} f(z)$$

$$\text{i.e. } \int_C f(z) dz = 2\pi i \left(\text{Sum of residues at the poles within } C \right)$$

Ex 1

Evaluate the following integral using Residue Theorem.

$$\oint_C \frac{4-3z}{z(z-1)(z-2)} dz$$

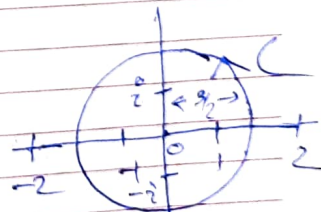
where $C: |z| = \frac{3}{2}$.

Sol

For singularity

$$z(z-1)(z-2) = 0$$

$$\Rightarrow z = 0, 1, 2$$



Here $z_0 = 0, 1, 2$ are simple poles.
 $z = 2$ lies outside C .

$$\text{So, } R_1 = \text{Res (at } z=2) = 0$$

$$\text{Now, } R_2 = \text{Res (at } z=0) = \lim_{z \rightarrow 0} z f(z)$$

$$= \lim_{z \rightarrow 0} z \frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z \rightarrow 0} \frac{4-3z}{(z-1)(z-2)}$$

$$= 2$$

&

$$R_3 = \text{Res (at } z=1)$$

$$= \lim_{z \rightarrow 1} (z-1) f(z)$$

$$= \lim_{z \rightarrow 1} (z-1) \frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z \rightarrow 1} \frac{4-3z}{z(z-2)}$$

$$= \frac{-1}{1} = -1$$

$$\text{Now, } \oint_C \frac{4-3z}{z(z-1)(z-2)} dz = 2\pi i (R_1 + R_2 + R_3)$$

$$\Rightarrow \oint_C \frac{4-3z}{z(z-1)(z-2)} dz = 2\pi i (0 + 2 + (-1)) = 2\pi i$$

Q Evaluate

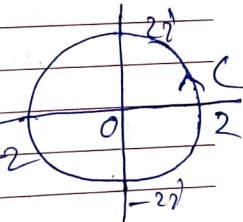
$$\oint_C \frac{z^2 \sinh z}{4z^2 - 1} dz \quad C: |z|=2$$

using Residue

Solⁿ For Singularity $4z^2 - 1 = 0$
 $\Rightarrow z = \pm \frac{1}{2}$

$z = \frac{1}{2}$ and $z = -\frac{1}{2}$ both

are simple pole and are inside C.



$$\text{Now, } R_1 = \text{Res (at } z = \frac{1}{2})$$

$$= \lim_{z \rightarrow \frac{1}{2}} (z - \frac{1}{2}) f(z)$$

$$= \lim_{z \rightarrow \frac{1}{2}} (z - \frac{1}{2}) \frac{z^2 \sinh z}{4(z - \frac{1}{2})(z + \frac{1}{2})}$$

$$= \lim_{z \rightarrow \frac{1}{2}} \frac{z^2 \sinh z}{4(z + \frac{1}{2})}$$

$$= \frac{\frac{1}{4} \sinh \frac{1}{2}}{4}$$

$$= \sinh \frac{1}{2}$$

$$R_2 = \text{Res (at } z = -\frac{1}{2})$$

$$= \lim_{z \rightarrow -\frac{1}{2}} (z + \frac{1}{2}) \frac{z^2 \sinh z}{4(z + \frac{1}{2})(z - \frac{1}{2})}$$

$$= \lim_{z \rightarrow -\frac{1}{2}} \frac{z^2 \sinh z}{4(z - \frac{1}{2})}$$

$$= \frac{\frac{1}{4} \sinh(-\frac{1}{2})}{-4}$$

$$= \sinh \frac{1}{2}$$

$$\begin{aligned} \text{Thus } \oint_C \frac{z^2 \sinh z}{4z^2 - 1} dz &= 2\pi i (R_1 + R_2) \\ &= 2\pi i (\sinh 1 + \sinh 1) \\ &= 4\pi i \sinh 1 \end{aligned}$$