

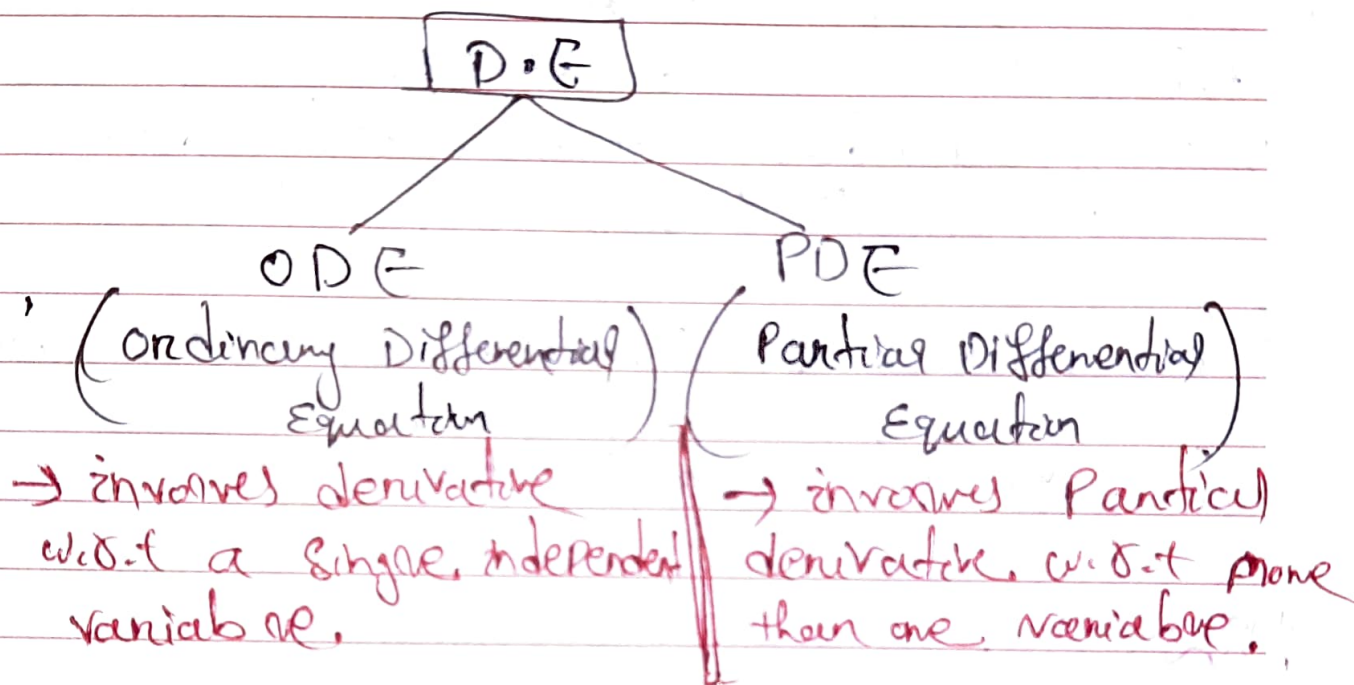
Differential Equation

Defⁿ:

An equation that contains one or more functions with it's derivatives is called differential equation.

$$\text{e.g. } \frac{dy}{dx} + \frac{x}{y} = 0.$$

i.e. An equation involving dependent variables, independent variables and derivative of dependent variables is called an differential equation.



Order &

The order of the highest order derivative in a D.E is called the order of the D.E.

$$\text{e.g. } \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = x^2 \text{ is 2nd order D.E.}$$

Degree's

The degree of the highest derivative is called the degree of the D.E.

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if $\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + y = 0$ is 2nd degree D.E

N.B.

* A D.E contains following types of terms

$$\log\left(\frac{dy}{dx}\right), e^{\left|\frac{dy}{dx}\right|}, \sin\left(\frac{dy}{dx}\right), \cos\left(\frac{dy}{dx}\right)$$

then degree of this D.E doesn't exist.

Linear Differential Equation is

The D.E is called linear if

- ① Every dependent variable & every derivative occurs in 1st degree only.
- ② No product of dependent variables & derivatives

$$\text{e.g. } \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = \sin x$$

Non-Linear Differential Equation is

The D.E which is not linear is called non-linear D.E.

$$\text{e.g. } y\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} = \sin x$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right) = 0$$

Solution of Differential Equation is WEDNESDAY

Any resⁿ betⁿ dependent variables & independent variables, when substituted in the D.E, reduce it to an identity is called a solution of D.E.

General Solution is

It is the relation between the dependent & independent variables which is obtained after removing the derivatives (i.e. integration) where the relation contains arbitrary constant to denote the order of the equation.

$$\text{e.g. } y = ce^{2x} \text{ is the general solⁿ of } \frac{dy}{dx} = 2y.$$

Particular Solution is

A solution obtained from general solution by giving particular value of arbitrary constant is called Particular solution.

Singular Solution is

A solution which can't be obtained from any general solution by any choice of constant.

$$\text{e.g. } y = 0 \text{ is a singular solution of } \left(\frac{dy}{dx}\right)^2 - y = 0.$$

Equations of 1st order & 1st Degree is

General form is $\frac{dy}{dx} = f(x, y)$ or $M dx + N dy = 0$.

Methods to solve 1st order & 1st degree D.E.

(1) Variable Separable form

Step-1: An eqⁿ whose variables are separable
Put in the form $g(x)dx + h(y)dy = 0$.

This eqⁿ is called an eqⁿ of variable separable form.

Step-1) Integrate both sides.

$$\int g(x) dx + \int h(y) dy = C$$

where C is an arbitrary constant.
This is the general solution of the D.E.

Ex: $\frac{dy}{dx} = \frac{1-x}{y}$

$$\Rightarrow y dy = (1-x) dx$$

Integrations, $\int y dy = \int (1-x) dx$

$$\Rightarrow \frac{y^2}{2} = x - \frac{x^2}{2} + C$$

$$\Rightarrow y^2 = 2x - x^2 + C'$$

Eqⁿ Reducible to variable separable form

Let the D.E be $\frac{dy}{dx} = f(ax+by+c)$
To solve, put $ax+by+c = z$
D.w.r.t x

$$a + b \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b} \left(\frac{dz}{dx} - a \right)$$

Putting it in (1), $\frac{1}{b} \left(\frac{dz}{dx} - a \right) = f(z)$

$$\Rightarrow \frac{dz}{dx} = bf(z) + a$$

which is of the type "variable separable"
or by putting $y = vx$ in some questions where $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$
can be reduced to variable separable form.

Ex: $\frac{dy}{dx} = (x+y)^2$

Let $x+y = v$

Differentiating w.r.t x we have

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\Rightarrow \frac{dv}{dx} - 1 = v^2 \Rightarrow \frac{dv}{dx} = v^2 + 1$$

$$\Rightarrow \int \frac{dv}{v^2+1} = \int dx$$

$$\Rightarrow \tan^{-1} v = x + C$$

$$\Rightarrow \tan^{-1}(x+y) = x + C$$

$$\Rightarrow x+y = \tan(x+C)$$

$$\Rightarrow y = \tan(x+C) - x$$

Homogenous Function :-

Homogeneous Function
A funⁿ $f(x, y)$ is said to be homogeneous
funⁿ of n^{th} degree in x and y if it

Can be put in the form $a^2 + \left(\frac{y}{a}\right)^2$

ex $f(x,y) = x^4 + x^2y^2 + xy^3 + y^4$ is a homogeneous funⁿ of degree 4.

* An equation in x and y is said to be homogeneous equation if it can be put in the form.

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

where $f(x, y)$ and $\phi(x, y)$ are both homogeneous function of same degree in x, y .

Methods to solve:

Step-1 : To solve $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ — (1)

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = r + n \frac{dr}{dx}$$

Step-4 $\rightarrow$ Substitute in ① and solve by variable separable method.

egs solve $x^2y dx - (x^3 + y^3) dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3} \quad \text{--- (1)}$$

Here xy & x^3+y^3 are homogeneous funⁿ in xy of degree 3.

net $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + u \frac{dv}{du}$$

from (1),

$$v + x \frac{dv}{dx} = \frac{vx^3}{x^3 + vx^3} = \frac{v}{1+v^3}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v}{1+v^3} - v = -\frac{v^4}{1+v^3}$$

$$\Rightarrow \int \frac{1+r^3}{v^4} dv = - \int \frac{dx}{x}$$

$$\Rightarrow \int \left(\frac{1}{u} + \frac{1}{v} \right) dv = -\log u + C$$

$$\Rightarrow -\frac{1}{3v^3} + \log v = -\log u + C$$

$$\Rightarrow -\frac{1}{3x^3} + \log vx = C$$

$$\Rightarrow -\frac{1}{3} \frac{x^3}{y^3} + \log y = C$$

which is the required solution.

Equation reducible to Homogeneous Equation:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \quad \text{--- (1)}$$

Case-1 If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, we have

Linear Differential Equation of 1st orderThe general form is $\frac{dy}{dx} + P(x)y = Q(x)$.If $Q(x) = 0$

→ It is called homogeneous L.D.E.

If $Q(x) \neq 0$

→ It is called non-homogeneous L.D.E.

* The coefficient of $\frac{dy}{dx}$ should be 1.
Form-1 (Linear in y).

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{--- (1)}$$

Step-1 Find I.F (Integrating Factor)

$$= e^{\int P(x) dx}$$

Step-2 Find solution,

$$y(I.F) = \int Q(x) \cdot (I.F) dx$$

Form-2 (Linear in x)

$$\frac{dx}{dy} + P(y)x = Q(y)$$

Step-1 Find I.F = $e^{\int P(y) dy}$ Step-1 Find solution

$$x(I.F) = \int Q(y) \cdot (I.F) dy$$

Q1 Solve $(1+x^2) \frac{dy}{dx} + 2xy - 4x^2 = 0, y(0) = 0$.Sol This given D.E can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{4x^2}{1+x^2} \quad \text{--- (1)}$$

$$\text{Here } P(x) = \frac{2x}{1+x^2}, \quad Q(x) = \frac{4x^2}{1+x^2}$$

$$\text{Now, I.F} = e^{\int P(x) dx}$$

$$= e^{\int \frac{2x}{1+x^2} dx}$$

$$= e^{\log(1+x^2)} = (1+x^2)$$

The solution is

$$y(1+x^2) = \int \frac{4x^2}{1+x^2} \cdot (1+x^2) dx$$
$$= \int 4x^2 dx$$

$$\Rightarrow y(1+x^2) = \frac{4x^3}{3} + C \quad \text{--- (2)}$$

From (2) given $y(0) = 0$.

$$\Rightarrow 0 = C$$

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* No. of integrating factor of $Mdx + Ndy = 0$ is infinite.

Rules for finding I.F.

Rule-I $Mdx + Ndy = 0$

$$\text{if } \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \text{ only}$$

$$\text{Then I.F.} = e^{\int f(x) dx}$$

Rule-II $Mdx + Ndy = 0$

$$\text{if } \frac{\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}}{M} = f(y) \text{ only}$$

$$\text{Then I.F.} = e^{\int f(y) dy}$$

Rule-III

if the D.E $Mdx + Ndy = 0$ is of the form $y f_1(xy) dx + x f_2(xy) dy = 0$

$$\text{Then I.F.} = \frac{1}{Mx - Ny} \text{ where } Mx - Ny \neq 0$$

Rule-IV

if the D.E $Mdx + Ndy = 0$ is homogeneous in x and y of same degree.

$$\text{then } \frac{1}{Mx + Ny} = \text{I.F.}$$

where $Mx + Ny \neq 0$.

e.g. $2 \sin wy dx + w \cos wy dy = 0$, $y(0) = \frac{\pi}{2w}$

Here $M = 2 \sin wy$, $N = w \cos wy$

$$\frac{\partial M}{\partial y} = 2w \cos wy, \quad \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ This D.E is not exact

$$\text{Now, } \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2w \cos wy - 0}{w \cos wy} = 2$$

$$\text{I.F.} = e^{\int 2 dx} = e^{2x}$$

now multiplying the I.F. e^{2x} we have

$$2e^{2x} \sin wy dx + we^{2x} \cos wy dy = 0$$

Now the solution is

$$\int 2e^{2x} \sin wy dx + \int we^{2x} \cos wy dy = C$$

(y as const.) (terms independent of x)

$$\Rightarrow e^{2x} \sin wy = C$$

Given that $y(0) = \frac{\pi}{2w}$

$$\Rightarrow e^0 \sin(w \cdot \frac{\pi}{2w}) = C$$

$$\Rightarrow \sin \frac{\pi}{2} = C \Rightarrow 1 = C$$

Hence, $e^{2x} \sin wy = 1$

which is the required solution.

Q: Integrating factor of $(x^7y^2 + 3y)dx + (3x^8y - x)dy$ is $x^m y^n$, then $m = ?$, $n = ?$

Sol: If $x^m y^n$ is an I.F. of the D.E

$$(x^7y^2 + 3y)dx + (3x^8y - x)dy = 0$$

then $x^m y^n (x^7y^2 + 3y)dx + x^m y^n (3x^8y - x)dy = 0$ will be exact.

$$\Rightarrow (x^{7+m}y^{2+n} + 3x^m y^{1+n})dx + (3x^{8+m}y^{1+n} - x^{1+m}y^n)dy = 0$$

will be exact.

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow (2+n)x^{7+m+1+n}y^{2+n} + 3(1+n)x^m y^{1+n} = 3(8+m)x^{7+m}y^{1+n} - (1+m)x^{1+m}y^n$$

Now, comparing both sides, (coefficients)

$$2+n = 3(8+m) \quad \text{and} \quad 3(1+n) = -(1+m)$$

$$\Rightarrow n-3m = 22 \quad \text{--- ①} \quad \Rightarrow 3n+m = -4 \quad \text{--- ②}$$

solving ① & ② we have

$$\begin{array}{r} n-3m = 22 \\ 3 \times \text{eq ②} (+) \quad 9n+3m = -12 \\ \hline 10n = 10 \end{array}$$

$$10n = 10$$

$$\Rightarrow n=1 \quad \text{and} \quad m=-7$$

Hence $n=1$ and $m=-7$

Ans

Terms

Time (t)

Electric charge (q)

Current (i)

Resistance (R) $\pm \frac{V}{i} = \boxed{\text{ohm}}$

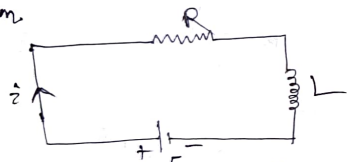
Inductance (L) $\pm \frac{V}{\frac{di}{dt}} = \boxed{\text{Henry (H)}}$

Capacitance (C) $= \frac{Q}{V}$

Electromotive force ($\mathcal{E}$).

R-L Circuit

Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) $\mathcal{E}$ as shown in diagram.



Let " i " be the current flowing in the circuit at any time " t ".

Then we know,

Voltage drop across resistance $R = Ri$

Voltage drop across inductance $L = L \frac{di}{dt}$

Applied voltage (e.m.f) $= \mathcal{E}$.

Kirchoff's Voltage Law

The summation of voltage drop across the closed circuit is equal to resultant emf in the circuit.

$$L \frac{di}{dt} + Ri = \mathcal{E} \quad \text{--- (1)}$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{\mathcal{E}}{L} \quad (\text{linear D.E in } i)$$
$$I.F = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$$

Now, the solution is,

$$i \cdot e^{\frac{Rt}{L}} = \int \frac{\mathcal{E}}{L} \cdot e^{\frac{Rt}{L}} dt$$
$$= \frac{\mathcal{E}}{L} \int e^{\frac{Rt}{L}} dt$$
$$= \frac{\mathcal{E}}{L} \cdot e^{\frac{Rt}{L}} \times \frac{1}{R} + A$$

$$\Rightarrow i \cdot e^{\frac{Rt}{L}} = \frac{\mathcal{E}}{R} e^{\frac{Rt}{L}} + A \quad \text{--- (2)}$$

multiplied by $e^{-\frac{Rt}{L}}$ to eqn (2) - we have

$$i = \frac{\mathcal{E}}{R} + A e^{-\frac{Rt}{L}}$$

Initially, $t=0$, $i=0$.

$$0 = \frac{\mathcal{E}}{R} + A \cdot e^0$$

$$\Rightarrow A = -\frac{\mathcal{E}}{R}$$

$$\text{Thus } i = \frac{\mathcal{E}}{R} - \frac{\mathcal{E}}{R} e^{-\frac{Rt}{L}}$$

$$\Rightarrow i = \frac{\mathcal{E}}{R} \left[1 - e^{-\frac{Rt}{L}} \right]$$

This is the expression of current at any time t .

T Q.1 A resistance of 100 ohms, an inductance of 0.5 Henry & one connected in series with a battery of 20 volts. Find the current in a circuit as a function of t .

Solⁿ

$$R = 100 \Omega$$

$$L = 0.5 H$$

$$E = 20 V$$

$$I = ?$$

$$I = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}} \right)$$

$$= \frac{20}{100} \left(1 - e^{-\frac{100t}{0.5}} \right)$$

$$\Rightarrow I = \frac{1}{5} \left(1 - e^{-200t} \right)$$

Q.2 An electric circuit contains an inductance $L = 0.5$ Henry & Resistance $R = 10 \Omega$ in series with emf of 20 volts. Find the current at any time t .

Solⁿ

$$L = 0.5 H$$

$$R = 10 \Omega$$

$$E = 20 V$$

$$I = ?$$

$$I = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}} \right)$$

$$= \frac{20}{10} \left(1 - e^{-\frac{10t}{0.5}} \right)$$

$$\Rightarrow I = 2 \left(1 - e^{-20t} \right)$$

current at any time t .

Newton's Law of Cooling

The rate of change of temp. of a body (heat loss) is directly proportional to the difference in the temperature between the body and its surrounding.

$$\text{i.e. } \frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\Rightarrow \frac{d\theta}{dt} = -K(\theta - \theta_0)$$

(may be $\theta < \theta_0$)

Ex.1 Water at temp. $100^\circ C$ cools in 10 minutes to $88^\circ C$ in a room temp. $25^\circ C$. Find the temp. of water after 20 minutes.

Solⁿ

$\theta^\circ C$	$100^\circ C$	$88^\circ C$	$?$
t	0	10	20

Here $\theta_0 = 25^\circ C$.

Now by Newton's Law of Cooling,

$$\frac{d\theta}{dt} = -K(\theta - 25)$$

$$\Rightarrow \frac{d\theta}{\theta - 25} = -K dt$$

Integrating both sides,

$$\int \frac{d\theta}{\theta - 25} = - \int K dt$$

$$\Rightarrow \int_{100}^{88} \frac{d\theta}{\theta - 25} = - \int_0^{10} K dt$$

$$\Rightarrow \log(\theta - 25) \Big|_{100}^{88} = -K [t]_0^{10}$$

$$\Rightarrow \log 63 - \log 75 = -10K$$

$$\Rightarrow \log \left(\frac{63}{75} \right) = -10K$$

$$\Rightarrow K = -\frac{1}{10} \log \left(\frac{63}{75} \right)$$

Now

$$\int_{100}^{\theta} \frac{d\theta}{\theta - 25} = - \int_0^{20} K dt$$

$$\Rightarrow \log(\theta - 25) \Big|_{100}^{\theta} = -K [t]_0^{20}$$

$$\Rightarrow \log(\theta - 25) - \log 75 = -20K$$

$$\Rightarrow \log\left(\frac{\theta - 25}{75}\right) = -20 \left[-\frac{1}{10} \log\left(\frac{63}{75}\right) \right]$$

$$\Rightarrow \log\left(\frac{\theta - 25}{75}\right) = 2 \log\left(\frac{63}{75}\right)$$

$$\Rightarrow \frac{\theta - 25}{75} = \frac{63^2}{75^2}$$

$$\Rightarrow \theta - 25 = \frac{63 \times 63 \times 75}{75 \times 75}$$

$$\Rightarrow \boxed{\theta = \frac{63 \times 63}{75} + 25} = 77.92^\circ\text{C}$$