

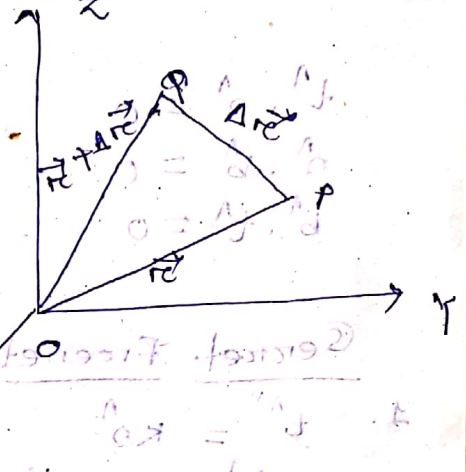
DIFFERENTIAL GEOMETRY

04/01/2019

A space curve may be defined as the locus of a point whose position vector $\vec{r}$ relative to fixed origin may be expressed as a function of single parameter then its cartesian co-ordinates are also funⁿ of same parameter.

If $\vec{r}$ is a position vector of point P in 3D euclidean space. then the eqⁿ of space curve is given by

$$\vec{r}(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$$



Tangent :-

The vector eqⁿ of the tangent at P is given by

$$\vec{R} = \vec{r} + \lambda \hat{t}$$

where $\vec{R}$ is the current pt on the tangent

$\vec{r}$ is the position vector at point P

λ is a scalar variable

The vector eqⁿ of the principal normal at P is given by

$$\vec{R} = \vec{r} + \lambda \hat{n}$$

where $\vec{R}$ is the current point on the normal

$\vec{r}$ is a position vector at point P

λ is a scalar variable

The vector equation of Binormal at P. is

given by

$$\vec{R} = \vec{r} + \lambda \hat{b}$$

$$\hat{b} = \hat{t} \times \hat{n}$$

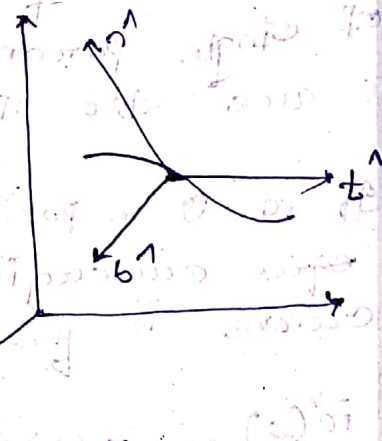
$$\hat{t} = \hat{n} \times \hat{b}$$

$$\hat{n} = \hat{b} \times \hat{t}$$

$$\hat{t} \cdot \hat{n} = 0$$

$$\hat{n} \cdot \hat{b} = 0$$

$$\hat{b} \cdot \hat{t} = 0$$



Serret Frenet Formulae-

1. $\hat{t}' = \kappa \hat{n}$ κ - curvature

2. $\hat{n}' = -\tau \hat{b} - \kappa \hat{t}$ τ - Torsion

3. $\hat{b}' = -\tau \hat{n}$

Proof - We know that

1. $\hat{t}' = \hat{r}'$

$$\hat{t}' = \hat{r}'' = \frac{d\hat{t}}{ds} = \kappa \hat{n} \quad (\because \frac{d\hat{t}}{ds} \text{ Parallel to } \hat{n})$$

$$\boxed{\hat{t}' = \kappa \hat{n}}$$

3- we know that $\hat{b}$ is perpendicular to $\hat{t}$

$$\text{i.e. } \hat{b} \cdot \hat{t} = 0$$

$\Rightarrow$ differentiate w.r.t s

$$\frac{d}{ds} (\hat{b} \cdot \hat{t}) = 0$$

$$\Rightarrow \frac{d\hat{b}}{ds} \cdot \hat{t} + \hat{b} \cdot \frac{d\hat{t}}{ds} = 0$$

$$\frac{d}{ds} \cdot \hat{t} + b \cdot (k \hat{n}) = 0$$

$$\Rightarrow \frac{db}{ds} \cdot \hat{t} + (b \cdot n) k = 0$$

$$\Rightarrow \frac{db}{ds} \cdot \hat{t} = 0$$

$$\Rightarrow b \cdot \hat{t} = 0 \quad (b \text{ is perpendicular to } \hat{t})$$

b is ^{also} parallel to $\hat{n}$

$$b = -\tau \hat{n} \quad (\text{where } \tau \text{ is called Torsion})$$



$$\textcircled{2} \quad \hat{n} = \hat{b} \times \hat{t}$$

differentiating with respect to s

$$\frac{d\hat{n}}{ds} = \frac{db}{ds} \times \hat{t} + \hat{b} \times \frac{d\hat{t}}{ds}$$

$$\Rightarrow \hat{n}' = (-\tau \hat{n}) \times \hat{t} + \hat{b} \times (k \hat{n})$$

$$\boxed{\hat{n}' = \tau \hat{b} - k \hat{t}}$$

Find out the expression for $\hat{r}', \hat{r}'', \hat{r}'''$ and prove $[\hat{r}', \hat{r}'', \hat{r}'''] = k \hat{r}$

$$\hat{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$\hat{r}' = \frac{d\hat{r}}{ds} = \hat{t} = \frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} + \frac{dz}{ds} \hat{k}$$

$$\hat{r}'' = \frac{d\hat{r}'}{ds} = \frac{d\hat{t}}{ds} = k \hat{n}$$

$$\hat{r}''' = \frac{d}{ds} (k \hat{n})$$

$$= k' \hat{n} + k \frac{d\hat{n}}{ds}$$

$$= k' \hat{n} + k (\tau \hat{b} - k \hat{t})$$

$$[\hat{r}', \hat{r}'', \hat{r}''']$$

$$= \hat{r}' \cdot (\hat{r}'' \times \hat{r}''')$$

$$= \hat{r}' \cdot (k \hat{n} \times (k' \hat{n} + k (\tau \hat{b} - k \hat{t})))$$

$$\vec{r}'' \times \vec{r}''' = k \hat{n} \times (k' \hat{n} + k \tau \hat{b} - k^2 \hat{t})$$

$$= k^2 \hat{t} + k^3 \hat{b}$$

$$\vec{r}' \cdot (\vec{r}'' \times \vec{r}''') = \hat{t} \cdot (k^2 \hat{t} + k^3 \hat{b})$$

$$= k^2$$

$$\begin{aligned} \vec{r}'''' &= \frac{d}{ds} (k' \hat{n} + k \tau \hat{b} - k^2 \hat{t}) \\ &= k'' \hat{n} + k' \frac{d\hat{n}}{ds} + k \tau \frac{d\hat{b}}{ds} + k' \tau \hat{b} + k \tau' \hat{b} - 2k k' \hat{t} - k^2 \frac{d\hat{t}}{ds} \\ &= k'' \hat{n} + k' (\tau \hat{b} - k \hat{t}) + k \tau (-\tau \hat{n}) + k' \tau \hat{b} + k \tau' \hat{b} - 2k k' \hat{t} - k^2 (-k \hat{n}) \\ &= k'' \hat{n} + k' \tau \hat{b} - k' k \hat{t} - \tau^2 k \hat{n} + k' \tau \hat{b} + k \tau' \hat{b} - 2k k' \hat{t} - k^3 \hat{n} \\ &= (k'' - \tau^2 k - k^3) \hat{n} + (k' \tau + 2k \tau') \hat{b} - 3k k' \hat{t} = \vec{t}'''' \end{aligned}$$

Q. Prove that the usual notation

$$[\hat{t}' \hat{t}'' \hat{t}'''] = [\vec{r}'' \vec{r}''' \vec{r}'''] = k^5 \left(\frac{\tau}{k} \right)'$$

$$\vec{r}'' \times \vec{r}''' = k^2 \hat{t} + k^3 \hat{b}$$

$$\text{Now } [\vec{r}'' \vec{r}''' \vec{r}''']$$

$$= (\vec{r}'' \times \vec{r}''') \cdot \vec{r}''''$$

$$= (k^2 \hat{t} + k^3 \hat{b}) \cdot \{ (k'' - \tau^2 k - k^3) \hat{n} + (k' \tau + 2k \tau') \hat{b} - 3k k' \hat{t} \}$$

$$= (-3x^3x'z + x^3(xz' + 2x'z)) - 3x^3x'z + x^4z' + 2x^3x'z$$

$$= x^4z' - 3x^3x'z$$

$$= x^3(xz' - x'z)$$

$$= x^3 \frac{(xz' - x'z)}{k^2}$$

$$= k^5 \frac{d}{ds} \left(\frac{z}{k} \right)$$

$$= k^5 \left(\frac{z}{k} \right)'$$

Now $(t^I \times t^{II}) = k^2 t^I + x^3 b^I$

Now $[t^I, t^{II}, t^{III}] = (k^5 \left(\frac{z}{k} \right)') b^I$

Q. Prove with vector notation $[b^I, b^{II}, b^{III}] = z \left(\frac{z}{k} \right)'$

Proof:-

$$b^I = \frac{db^I}{ds} = (-zn^I)$$

$$b^{II} = \frac{d^2 b^I}{ds^2} = \frac{d}{ds} (-zn^I)$$

$$= -z \left(\frac{dn^I}{ds} \right) - n^I \frac{dz}{ds}$$

$$= -z (zn^I - kt^I) - n^I z'$$

$$= -z^2 n^I + zkt^I - n^I z'$$

$$b^{III} = \frac{d}{ds} (-z^2 n^I + zkt^I - n^I z')$$

$$= -2zz'n^I - z^2 (zn^I)' + zK(kn^I) + zkt^I + kt^I z'$$

$$= -2zz'n^I + z^3 n^I + zK^2 n^I + zkt^I + kt^I z'$$

$$- n^I z'' - z z' n^I + K z t^I$$

$$= -3zz'n^I + n^I (z^3 + zK^2 - z'') + zt^I (2Kz' + z'K)$$

$$\begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} \times \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} = -\tau^1 \times \begin{pmatrix} -\tau^2 b^1 + \tau^1 \kappa \tau^1 - \tau^1 \tau^1 \end{pmatrix}$$

$$= \tau^3 \tau^1 + \tau^2 \kappa b^1$$

$$\begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} \times \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} \cdot b^1$$

$$= (\tau^3 \tau^1 + \tau^2 \kappa b^1) \cdot (-\tau^2 \tau^1 b^1 + \tau^1 \tau^1 (-\tau^3 + \kappa^2 \tau^1 - \tau^1))$$

$$+ \tau^1 (2\kappa \tau^1 + \kappa^1 \tau^1)$$

$$= \tau^3 (2\kappa \tau^1 + \kappa^1 \tau^1) - 3\tau^3 \kappa \tau^1$$

$$= 2\kappa \tau^3 \tau^1 + \tau^4 \kappa^1 - 3\tau^3 \kappa \tau^1$$

$$= \tau^4 \kappa^1 - \tau^3 \kappa \tau^1$$

$$= \tau^3 (\kappa^1 \tau^1 - \kappa \tau^1)$$

$$= \tau^5 \frac{(\kappa^1 \tau^1 - \kappa \tau^1)}{\tau^2}$$

$$= \tau^5 \frac{d}{ds} \left(\frac{\kappa}{\tau} \right)$$

$$= \tau^5 \left(\frac{\kappa}{\tau} \right)'$$

Curvature and Torsion of a

curvature and torsion of a curve is given by

$$\vec{r} = \vec{r}(s)$$

$$\vec{r}' = \tau^1 \left(\tau^1 b^1 - \tau^1 \kappa \tau^1 + \tau^1 \tau^1 \right) \frac{b^1}{\tau^1} = \tau^1 b^1$$

$$\vec{r}'' = \kappa \tau^1 b^1$$

$$\vec{r}''' = \kappa^1 \tau^1 + \kappa \tau^1 \tau^1$$

$$= \kappa^1 \tau^1 + \kappa (\tau^1 b^1 - \kappa \tau^1)$$

$$[\vec{r}', \vec{r}'', \vec{r}'''] = \kappa^2 \tau^1$$

$$\text{curvature } (\kappa) = |\vec{r}''|$$

$$\text{Torsion } (\tau) = \frac{1}{\kappa^2} [\vec{r}' \cdot \vec{r}'' \cdot \vec{r}''']$$

For the curve $\vec{r} = [3u - u^3, 3au^2, a(3u + u^3)]$

Show the curvature & torsion are equal.

$$\vec{r}' = [3 - 3u^2, 6au, a(3 + 3u^2)]$$

$$\vec{r}'' = [-6u, 6a, 6u]$$

$\vec{t}$ is a unit vector whose magnitude is 1.
Therefore

$$\vec{t} = \frac{\vec{r}'}{|\vec{r}'|} = \frac{[3 - 3u^2, 6au, a(3 + 3u^2)]}{\sqrt{9(1 - u^2)^2 + 36a^2u^2 + 9a^2(1 + u^2)^2}}$$

$$|\vec{r}'|^2 = 9a^2 \{ (1 - u^2)^2 + 4u^2 + (1 + u^2)^2 \}$$

$$\left(\frac{ds}{du} \right)^2 = 9a^2 \{ 2(1 + u^4) + 4u^2 \}$$

$$= 9a^2 \cdot 2(1 + u^2)^2$$

$$18a^2 (1 + u^2)^2$$

$$\frac{ds}{du} = 3\sqrt{2}a(1 + u^2)$$

Substituting - (2) in (1) we get

$$\vec{r}'' = \frac{[-6u, 6a, 6u]}{3\sqrt{2}a(1 + u^2)}$$

$$= \left[\frac{-2u}{\sqrt{2}(1 + u^2)}, \frac{2}{\sqrt{2}(1 + u^2)}, \frac{2u}{\sqrt{2}(1 + u^2)} \right]$$

$$|\vec{r}''| = \frac{1}{\sqrt{2}} \sqrt{\frac{1 - u^2}{1 + u^2} + \frac{4u^2}{1 + u^2} + \frac{4u^2}{1 + u^2}}$$

$$= \frac{1}{\sqrt{2}} \cos \left[\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}, \frac{2 \tan \theta}{1 + \tan^2 \theta}, 1 \right] \quad \text{let } u = \tan \theta$$

$$t^{\wedge} = \frac{1}{\sqrt{2}} [\cos 2\theta, \sin 2\theta, 1]$$

differentiating with respect to s

$$\Rightarrow \frac{dt^{\wedge}}{ds} = \frac{1}{\sqrt{2}} [-2 \sin 2\theta, 2 \cos 2\theta, 0] \frac{d\theta}{ds}$$

$$\Rightarrow k n^{\wedge} = \sqrt{2} [-\sin 2\theta, \cos 2\theta, 0] \frac{d\theta}{ds} \quad \text{--- (3)}$$

$$\Rightarrow |k| = \sqrt{2} \left| \frac{d\theta}{ds} \right|$$

$$\Rightarrow k = \sqrt{2} \left| \frac{d\theta}{ds} \right|$$

Putting the value of k in (3)

$$\Rightarrow n^{\wedge} = [-\sin 2\theta, \cos 2\theta, 0]$$

$$\text{Now } b^{\wedge} = t^{\wedge} \times n^{\wedge}$$

$$= \frac{1}{\sqrt{2}} \cos^2 2\theta k^{\wedge} + \frac{1}{\sqrt{2}} \sin^2 2\theta k^{\wedge} + \frac{1}{\sqrt{2}} (-\sin 2\theta \cos 2\theta k^{\wedge})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k^{\wedge} \\ \frac{1}{\sqrt{2}} \cos 2\theta & \frac{1}{\sqrt{2}} \sin 2\theta & \frac{1}{\sqrt{2}} \\ -\sin 2\theta & \cos 2\theta & 0 \end{vmatrix}$$

$$= \hat{i} \left(-\frac{1}{\sqrt{2}} \cos 2\theta \right) - \hat{j} \left(\frac{1}{\sqrt{2}} \sin 2\theta \right) + k^{\wedge} \left(\frac{1}{\sqrt{2}} \cos^2 2\theta + \frac{1}{\sqrt{2}} \sin^2 2\theta \right)$$

$$= \left[\frac{1}{\sqrt{2}} (-\cos 2\theta) \hat{i} - \frac{1}{\sqrt{2}} \sin 2\theta \hat{j} + \hat{k} \right]$$

$$b^{\wedge} = \left[\frac{1}{\sqrt{2}} (-\cos 2\theta) \hat{i} - \frac{1}{\sqrt{2}} \sin 2\theta \hat{j} + \hat{k} \right]$$

$$= \frac{1}{\sqrt{2}} [-\cos 2\theta, -\sin 2\theta, 1]$$

$$-z_0^1 = \sqrt{2} \begin{bmatrix} \sin 2\theta & -\cos 2\theta & 0 \end{bmatrix}$$

$$\Rightarrow z_0^1 = \sqrt{2} \begin{bmatrix} -\sin 2\theta & \cos 2\theta & 0 \end{bmatrix}$$

$$\Rightarrow z = \sqrt{2} \left| \frac{d\theta}{ds} \right|$$

$$\begin{aligned} \text{Now } x &= \sqrt{2} \frac{d\theta}{ds} \\ &= \sqrt{2} \frac{d\theta}{du} \cdot \frac{du}{ds} \\ &= \sqrt{2} \frac{d\theta}{du} \cdot \frac{1}{3\sqrt{2}a(1+u^2)} \\ &= \frac{1}{3a(1+u^2)^2} \end{aligned}$$

$$u = \tan \theta$$

$$\sec^2 \theta d\theta = du$$

$$(1+u^2) = \frac{du}{d\theta}$$

$$\Rightarrow \frac{d\theta}{du} = \frac{1}{1+u^2}$$

$$\begin{aligned} \text{Now } x &= \sqrt{2} \frac{d\theta}{ds} \\ &= \sqrt{2} \cdot \frac{d\theta}{du} \cdot \frac{du}{ds} \\ &= \sqrt{2} \cdot \frac{1}{(1+u^2)} \cdot \frac{1}{3\sqrt{2}a(1+u^2)} \\ &= \frac{1}{3a(1+u^2)^2} \end{aligned}$$

$$z = \sqrt{2} \frac{d\theta}{ds} = \frac{1}{3a(1+u^2)^2}$$

Example 1.5 -

Show that for the curve $x = 4a \cos^3 u$, $y = 4a \sin^3 u$

$$z = 3c \cos 2u, \quad t = \frac{a}{6(a^2+c^2)\sin 2u}$$

Proof:-

$$r = [4a \cos^3 u \quad 4a \sin^3 u \quad 3c \cos 2u]$$

$$t^1 = \frac{dr}{ds} = \left[12a \cos^2 u \cdot (\sin u), \quad 12a \sin^2 u \cdot (\cos u), \quad -6c \sin 2u \right] \frac{du}{ds} \quad (1)$$

$$\Rightarrow t^1 = \left[-12a(\sin^2 u \cdot \sin 2u + \cos^2 u \cdot \cos 2u), \right.$$

$$\left. 12a(\cos^2 u \cdot \sin 2u + \sin^2 u \cdot \cos 2u) \right]$$

Since t^1 is a unit vector and whose square is unity.

$$\Rightarrow 1 = (144a^2 \cos^4 u \cdot \sin^2 u + 144a^2 \sin^4 u \cdot \cos^2 u + 36c^2 \sin^2 u \cos^2 u) \left(\frac{du}{ds}\right)^2$$

$$\Rightarrow 1 = \{144a^2 \sin^2 u \cos^2 u (\cos^2 u + \sin^2 u) + 36c^2 \sin^2 u \cos^2 u\} \left(\frac{du}{ds}\right)^2$$

$$\Rightarrow 1 = 144 \sin^2 u \cos^2 u (a^2 + c^2) \left(\frac{du}{ds}\right)^2$$

$$\Rightarrow \left(\frac{ds}{du}\right)^2 = 144 \sin^2 u \cos^2 u (a^2 + c^2)$$

$$\Rightarrow \frac{ds}{du} = 12 \sin u \cos u \sqrt{a^2 + c^2}$$

Now substituting this value in — (1)

$$t' = \frac{[12a \cos^2 u \cdot (-\sin u), 12a \sin^2 u \cdot \cos u, -6c \sin 2u]}{12 \sin u \cos u \sqrt{a^2 + c^2}}$$

$$t' = \left[\frac{a}{\sqrt{a^2 + c^2}} (-\cos u), \frac{a}{\sqrt{a^2 + c^2}} \sin u, \frac{-c}{\sqrt{a^2 + c^2}} \right]$$

$$\frac{dt'}{ds} = \left[\frac{a}{\sqrt{a^2 + c^2}} \sin u, \frac{a}{\sqrt{a^2 + c^2}} \cos u, 0 \right] \frac{du}{ds}$$

$$\Rightarrow k' = \left[\frac{a}{\sqrt{a^2 + c^2}} \sin u, \frac{a}{\sqrt{a^2 + c^2}} \cos u, 0 \right] \frac{du}{ds}$$

since n' is a unit vector, whose square is unity.

$$\Rightarrow |k| = \sqrt{\left\{ \frac{a^2}{(a^2 + c^2)} \sin^2 u + \frac{a^2}{(a^2 + c^2)} \cos^2 u \right\} \left(\frac{du}{ds}\right)^2}$$

$$\Rightarrow |k| = \frac{a}{\sqrt{a^2 + c^2}} \left| \frac{du}{ds} \right| \quad \text{--- (2)}$$

Putting the value of $\frac{du}{ds}$ in — (2)

$$k = \frac{a}{\sqrt{a^2 + c^2}} \frac{1}{12 \sin u \cos u \sqrt{a^2 + c^2}}$$

$$\kappa = \frac{a}{6 \sin 2\pi (a^2 c^2)} \quad \text{proved}$$

Curvature and Torsion of a curve:

$$r = r(t)$$

$$\kappa = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$$

$$\tau = \frac{[\dot{r} \ \ddot{r} \ (\ddot{r} \times \ddot{r})]}{|\dot{r} \times \ddot{r}|^2}$$

Q For the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$ show that $\rho = r$ where $\rho = \frac{1}{\kappa}$, $r = \frac{1}{\tau} = \rho \rightarrow$ Radius of curvature

Proof

$$r(t) = [3t, 3t^2, 2t^3]$$

$$\dot{r}(t) = [3, 6t, 6t^2]$$

$$\ddot{r} = [0, 6, 12t]$$

$$\ddot{r}$$

$$\dot{r} \times \ddot{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 6t & 6t^2 \\ 0 & 6 & 12t \end{vmatrix} = \hat{i}(-72t^3) - \hat{j}(36t^2) + \hat{k}(36t)$$

$$= -72t^3 \hat{i} - 36t^2 \hat{j} + 36t \hat{k} = 36(-2t^3 \hat{i} - t^2 \hat{j} + t \hat{k})$$

$$|\dot{r} \times \ddot{r}| = \sqrt{1296(4t^6 + t^4)} = 36\sqrt{4t^6 + t^4} = 36t^2\sqrt{4 + t^2}$$

$$\kappa = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} = \frac{36t^2\sqrt{4 + t^2}}{(9 + 36t^2 + 36t^4)^{3/2}}$$

$$\vec{r} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6t & 6t^2 \\ 0 & 6 & 12t \end{vmatrix}$$

$$= \hat{i} (72t^2 - 36t^2) - \hat{j} (36t - 0) + \hat{k} (18 - 36t)$$

$$= 36t^2 \hat{i} - 36t \hat{j} + 18 \hat{k}$$

$$= 18 (2t^2 \hat{i} - 2t \hat{j} + \hat{k})$$

$$|\vec{r} \times \ddot{\vec{r}}| = \sqrt{324 (4t^4 + 4t^2 + 1)}$$

$$= 18 \sqrt{4t^4 + 4t^2 + 1}$$

$$= 18 (2t^2 + 1)$$

$$k = \frac{|\vec{r} \times \ddot{\vec{r}}|}{|\vec{r}|^3} = \frac{18 (2t^2 + 1)}{\sqrt{9 + 36t^2 + 36t^4}^3}$$

$$= \frac{18 (2t^2 + 1)}{[3 \sqrt{1 + 4t^2 + 4t^4}]^3}$$

$$= \frac{18 (2t^2 + 1)}{[3 (2t^2 + 1)]^3} = \frac{18 (2t^2 + 1)}{27 (2t^2 + 1)^3}$$

$$= \frac{2}{3 (2t^2 + 1)^2}$$

$$\boxed{\tau = \frac{1}{6}}$$

$$(\vec{r} \times \ddot{\vec{r}}) \cdot \ddot{\vec{r}} = (36t^2 \hat{i} - 36t \hat{j} + 18 \hat{k}) \cdot 12 \hat{k}$$

$$= 18 \times 12$$

$$\tau = \frac{(\vec{r} \times \ddot{\vec{r}}) \cdot \ddot{\vec{r}}}{|\vec{r} \times \ddot{\vec{r}}|^2}$$

$$= \frac{18 \times 12}{324 (2t^2 + 1)^2}$$

$$= \frac{2}{3 (2t^2 + 1)^2}$$

$$\beta = \frac{3}{2}(at^2+1)^2 = \sigma$$

Fundamental Planes:-

The plane contain the tangent and principal normal is called oscillating plane.

The eqⁿ of oscillating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

Find the eqⁿ of the oscillating plane. $x = at$,
 $y = bt^2$, $z = ct^3$

solⁿ $(x, y, z) = (at, bt^2, ct^3)$

$$\dot{x} = a \quad \dot{y} = 2bt \quad \dot{z} = 3ct^2$$

$$\ddot{x} = 0 \quad \ddot{y} = 2b \quad \ddot{z} = 6ct$$

$$\begin{vmatrix} X-at & Y-bt^2 & Z-ct^3 \\ a & 2bt & 3ct^2 \\ 0 & 2b & 6ct \end{vmatrix} = 0$$

$$\Rightarrow (X-at)(2bct^2 - 6bct^3) - (Y-bt^2)(6act-0) + (Z-ct^3)(2ab) = 0$$

$$\Rightarrow (X-at)6bct^2 - (Y-bt^2)6act + (Z-ct^3)(2ab) = 0$$

$$\Rightarrow 6bct^2[X - 6ab/ct^3] - 6actY + 6a/ct^3 + 2abZ - 2abcct^3 = 0$$

$$\Rightarrow 3b[at^2X] - 3actY + abZ - abct^3 = 0$$

i.e required eqⁿ of oscillating plane.

Theorem :-

The necessary and sufficient condⁿ that
(i) A curve be a straightline is $\kappa = 0$ and

(ii) A curve lies in a plane is $\tau = 0$

$\Rightarrow$ Determine the funⁿ $\phi(u)$ so that the curve
 $\vec{r} = (a \cos u, a \sin u, \phi(u))$ shall lie in a
plane.

Solⁿ :-

$$\vec{r} = (a \cos u, a \sin u, \phi(u))$$

$$\vec{r}' = (-a \sin u, a \cos u, \phi'(u))$$

$$\vec{r}'' = (-a \cos u, -a \sin u, \phi''(u))$$

$$\vec{r}''' = (a \sin u, -a \cos u, \phi'''(u))$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin u & a \cos u & \phi'(u) \\ -a \cos u & -a \sin u & \phi''(u) \end{vmatrix}$$

$$= \hat{i} (a \cos u \phi''(u) + a \sin u \phi'(u))$$

$$+ \hat{j} (-a \sin u \phi''(u) + a \cos u \phi'(u)) +$$

$$\hat{k} (a^2 \sin^2 u + a^2 \cos^2 u)$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{[a \cos u \phi''(u) + a \sin u \phi'(u)]^2 + [-a \sin u \phi''(u) + a \cos u \phi'(u)]^2 + a^4}$$

$$= \sqrt{a^2 \cos^2 u (\phi''(u))^2 + a^2 \sin^2 u (\phi''(u))^2 + a^2 \cos^2 u \phi'(u)^2 + a^2 \sin^2 u \phi'(u)^2 + a^4}$$

$$= \sqrt{a^2 (\cos^2 u + \sin^2 u) (\phi''(u))^2 + a^2 (\cos^2 u + \sin^2 u) \phi'(u)^2 + a^4}$$

$$= \sqrt{a^2 (\ddot{\phi}(u))^2 + a^2 (\dot{\phi}(u))^2 + a^4} = \frac{(u) \ddot{\phi}(u)}{u \dot{\phi}(u)}$$

$$\begin{aligned} (\ddot{\vec{r}} \times \ddot{\vec{r}}) \cdot \ddot{\vec{r}} &= a \sin u (a \cos u \ddot{\phi}(u) + a \sin u \dot{\phi}(u)) \\ &\quad + a \cos u (-a \sin u \ddot{\phi}(u) + a \cos u \dot{\phi}(u)) \\ &\quad + \ddot{\phi}(u) a^2 \\ &= a^2 \sin u \cos u \ddot{\phi}(u) + a^2 \sin^2 u \dot{\phi}(u) \\ &\quad - a^2 \cos u \sin u \ddot{\phi}(u) + a^2 \cos^2 u \dot{\phi}(u) \\ &\quad + \ddot{\phi}(u) a^2 \\ &= a^2 \dot{\phi}(u) + \ddot{\phi}(u) a^2 = (u) \ddot{\phi}(u) \end{aligned}$$

$$\tau = \frac{[\ddot{\vec{r}} \times \ddot{\vec{r}}] \cdot \ddot{\vec{r}}}{|\ddot{\vec{r}} \times \ddot{\vec{r}}|^2} = \frac{(u) \ddot{\phi}(u)}{(u) \ddot{\phi}(u)}$$

$$\tau = \frac{a^2 \dot{\phi}(u) + \ddot{\phi}(u) a^2}{a^2 \dot{\phi}(u)^2 + a^2 (\ddot{\phi}(u))^2 + a^4}$$

Since $\tau = 0$

$$\Rightarrow a^2 \dot{\phi}(u) + a^2 \ddot{\phi}(u) = 0$$

$$\Rightarrow \dot{\phi}(u) + \ddot{\phi}(u) = 0$$

$$\Rightarrow \phi(u) + \dot{\phi}(u) = a$$

$$\Rightarrow \dot{\phi}(u) = a - \phi(u)$$

$$\Rightarrow 2 \dot{\phi}(u) \ddot{\phi}(u) = 2a \dot{\phi}(u) - 2 \phi(u) \dot{\phi}(u)$$

$$\Rightarrow 2 \int \dot{\phi}(u) \ddot{\phi}(u) = 2 \int (a - \phi(u)) \dot{\phi}(u)$$

$$\Rightarrow 2 \int \dot{\phi}(u) \ddot{\phi}(u) = 2 \int (a - \phi(u)) \dot{\phi}(u)$$

$$\Rightarrow \frac{2 (\dot{\phi}(u))^2}{2} = -2 \frac{(a - \phi(u))^2}{2} + b^2$$

$$\Rightarrow (\dot{\phi}(u))^2 = (a - \phi(u))^2 + b^2$$

$$\Rightarrow f(u) = \sqrt{b^2 - (a - f(u))^2}$$

$$\Rightarrow \int \frac{df(u)}{du} = \int \sqrt{b^2 - (a - f(u))^2} \cdot (-f'(u)) du$$

$$\Rightarrow \frac{df(u)}{du} = -\sqrt{b^2 - (a - f(u))^2}$$

$$\Rightarrow \int \frac{df(u)}{\sqrt{b^2 - (a - f(u))^2}} = \int du$$

$$\Rightarrow -\sin^{-1} \left(\frac{a - f(u)}{b} \right) = u - c$$

$$\Rightarrow \left(\frac{a - f(u)}{b} \right) = \sin(c - u)$$

$$\Rightarrow a - f(u) = b \sin(c - u)$$

$$\Rightarrow \boxed{f(u) = a - b \sin(c - u)}$$

Theorem:-

at conjugate point
The principal normal do not intersect at $z=0$

Helices:-

A curve traced on a surface of cylinder and cutting the generator on a constant angle
thus the tangent to helices inclined at a constant angle to a fixed direction.

As $t^{\wedge}$ is the unit tangent to a helices and $a^{\wedge}$ is a constant vector parallel to the generator of cylinder we have

$$t^{\wedge} \cdot a^{\wedge} = \text{constant}$$

$$\Rightarrow t^{\wedge} \cdot \dot{a} = 0$$

$$\boxed{t^{\wedge} \cdot \dot{a} = 0}$$

Q Find the curvature and torsion of helices

$$\vec{r} = (a \cos \theta, a \sin \theta, b \theta)$$

Sol: $\vec{r}' = (-a \sin \theta, a \cos \theta, b) \frac{d\theta}{ds}$

$$t^{\wedge} = (-a \sin \theta, a \cos \theta, b) \frac{d\theta}{ds}$$

Since $t^{\wedge}$ is a unit vector whose square is a unity.

$$1 = (a^2 \sin^2 \theta + a^2 \cos^2 \theta + b^2) \left(\frac{d\theta}{ds} \right)^2$$

$$\Rightarrow 1 = (a^2 + b^2) \left(\frac{d\theta}{ds} \right)^2$$

$$\Rightarrow \frac{1}{\sqrt{a^2 + b^2}} = \frac{d\theta}{ds}$$

$$t^{\wedge} = \left(\frac{-a \sin \theta}{\sqrt{a^2 + b^2}}, \frac{a \cos \theta}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right)$$

diff. w.r.t θ both sides

$$t^{\wedge} = \left(\frac{-a \cos \theta}{\sqrt{a^2 + b^2}}, \frac{-a \sin \theta}{\sqrt{a^2 + b^2}}, 0 \right) \frac{d\theta}{ds}$$

$$\Rightarrow n^{\wedge} = \left(\frac{-a \cos \theta}{\sqrt{a^2 + b^2}}, \frac{-a \sin \theta}{\sqrt{a^2 + b^2}}, 0 \right) \frac{d\theta}{ds}$$

$$\Rightarrow k = \left(\frac{a^2 \cos^2 \theta}{a^2 + b^2} + \frac{a^2 \sin^2 \theta}{a^2 + b^2} + 0 \right) \frac{d\theta}{ds}$$

$$k = \frac{a}{a^2 + b^2} (1)$$

$$n^{\wedge} = (\cos \theta, \sin \theta, 0)$$

$$s^{\wedge} = -n^{\wedge} \times t^{\wedge} = \left(\sin \theta, -\cos \theta, 0 \right) \frac{d\theta}{ds}$$

$$\hat{s} = \hat{t} \times \hat{n}$$

$$\vec{b} = \vec{t} \times \vec{n}$$

$$= \hat{i} \begin{vmatrix} \hat{j} & \hat{k} \\ -\frac{a \sin \theta}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \end{vmatrix} - \hat{j} \begin{vmatrix} \hat{i} & \hat{k} \\ \frac{a \cos \theta}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \end{vmatrix} + \hat{k} \begin{vmatrix} \hat{i} & \hat{j} \\ \frac{a \cos \theta}{\sqrt{a^2+b^2}} & \frac{a \sin \theta}{\sqrt{a^2+b^2}} \end{vmatrix}$$

$$= \hat{i} \left(-\frac{b \sin \theta}{\sqrt{a^2+b^2}} \right) - \hat{j} \left(-\frac{b \cos \theta}{\sqrt{a^2+b^2}} \right) + \hat{k} \left(-\frac{a \sin^2 \theta}{\sqrt{a^2+b^2}} - \frac{a \cos^2 \theta}{\sqrt{a^2+b^2}} \right)$$

$$= \left(-\frac{b \sin \theta}{\sqrt{a^2+b^2}}, \frac{b \cos \theta}{\sqrt{a^2+b^2}}, -\frac{a}{\sqrt{a^2+b^2}} \right)$$

$$\Rightarrow \vec{b}' = \left(-\frac{b \cos \theta}{\sqrt{a^2+b^2}}, \frac{b \sin \theta}{\sqrt{a^2+b^2}}, 0 \right) \frac{d\theta}{ds}$$

$$\Rightarrow -\vec{t}' = -\frac{b}{\sqrt{a^2+b^2}} \begin{pmatrix} \cos \theta & \sin \theta & 0 \end{pmatrix} \frac{d\theta}{ds}$$

$$\Rightarrow |\vec{t}'| = \frac{b}{\sqrt{a^2+b^2}} \sqrt{\cos^2 \theta + \sin^2 \theta} \left| \frac{d\theta}{ds} \right|$$

$$\Rightarrow |\vec{t}'| = \frac{b}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \kappa = \frac{b}{a^2+b^2}$$

$$\begin{aligned} x &= a \cos \theta & y &= a \sin \theta & z &= b\theta \\ \dot{x} &= -a \sin \theta & \dot{y} &= a \cos \theta & \dot{z} &= b \\ \ddot{x} &= -a \cos \theta & \ddot{y} &= -a \sin \theta & \ddot{z} &= 0 \end{aligned}$$

$$\begin{vmatrix} x & y & z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = \begin{vmatrix} a \cos \theta & a \sin \theta & b\theta \\ -a \sin \theta & a \cos \theta & b \\ -a \cos \theta & -a \sin \theta & 0 \end{vmatrix} = 0$$

$$\Rightarrow (x - a \cos \theta) a b \sin \theta - (y - a \sin \theta) (a b \cos \theta) + (z - b \theta) (a^2 \sin^2 \theta + a^2 \cos^2 \theta) = 0$$

$$\Rightarrow a b x \sin \theta - a^2 b \sin \theta \cos \theta - a b y \cos \theta + a^2 b \sin \theta \cos \theta + (z - b \theta) a^2 = 0$$

$$\Rightarrow a b x \sin \theta - a b y \cos \theta + z a^2 - a^2 b \theta = 0$$

Theorem:-

The necessary and sufficient condition for a curve to be a helix is that $\frac{\kappa}{\tau}$ is constant.

Q. Prove that the curve $x = au, y = bu^2, z = cu^3$ is a helix provided $4ac = 3b^2$.

Proof:-

$$\vec{r} = [au, bu^2, cu^3]$$

$$\vec{r}' = \frac{d\vec{r}}{du} = [a, 2bu, 3cu^2]$$

$$\vec{r}'' = [0, 2b, 6cu]$$

$$\vec{r}''' = [0, 0, 6c]$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & 2bu & 3cu^2 \\ 0 & 2b & 6cu \end{vmatrix}$$

$$= \hat{i} (12bcu^2 - 0) - \hat{j} (6acu - 0) + \hat{k} (2ab)$$

$$= 6bcu^2 \hat{i} - 6acu \hat{j} + 2ab \hat{k}$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{36b^2c^2u^4 + 36a^2c^2u^2 + 4a^2b^2}$$

$$= 2 \sqrt{9b^2c^2u^4 + 9a^2c^2u^2 + a^2b^2}$$

$$|\vec{r}'| = \sqrt{a^2 + 4b^2u^2 + 9c^2u^4}$$

$$(\vec{r}' \times \vec{r}'') \cdot \vec{r}''' = 12abc$$

$$\kappa = \frac{2 \sqrt{9b^2c^2u^4 + 9a^2c^2u^2 + a^2b^2}}{(a^2 + 4b^2u^2 + 9c^2u^4)^{3/2}}$$

$$= \frac{12abc}{(a^2 + 4b^2u^2 + 9c^2u^4)^{3/2}}$$

$$r = \frac{3abc}{4(a^2b^2c^2 + a^2c^2b^2 + a^2b^2c^2)}$$

$$\frac{x}{c} = \frac{2\sqrt{a^2b^2c^2 + a^2c^2b^2 + a^2b^2c^2}}{(a^2 + 4b^2c^2 + 9c^2a^2)^{3/2}} \cdot \frac{a^2b^2c^2 + 9a^2b^2 + 4a^2c^2}{3abc}$$

$$= \frac{2\sqrt{a^2b^2c^2 + 4b^4c^2 + 9a^2b^2}}{(a^2 + 4b^2c^2 + 9c^2a^2)^{3/2}} \cdot \frac{1}{2b^3}$$

$$= \frac{2b^3(a^2c^2 + 4b^2c^2 + a^2)^{3/2}}{(a^2 + 4b^2c^2 + 9c^2a^2)^{3/2}} \cdot \frac{1}{2b^3}$$

$$= 1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

$$\frac{1}{x^2} = x^{-2}$$

$$f'(x) = -2x^{-3}$$

$$f'(x) = -\frac{2}{x^3}$$

$$f'(x) = -\frac{2}{x^3}$$

$$f'(x) = -\frac{2}{x^3}$$

Exercise - 1 (a)

1. Find the curvature and torsion of the curve given by $r = (2abt, a^2 \log t, b^2 t^2)$

$$\text{Sol: } r = (2abt, a^2 \log t, b^2 t^2)$$

$$\dot{r} = t^1 = (2ab, \frac{a^2}{t}, 2b^2 t)$$

Since t^1 is a unit vector whose square is a unity.

$$1 = \sqrt{4a^2b^2 + \frac{a^4}{t^2} + 4b^4 t^2} \left(\frac{dt}{ds} \right)^2$$

$$\Rightarrow 1 = \left(4a^2b^2 + \frac{a^4}{t^2} + 4b^4 t^2 \right) \left(\frac{dt}{ds} \right)^2$$

$$\Rightarrow \frac{dt}{ds} = \frac{1}{\sqrt{4a^2b^2 + \frac{a^4}{t^2} + 4b^4 t^2}}$$

$$\ddot{r} = (0, -\frac{a^2}{t^2}, 2b^2)$$

$$\ddot{r} = (0, \frac{2a^2}{t^3}, 0)$$

$$\ddot{r} \times \ddot{r} =$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{2a^2}{t^3} & 0 \\ 0 & -\frac{a^2}{t^2} & 2b^2 \end{vmatrix}$$

$$= \hat{i} \left(\frac{2a^2b^2}{t} + \frac{2a^2b^2}{t} \right) - \hat{j} (4ab^3 - 0) + \hat{k} (-2a^3b)$$

$$= \frac{4a^2b^2}{t} \hat{i} - 4ab^3 \hat{j} - \frac{2a^3b}{t^2} \hat{k}$$

$$|\ddot{r} \times \ddot{r}| = \sqrt{\frac{16a^4b^4}{t^2} + 16a^4b^6 + \frac{4a^6b^2}{t^4}}$$

$$= 2a \sqrt{\frac{4a^2b^4}{t^2} + 4b^6 + \frac{a^4b^2}{t^4}}$$

$$= 2a \sqrt{\left(\frac{a^2b}{t^2} + 2b^3 \right)^2}$$

$$= 2a \left(\frac{a^2b}{t^2} + 2b^3 \right)$$

$$|\ddot{r}| = \sqrt{4a^2b^2 + \frac{a^4}{t^2} + 4b^4 t^2}$$

$$= \sqrt{\left(\frac{a^2}{t} + 2b^2 t \right)^2} = \frac{a^2}{t} + 2b^2 t$$

Curvature $k = \frac{|\ddot{r} \times \ddot{r}|}{|\ddot{r}|^3}$

$$= \frac{2a \left(\frac{a^2 b}{t^2} + 2b^3 \right)}{\left(\frac{a^2}{t} + 2b^2 t \right)^3}$$

$$= \frac{2ab \left(\frac{a^2}{t^2} + 2b^2 \right)}{\left(\frac{a^2}{t} + 2b^2 t \right)^3}$$

$$= \frac{2ab}{t} \left(\frac{a^2}{t} + 2b^2 t \right) \cdot \frac{1}{\left(\frac{a^2}{t} + 2b^2 t \right)^3}$$

$$= \frac{2ab}{t} \cdot \frac{1}{a^2 + 2b^2 t^2}$$

Torsion

Now $(\ddot{r} \times \ddot{r}) \cdot \ddot{r}$

$$= -4ab^3 \cdot \frac{2a^2}{t^3} = -\frac{8a^3 b^3}{t^3}$$

Torsion $\tau = \frac{(\ddot{r} \times \ddot{r}) \cdot \ddot{r}}{|\ddot{r} \times \ddot{r}|^2}$

$$= \frac{-\frac{8a^3 b^3}{t^3}}{\left(\frac{a^2}{t} + 2b^2 t \right)^2} = -\frac{8a^3 b^3}{t^3} \times \frac{t^2}{(a^2 + 2b^2 t^2)^2}$$

$$= \frac{-8a^3 b^3}{t (a^2 + 2b^2 t^2)^2}$$

$$= \frac{-2ab (a^2 + 2b^2 t^2)}{t^3}$$

$$= \frac{2ab t}{(a^2 + 2b^2 t^2)^2}$$

$(\ddot{r} \times \ddot{r}) \cdot \ddot{r}$

$$= -\frac{8a^3 b^3}{t^3}$$

$$\tau = \frac{-\frac{8a^3 b^3}{t^3}}{4a^2 \left(\frac{a^2 b}{t^2} + 2b^3 \right)^2} = \frac{-\frac{8a^3 b^3}{t^3}}{4a^2 \left(\frac{a^2}{t} + 2b^2 t \right)^2}$$

$$= \frac{-2abt}{(a^2 + 2b^2z^2)^2}$$

(2) Find the radii of curvature and torsion of the curve given by

$$r = (a \cos \theta, a \sin \theta, a \cos 2\theta) \text{ at } \theta = \pi/4$$

$$\text{Sol: } r = (a \cos \theta, a \sin \theta, a \cos 2\theta)$$

$$t = \frac{dr}{ds} = (-a \sin \theta, a \cos \theta, -2a \sin 2\theta) \frac{d\theta}{ds}$$

t is a unit vector whose square is a unity

$$1 = (a^2 \sin^2 \theta + a^2 \cos^2 \theta + 4a^2 \sin^2 2\theta) \left(\frac{d\theta}{ds}\right)^2$$

$$1 = (a^2 + 4a^2 \sin^2 2\theta) \left(\frac{d\theta}{ds}\right)^2$$

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{a^2 + 4a^2 \sin^2 2\theta}}$$

$$\left. \frac{d\theta}{ds} \right|_{\theta = \pi/4} = \frac{1}{\sqrt{a^2 + 4a^2}} = \frac{1}{\sqrt{5}a}$$

$$t = (-\sin \theta, \cos \theta, -2 \sin 2\theta) \frac{1}{\sqrt{5}a} = \frac{1}{\sqrt{5}} (-\sin \theta, \cos \theta, -2 \sin 2\theta)$$

$$t' = (-\cos \theta, -\sin \theta, -4 \cos 2\theta) \frac{1}{\sqrt{5}a} \frac{d\theta}{ds}$$

$$k n = (-\cos \theta, -\sin \theta, -4 \cos 2\theta) \frac{1}{\sqrt{5}a} \frac{1}{\sqrt{5}a}$$

$$|k| = \frac{1}{5a} \sqrt{\cos^2 \theta + \sin^2 \theta + 16 \cos^2 2\theta}$$

$$= \frac{1}{5a} \sqrt{1 + 16 \cos^2 2\theta}$$

$$\text{at } \theta = \frac{\pi}{4} \quad |k| = \frac{1}{5a}$$

Radius of curvature $\rho = \frac{1}{|k|} = 5a$

$$b = [t' \times t] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{1}{\sqrt{5}} \cos \theta & -\frac{1}{\sqrt{5}} \sin \theta & -\frac{4}{\sqrt{5}} \cos 2\theta \\ \frac{1}{\sqrt{5}} \sin \theta & \frac{1}{\sqrt{5}} \cos \theta & -\frac{2}{\sqrt{5}} \sin 2\theta \end{vmatrix}$$

$$= \frac{1}{5} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos \theta \cos 2\theta - 2 \sin^2 2\theta & \cos \theta \cos 2\theta - 2 \sin \theta \sin 2\theta & \cos \theta \sin 2\theta + \sin \theta \cos 2\theta \end{vmatrix}$$

$$= \frac{1}{5} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos \theta \cos 2\theta - 2 \sin^2 2\theta & \cos \theta \cos 2\theta - 2 \sin \theta \sin 2\theta & \cos \theta \sin 2\theta + \sin \theta \cos 2\theta \end{vmatrix}$$

$$\begin{aligned} \mathbf{b}^1 &= \frac{1}{2} \left(\frac{-4}{\sqrt{5}} \cos \theta (1 - 2 \sin^2 \theta) - \frac{4 \sin^2 \theta \cos \theta}{\sqrt{5}} \right) \\ &\quad - j \left(\frac{4}{\sqrt{5}} \sin \theta (2 \cos^2 \theta - 1) - \frac{4 \cos^2 \theta \sin \theta}{\sqrt{5}} \right) \\ &\quad + \frac{1}{\sqrt{5}} \mathbf{k}^1 \\ &= \frac{1}{2} \left(\frac{-4}{\sqrt{5}} \cos \theta + \frac{4 \sin^2 \theta \cos \theta}{\sqrt{5}} \right) - j \left(\frac{4}{\sqrt{5}} \sin \theta - \frac{4 \cos^2 \theta \sin \theta}{\sqrt{5}} \right) \\ &\quad + \frac{1}{\sqrt{5}} \mathbf{k}^1 \end{aligned}$$

$$\mathbf{b}^1 = \left(\frac{-4}{\sqrt{5}} \cos^3 \theta, \frac{4}{\sqrt{5}} \sin^3 \theta, \frac{1}{\sqrt{5}} \right)$$

$$\mathbf{b}^1 = \left(\frac{-4}{\sqrt{5}} \cos^3 \theta, \frac{4}{\sqrt{5}} \sin^3 \theta, \frac{1}{\sqrt{5}} \right)$$

$$\begin{aligned} \Rightarrow \nabla \tau^1 &= \frac{4}{\sqrt{5}} \left(-3 \cos^2 \theta \sin \theta, 3 \sin^2 \theta \cos \theta, 0 \right) \cdot \frac{1}{\sqrt{5a}} \\ &= \frac{4}{5a} \left(-3 \cos^2 \theta \sin \theta, 3 \sin^2 \theta \cos \theta, 0 \right) \end{aligned}$$

$$\begin{aligned} \Rightarrow |\tau| &= \frac{4}{5a} \sqrt{9 \cos^4 \theta \sin^2 \theta + 9 \sin^4 \theta \cos^2 \theta} \\ &= \frac{4}{5a} \sqrt{9 \sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta)} \end{aligned}$$

$$= \frac{4}{5a} \cdot 3 \sin \theta \cos \theta$$

$$\theta = \frac{\pi}{4} \quad |\tau| = \frac{4}{5a} \cdot \frac{3}{2} = \frac{6}{5a}$$

$$\sigma = \frac{1}{\tau} = \frac{5a}{6}$$

③. Find the curvature and torsion of the curve $x = a(u - \sin u)$, $y = a(1 - \cos u)$, $z = bu$

$$\vec{r} = [a(u - \sin u), a(1 - \cos u), bu]$$

$$\Rightarrow \dot{\mathbf{r}} = \left[a(1 - \cos u), a \sin u, b \right] \frac{du}{ds}$$

$$\Rightarrow 1 = \left(a^2(1 - \cos u)^2 + a^2 \sin^2 u + b^2 \right) \left(\frac{du}{ds} \right)^2$$

$$\Rightarrow 1 = (a^2 + a^2 - 2a^2 \cos u + b^2) \left(\frac{du}{ds} \right)^2$$

$$\Rightarrow 1 = \sqrt{(2a^2(1-\cos u) + b^2)} \left(\frac{du}{ds}\right)^2$$

$$\Rightarrow 1 = 2a^2$$

$$\Rightarrow \left(\frac{du}{ds}\right)^2 = \frac{1}{2a^2(1-\cos u) + b^2}$$

$$\Rightarrow \left(\frac{du}{ds}\right)^2 = \frac{1}{2a^2(1-\cos u)}$$

③ Find the curvature and torsion of the curve
 $x = a(u - \sin u)$, $y = a(1 - \cos u)$, $z = bu$.

$$\text{Sol: } \vec{r} = [a(u - \sin u), a(1 - \cos u), bu]$$

$$\Rightarrow \vec{r}' = [a(1 - \cos u), a \sin u, b]$$

$$\Rightarrow \vec{r}'' = [a \sin u, a \cos u, 0]$$

$$\Rightarrow \vec{r}''' = [a \cos u, -a \sin u, 0]$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a(1 - \cos u) & a \sin u & b \\ a \sin u & a \cos u & 0 \end{vmatrix}$$

$$= \hat{i}(-ab \cos u) - \hat{j}(-ab \sin u) + \hat{k}(a^2 \cos u - a^2 \sin^2 u)$$

$$= -ab \cos u \hat{i} + ab \sin u \hat{j} + (a^2 \cos u - a^2 \sin^2 u) \hat{k}$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{a^2 b^2 \cos^2 u + a^2 b^2 \sin^2 u + a^4 (\cos u + 1)^2}$$

$$= \sqrt{a^2 b^2 + a^4 (\cos u + 1)^2}$$

$$= a \sqrt{b^2 + a^2 (\cos u + 1)^2}$$

$$\vec{r}' = \sqrt{a^2 (1 - \cos u)^2 + a^2 \sin^2 u + b^2}$$

$$= \sqrt{a^2 + a^2 - 2a^2 \cos u + b^2}$$

$$= \sqrt{2a^2(1 - \cos u) + b^2} = \frac{b}{2(1 - \cos u)}$$

$$K = \frac{a \sqrt{b^2 + a^2 (\cos u + 1)^2}}{(2a^2(1 - \cos u) + b^2)^{3/2}} = \frac{ab}{2b^2}$$

$$\left(\frac{\partial \vec{r}'}{\partial u} \cdot \frac{\partial \vec{r}'}{\partial u} \right) = 1$$

$$\left(\frac{\partial \vec{r}'}{\partial u} \cdot \frac{\partial \vec{r}'}{\partial u} \right) = 1$$

$$\begin{aligned}
 & 1000 (\ddot{r} \times \ddot{r}) \cdot \ddot{r} \\
 &= (-abc \cos u)(a \cos u) + (abs \sin u)(-a \sin u) + 0 \\
 &= -a^2b \cos^2 u - a^2b \sin^2 u \\
 &= -a^2b (\cos^2 u + \sin^2 u) \\
 &= -a^2b
 \end{aligned}$$

$$\begin{aligned}
 \tau &= \frac{-a^2b}{a^2(b^2 + a^2(1 - \cos^2 u))} \\
 &= \frac{-b}{[b^2 + a^2(1 - \cos^2 u)]}
 \end{aligned}$$

④. Find the curvature and torsion of the helix
 $x = a \cos \theta$, $y = a \sin \theta$, $z = a \tan \theta$

⑤. Show that the radius of curvature of the
 twisted curve $x = \lambda \cos \theta$, $y = \lambda \sin \theta$,
 $z = \sqrt{2} \theta$ is $\rho = \sqrt{2} \operatorname{cosec} 2\theta$

Proof: $\vec{r} = [\lambda \cos \theta, \lambda \sin \theta, \sqrt{2} \theta]$

$$t^1 = \vec{r}' = \left[-\lambda \sin \theta, \lambda \cos \theta, \sqrt{2} \right] \frac{d\theta}{ds}$$

$$\Rightarrow 1 = \sqrt{\lambda^2 \sin^2 \theta + \lambda^2 \cos^2 \theta + 2} \left(\frac{d\theta}{ds} \right)^2$$

$$\Rightarrow 1 = \frac{\lambda^2 \sin^2 \theta + \lambda^2 \cos^2 \theta + 2}{\lambda^2} \left(\frac{d\theta}{ds} \right)^2$$

$$\Rightarrow 1 = \frac{(\lambda^2 \sin^2 \theta + \lambda^2 \cos^2 \theta + 2)}{\lambda^2} \left(\frac{d\theta}{ds} \right)^2$$

$$\Rightarrow \frac{\lambda^2}{(\lambda^2 \sin^2 \theta + \lambda^2 \cos^2 \theta + 2)} = \left(\frac{d\theta}{ds} \right)^2$$

$$\Rightarrow \frac{d\theta}{ds} = \frac{\lambda}{\sqrt{\lambda^2 \sin^2 \theta + \lambda^2 \cos^2 \theta + 2}}$$

$$t^1 = \left(\frac{-\lambda \sin \theta}{1 + \lambda^2}, \frac{\lambda \cos \theta}{1 + \lambda^2}, \frac{\sqrt{2} \lambda}{1 + \lambda^2} \right)$$

$$\Rightarrow t^1 = (-\sin \theta, \cos \theta, \sqrt{2} \sin \theta \cos \theta)$$

$$\Rightarrow \mathbf{r}'' = \mathbf{t}' = (-8 \sin 2\theta, -8 \sin 2\theta \sqrt{2} \cos 2\theta) \cdot \frac{\sin \theta}{\cos \theta}$$

$$\Rightarrow \mathbf{r}'' = (-2 \sin^2 \theta \cdot \cos^2 \theta, -2 \sin^2 \theta \cdot \cos^2 \theta, (2\sqrt{2} \cos^2 \theta - \sqrt{2} \sin^2 \theta \cos \theta))$$

$$\begin{aligned} \Rightarrow |\mathbf{r}''| &= \sqrt{4 \sin^4 \theta \cdot \cos^4 \theta + 4 \sin^4 \theta \cdot \cos^4 \theta + 2 \cos^2 \theta (2 \cos^2 \theta - \sin^2 \theta)^2} \\ &= \sqrt{8 \sin^4 \theta \cdot \cos^4 \theta + 2 \cos^2 \theta (4 \cos^4 \theta + \sin^2 \theta + 4 \cos^2 \theta \cdot \sin \theta)} \\ &= \sqrt{8 \sin^4 \theta \cdot \cos^4 \theta + 8 \cos^6 \theta + 2 \cos^2 \theta \sin^2 \theta + 8 \cos^4 \theta \cdot \sin \theta} \end{aligned}$$

$$\Rightarrow \mathbf{r}'' = (-\sin 2\theta \cdot \sin \theta \cdot \cos \theta, -\sin 2\theta \cdot \sin \theta \cdot \cos \theta, \sqrt{2} \cos 2\theta \cdot \sin \theta \cdot \cos \theta)$$

$$= \left[\frac{-(\sin 2\theta)^2}{2}, \frac{-(\sin 2\theta)^2}{2}, \frac{\sqrt{2} \cos 2\theta \cdot \sin 2\theta}{2} \right]$$

$$\Rightarrow |\mathbf{r}''| = \sqrt{\frac{8 \sin^4 2\theta}{4} + \frac{8 \sin^4 2\theta}{4} + \frac{2 \cos^2 2\theta \cdot \sin^2 2\theta}{4}}$$

$$= \sqrt{\frac{2 \sin^4 2\theta}{4} + \frac{2 \cos^2 2\theta \cdot \sin^2 2\theta}{4}}$$

$$= \sqrt{\frac{2 \sin^2 2\theta}{4} (\sin^2 2\theta + \cos^2 2\theta)}$$

$$= \frac{\sin 2\theta}{\sqrt{2}}$$

$$\therefore \text{Radius of curvature } \rho = \frac{1}{|\mathbf{r}''|} = \sqrt{2} \csc 2\theta$$

Proved

⑥ For the curve $\mathbf{r} = (\sqrt{6} at^3, a(1+3t^2), \sqrt{6} at)$

Show that $\rho = a(1+3t^2)^2$

Proof:- $\mathbf{r} = (\sqrt{6} at^3, a(1+3t^2), \sqrt{6} at)$

$$\dot{\mathbf{r}} = (\sqrt{6} 3at^2, 6at, \sqrt{6} a)$$

$$\ddot{\mathbf{r}} = (6\sqrt{6} at, 6a, 0)$$

$$\ddot{\mathbf{r}} = (6\sqrt{6} a, 0, 0)$$

∴

$$\vec{r} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{6}at^2 & 6at & \sqrt{6}a \\ 6\sqrt{6}at & 6a & 0 \end{vmatrix}$$

$$= \hat{i}(0 - 6\sqrt{6}a^2) - \hat{j}(0 - 36a^2t) + \hat{k}(16\sqrt{6}a^2t^2 - 36\sqrt{6}a^2t^2)$$

$$= -6\sqrt{6}a^2\hat{i} + 36a^2t\hat{j} - 18\sqrt{6}a^2t^2\hat{k}$$

$$(\vec{r} \times \ddot{\vec{r}}) \cdot \ddot{\vec{r}} = 6\sqrt{6}a \cdot -6\sqrt{6}a^2$$

$$= -216a^3$$

$$\tau = -216a^3$$

$$|\vec{r} \times \ddot{\vec{r}}| = \sqrt{216a^4 + 1296a^4t^2 + 324 \times 6a^4t^4}$$

$$= \sqrt{216(a^4 + 6a^4t^2 + 9a^4t^4)}$$

$$= 12\sqrt{6} \sqrt{(a^2 + 3a^2t^2)^2}$$

$$= 12\sqrt{6} (a^2 + 3a^2t^2)$$

$$\tau = \frac{-216a^3}{36(a^2 + 3a^2t^2)^2} = \frac{-6a^3}{a^4(1+3t^2)^2}$$

$$\tau = \frac{-216a^3}{36 \times 36 (a^2 + 3a^2t^2)^2}$$

$$= \frac{-a^3}{6a^4(1+3t^2)^2}$$

$$= \frac{-1}{6a(1+3t^2)^2}$$

$$\sigma = \frac{1}{\tau} = \frac{1}{\frac{-1}{6a(1+3t^2)^2}} = -6a(1+3t^2)^2$$

$$\tau = \frac{-216a^3}{216(a^2 + 3a^2t^2)^2}$$

$$= \frac{-a^3}{a^4(1+3t^2)^2}$$

$$\sigma = \frac{1}{\tau} = -a(1+3t^2)^2$$

Theorem :-

The necessary and sufficient condition that

- (i) a curve be a straightline is $\kappa=0$ and
- (ii) a curve lies in a plane $\tau=0$

Proof :-

(i) If the curve is a straightline, then the tangent vector t is constant and consequently $t' = 0$, hence $\kappa = 0$

conversely, given $\kappa = 0$ we must have $t' = \kappa n = 0$, so that t is a constant vector i.e. its dirⁿ gets fixed. Thus the curve is a straightline

(ii) Let the curve lie in a plane τ and we choose the origin in its plane. Since both r' and r'' are vectors which lie in this plane, both t & n lie in τ . This means that $(b = t \times n)$ is a constant vector, its dirⁿ being orthogonal to τ . Thus $b' = 0$ i.e. $\tau n = 0$, hence $\tau = 0$

conversely, given $\tau = 0$, we have $b' = 0$ and therefore b is a constant vector. Then if the curve is given by $r(c)$, then

$$(r \cdot b)' = t \cdot b + r \cdot b' \\ = 0 + 0$$

Go that $r \cdot b = \text{constant}$. This implies that the vectors from the origin to the various points of the curve are orthogonal to the fixed direction b . Hence the curve necessarily lies in a plane.

Theorem 1.2 :-

The principal normals at consecutive points do not intersect unless $\tau = 0$

Proof:- Let $P(x)$ and $Q(x+d)$ be two consecutive points on a space curve C (where the unit principal normals are n and $n+d$ respectively).

For the intersection of the principal normals, the necessary condition is that the three vectors dr , n and $n+d$ be coplanar.

This requires that their scalar triple product should vanish i.e.

$$[dr, n, n+d] = 0$$

$$\Rightarrow \tau [dr, n, b] = \kappa [dr, n, t] = 0$$

$$\Rightarrow \tau [dr, n, b] = 0$$

which holds only when τ vanishes because $[dr, n, b] = 1$.

Theorem 1.3 :-

The necessary and sufficient condition for a curve to be a helix is that κ/τ is constant.

Proof:- Necessary condition:-

Suppose that the curve is helix.

Then $\tau \cdot a = \text{constant}$

$$\Rightarrow \kappa n \cdot a = 0$$

$$\text{i.e. } n \cdot a = 0$$

Thus a is perpendicular to the plane of the curve. Thus a is expressible (as a linear combination) of t and b i.e. $a = t \cos \alpha + b \sin \alpha$

where α is the angle between the tangent and the axis of the curve.

$$0 = t \cos \alpha - z \sin \alpha$$

$$\text{i.e. } \tan \alpha = \frac{z}{t} = \text{constant}$$

$$0 = (d, p) \cdot F$$

$$0 = \frac{z}{t} \text{ and } 0 = \frac{z}{t}$$

Sufficient condition:-

Let $k/z = \text{constant} = c/t$ (say). Since $t^2 = k^2$ and $b^2 = z^2 - t^2$ we have $b^2 + c^2 t^2 = a^2$ — (1)

where a is a constant vector.

Forming a scalar product of each side of (1) with t , we have $t \cdot a = c$.

Then t is inclined at a constant angle to the fixed dir of a , therefore a helix.

To prove that the education envelope is

$$t = \frac{x}{c} + \frac{y}{c} + \frac{z}{c}$$

$$t \cdot \frac{x}{c} + t \cdot \frac{y}{c} + t \cdot \frac{z}{c} = (t \cdot a) = c$$

$$(1) - t = \frac{z}{c}$$

$$t \cdot \frac{z}{c} = \frac{z}{c}$$

$$t \cdot \frac{z}{c} = \frac{z}{c}$$

Two parameters a family of surfaces $F(a, b) = 0$ where a and b are independent parameters.

The equation of envelop of two parameter family of surface can be obtained by eliminating a and b from the following equations

$$F(a, b) = 0$$

$$\frac{\partial F}{\partial a} = 0 \quad \text{and} \quad \frac{\partial F}{\partial b} = 0$$

Q. Show that the envelop of a plane $\frac{x}{a} \cos \theta \sin \phi + \frac{y}{b} \sin \theta \sin \phi + \frac{z}{c} \cos \phi = 1$ where θ, ϕ are independent parameters is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

The eqn of family of given plane is given by $\frac{x}{a} \cos \theta \sin \phi + \frac{y}{b} \sin \theta \sin \phi + \frac{z}{c} \cos \phi = 1$ where θ, ϕ are independent parameters.

To prove that the equation envelop is ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$F(\theta, \phi) = \frac{x}{a} \cos \theta \sin \phi + \frac{y}{b} \sin \theta \sin \phi + \frac{z}{c} \cos \phi = 1 \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial F}{\partial a} = -\frac{1}{a^2} x \cos \theta \sin \phi$$

$$\frac{\partial F}{\partial b} = -\frac{1}{b^2} y \sin \theta \sin \phi$$

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$$\frac{\partial F}{\partial \phi} = -\frac{2}{a} \sin \alpha \cdot \sin \phi + \frac{y}{b} \cos \alpha \cdot \cos \phi - \frac{z}{c} \sin \phi = 0 \quad (2)$$

$$\frac{\partial F}{\partial \phi} = \left(\frac{x}{a} \cos \alpha \cdot \cos \phi + \frac{y}{b} \sin \alpha \cdot \cos \phi - \frac{z}{c} \sin \phi \right) = 0 \quad (3)$$

From eqn (2) $\frac{\partial F}{\partial \phi} = 0 = 0$

$$\sin \phi \left(\frac{y}{b} \cos \alpha - \frac{x}{a} \sin \alpha \right) = 0$$

$$\Rightarrow \sin \phi = 0 \quad \frac{y}{b} \cos \alpha - \frac{x}{a} \sin \alpha = 0 \quad (4)$$

$$\Rightarrow \phi = 0$$

Equating & adding respective sides of eqn (1), (3) & (4)

$$\left(\frac{x}{a} \cos \alpha \cdot \sin \phi + \frac{y}{b} \sin \alpha \cdot \sin \phi + \frac{z}{c} \cos \phi \right) + \left(\frac{y}{b} \cos \alpha - \frac{x}{a} \sin \alpha \right)^2 + \left(\frac{x}{a} \cos \alpha \cdot \cos \phi + \frac{y}{b} \sin \alpha \cdot \cos \phi - \frac{z}{c} \sin \phi \right)^2 = 1$$

$$\begin{aligned} & \Rightarrow \frac{x^2}{a^2} \cos^2 \alpha \cdot \sin^2 \phi + \frac{y^2}{b^2} \sin^2 \alpha \cdot \sin^2 \phi + \frac{z^2}{c^2} \cos^2 \phi + \\ & \frac{2xy}{ab} \sin^2 \phi \cdot \sin \alpha \cdot \cos \alpha + \frac{2yz}{bc} \sin \alpha \cdot \sin \phi \cdot \cos \phi \\ & + \frac{2xz}{ac} \cos \alpha \cdot \sin \phi \cdot \cos \phi + \frac{y^2}{b^2} \cos^2 \alpha + \frac{x^2}{a^2} \sin^2 \alpha \\ & - \frac{2xy}{ab} \sin \alpha \cdot \cos \alpha + \frac{x^2}{a^2} \cos^2 \alpha \cos^2 \phi + \frac{y^2}{b^2} \sin^2 \alpha \\ & \cos^2 \phi + \frac{z^2}{c^2} \sin^2 \phi + \frac{2xy}{ab} \cos \alpha \sin \alpha \cos \phi \\ & - \frac{2yz}{bc} \sin \alpha \cdot \cos \phi \cdot \sin \phi - \frac{2xz}{ac} \sin \phi \cdot \cos \phi \cos \alpha = 1 \end{aligned}$$

$$\begin{aligned} & \Rightarrow \frac{x^2}{a^2} \cos^2 \alpha + \frac{y^2}{b^2} \sin^2 \alpha + \frac{z^2}{c^2} + \frac{2xy}{ab} \cos \alpha \cdot \sin \alpha \\ & + \frac{y^2}{b^2} \cos^2 \alpha + \frac{x^2}{a^2} \sin^2 \alpha - \frac{2xy}{ab} \cos \alpha \sin \alpha \\ & \Rightarrow \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right] = 1 \end{aligned}$$

Q. The envelop of the family of surface $F(x, y, z, a, b) = 0$ in which the parameters a, b are connected by the eqⁿ $f(a, b) = 0$ is found by eliminating a & b from the eq^s $F = 0$, $F_a = 0$, $F_b = 0$, $\frac{F_a}{F_a} = \frac{F_b}{F_b}$

sol:- The eqⁿ of family of surface is $F(x, y, z, a, b) = 0$ — (1)

The parameter a and b are connected by the relation $f(a, b) = 0$ — (2)

~~So there is only one~~ So there is only one parameter. let us take a as independent parameter.

$$F(x, y, z, a, b) = 0$$

$$\frac{\partial}{\partial a} (F(x, y, z, a, b)) = 0$$

$$\Rightarrow F'(x, y, z, a, b)$$

$$\Rightarrow \frac{\partial F}{\partial a} \cdot \frac{\partial a}{\partial a} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial a} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial a} + \frac{\partial F}{\partial b} \cdot \frac{\partial b}{\partial a} = 0$$

$$\Rightarrow 0 + 0 + 0 + \frac{\partial F}{\partial b} \cdot \frac{\partial b}{\partial a} = 0$$

$$\Rightarrow \frac{\partial F}{\partial a} + \frac{\partial F}{\partial b} \cdot \frac{\partial b}{\partial a} = 0 \Rightarrow \frac{\partial b}{\partial a} = -\frac{F_a}{F_b} \quad (3)$$

$$f(a, b) = 0$$

$$\frac{\partial f}{\partial a} = -\frac{f_a}{f_b}$$

$$\frac{\partial f}{\partial a} \cdot \frac{\partial a}{\partial a} + \frac{\partial f}{\partial b} \cdot \frac{\partial b}{\partial a} = 0$$

$$\Rightarrow \frac{\partial b}{\partial a} = -\frac{\partial f / \partial a}{\partial f / \partial b} = -\frac{f_a}{f_b} \quad (4)$$

EXERCISE 1.1 (a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k) (l) (m) (n) (o) (p) (q) (r) (s) (t) (u) (v) (w) (x) (y) (z)

$$\frac{f_a}{F_b} = \frac{f_b}{F_a}$$

$$\Rightarrow \frac{F_a}{f_a} = \frac{F_b}{f_b}$$

Q The envelop of a plane the sum of the square of whose intercepts on the axes is constant.

$$x^2/3 + y^2/3 + z^2/3 = \text{constant}$$

Q1:- Let the eqⁿ of plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$0 = p(x-a) + (q)(b-y) + r(c-z)$$

$$0 = p(x-a) + q(b-y) + r(c-z)$$

$$0 = p(x-a) + q(b-y) + r(c-z)$$

$$0 = p(x-a) + q(b-y) + r(c-z)$$

$$ps = \frac{p^2}{r^2}$$

$$p =$$

$$p^2 - q^2 - r^2 = (x, y, z)$$

$$p^2 - q^2 - r^2 = (x, y, z)$$

$$p =$$

$$p =$$

Envelope of tangent plane

$$0 = p(x-a) + q(b-y) + r(c-z)$$

$$0 = p(x-a) + q(b-y) + r(c-z)$$

$$0 = p(x-a) + q(b-y) + r(c-z)$$

$$0 = p(x-a) + q(b-y) + r(c-z)$$

Find the edge of regression of the envelop of the family of planes $x \sin \theta - y \cos \theta + z = a$ θ being the parameter.

$$F(\theta) = F(x, y, z, \theta) = x \sin \theta - y \cos \theta + z - a = 0 \quad (1)$$

$$\frac{\partial F}{\partial \theta} = x \cos \theta + y \sin \theta - a = 0 \quad (2)$$

$$\frac{\partial^2 F}{\partial \theta^2} = -x \sin \theta + y \cos \theta = 0 \quad (3)$$

Squaring and adding the respective sides of (2) & (3)

$$x^2 \sin^2 \theta + y^2 \cos^2 \theta + z^2 + a^2 + x^2 \cos^2 \theta + y^2 \sin^2 \theta + a^2 + x^2 \sin^2 \theta + y^2 \cos^2 \theta + z^2 + a^2 = 0$$

$$(x \cos \theta + y \sin \theta - a)^2 + (-x \sin \theta + y \cos \theta)^2 = 0$$

$$\Rightarrow x^2 \cos^2 \theta + y^2 \sin^2 \theta + a^2 + 2xy \cos \theta \sin \theta - 2ax \cos \theta - 2ay \sin \theta + x^2 \sin^2 \theta + y^2 \cos^2 \theta - 2xy \sin \theta \cos \theta = 0$$

$$\Rightarrow x^2 + y^2 + a^2 - 2ax \cos \theta - 2ay \sin \theta = 0$$

From equation (3) $-x \sin \theta = -y \cos \theta$

$$\Rightarrow \frac{y}{x} = \tan \theta$$

$$\Rightarrow \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

From (1) & (3)

$$x - a \theta = 0$$

$$z = a \theta \Rightarrow \theta = \frac{z}{a}$$

$$\Rightarrow \tan^{-1} \left(\frac{y}{x} \right) = \frac{z}{a}$$

from (2) & (3)

$$x \sin \theta \cos \theta + y \sin^2 \theta = a \sin \theta$$

$$-x \cos \theta \sin \theta + y \cos^2 \theta = 0$$

$$y = a \sin \theta$$

$$-x \sin \theta + a \sin \theta \cos \theta = 0$$

$$\Rightarrow x = \frac{-a \sin \theta \cos \theta}{-\sin \theta} = a \cos \theta$$

$$x^2 + y^2 = a^2 (\cos^2 \theta + \sin^2 \theta) = a^2$$

$$\Rightarrow x^2 + y^2 = a^2 \Rightarrow a = \sqrt{x^2 + y^2}$$

$$\text{Now } \tan\left(\frac{y}{x}\right) = \frac{z}{a}$$

$$\Rightarrow \tan\left(\frac{y}{x}\right) = \frac{z}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow \frac{y}{x} = \tan\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$

$$\Rightarrow y = x \tan\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$

(14) The envelop of a

(15) Prove that the envelop of the surface $F(x, y, z, a, b, c) = 0$ where a, b, c are parameters connected by the relation $f(a, b, c) = 0$ is obtained by eliminating a, b, c from the eqns

$$F = 0, F = 0 \quad \frac{F_a}{f_a} = \frac{F_b}{f_b} = \frac{F_c}{f_c}$$

solⁿ Given that the eqⁿ of the surface is $F(x, y, z, a, b, c) = 0$ — (1)

The parameters (a, b, c) are connected by the relation $f(a, b, c) = 0$ — (2)

To prove that the envelop can be obtained by eliminating a, b, c from the equations

$$\frac{F_a}{f_a} = \frac{F_b}{f_b} = \frac{F_c}{f_c}$$

let us take c as dependent parameter
and a, b as independent parameter

$$F(x, y, z, a, b, c) = 0$$

$$\Rightarrow \frac{\partial}{\partial a} F(x, y, z, a, b, c) = 0$$

$$\Rightarrow \frac{\partial F}{\partial a} \cdot \frac{\partial a}{\partial a} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial a} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial a} + \frac{\partial F}{\partial a} \cdot \frac{\partial a}{\partial a} +$$

$$\frac{\partial F}{\partial b} \cdot \frac{\partial b}{\partial a} + \frac{\partial F}{\partial c} \cdot \frac{\partial c}{\partial a} = 0$$

$$\Rightarrow 0 + 0 + 0 + F_a + 0 + F_c \frac{\partial c}{\partial a} = 0$$

$$\Rightarrow \frac{\partial c}{\partial a} = -\frac{F_a}{F_c} \quad \text{--- (3)}$$

$$\frac{\partial}{\partial b} F(x, y, z, a, b, c) = 0$$

$$\Rightarrow \frac{\partial F}{\partial a} \frac{\partial a}{\partial b} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial b} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial b} + \frac{\partial F}{\partial a} \frac{\partial a}{\partial b} + \frac{\partial F}{\partial b} \frac{\partial b}{\partial b} + \frac{\partial F}{\partial c} \frac{\partial c}{\partial b} = 0$$

$$\Rightarrow 0 + 0 + 0 + 0 + F_b + F_c \frac{\partial c}{\partial b} = 0$$

$$\Rightarrow \frac{\partial c}{\partial b} = -\frac{F_b}{F_c} \quad \text{--- (4)}$$

$$F(x, b, c) = 0$$

$$\frac{\partial F}{\partial a} \frac{\partial a}{\partial c} + \frac{\partial F}{\partial b} \frac{\partial b}{\partial c} + \frac{\partial F}{\partial c} \frac{\partial c}{\partial c} = 0$$

$$\Rightarrow F_a + F_c \frac{\partial c}{\partial a} = 0 \quad \frac{\partial c}{\partial a} = -\frac{F_a}{F_c} \quad c = f, \quad 0 = 1$$

$$\Rightarrow \frac{\partial c}{\partial a} = -\frac{F_a}{F_c} \quad \text{--- (5)}$$

$$F(x, a, c) = 0$$

$$\frac{\partial}{\partial b} (F(x, a, c)) = 0$$

$$\Rightarrow \frac{\partial F}{\partial a} \frac{\partial a}{\partial b} + \frac{\partial F}{\partial b} \frac{\partial b}{\partial b} + \frac{\partial F}{\partial c} \frac{\partial c}{\partial b} = 0$$

$$\Rightarrow 0 + F_b + F_c \frac{\partial c}{\partial b} = 0$$

$$\Rightarrow \frac{\partial c}{\partial b} = -\frac{F_b}{F_c} \quad \text{--- (6)}$$

From equation (3) and (5)

$$-\frac{F_a}{F_c} = -\frac{f_a}{f_c}$$

$$\boxed{\frac{F_a}{f_a} = \frac{F_c}{f_c}} \quad \text{--- (7)}$$

From equation (4) & (6)

$$-\frac{F_b}{F_c} = -\frac{f_b}{f_c}$$

$$\Rightarrow \boxed{\frac{F_b}{f_b} = \frac{F_c}{f_c}} \quad \text{--- (8)}$$

From eqn (7) & (8)

$$\boxed{\frac{F_a}{f_a} = \frac{F_b}{f_b} = \frac{F_c}{f_c}} \quad \text{Proved}$$

(16) The envelop of the plane $lx + my + nz = p$ where $p^2 = a^2l^2 + b^2m^2 + c^2n^2$ is an ellipsoid

Proof: The equation of family plane is $lx + my + nz = p$ --- (1)

where l, m, n, p are parameters

The parameters l, m, n, p are connected by the relation $p^2 = a^2l^2 + b^2m^2 + c^2n^2$

Here $F(x, y, z, l, m, n, p) = lx + my + nz - p$

$$\Phi(l, m, n, p) = a^2l^2 + b^2m^2 + c^2n^2 - p^2$$

The eqn of the envelop can be obtained by eliminating l, m, n, p from the following equation

$$F = 0 \Rightarrow lx + my + nz - p = 0 \quad \text{--- (2)}$$

$$\Phi = 0 \Rightarrow a^2l^2 + b^2m^2 + c^2n^2 - p^2 = 0 \quad \text{--- (3)}$$

$$\frac{F_l}{\Phi_l} = \frac{F_m}{\Phi_m} = \frac{F_n}{\Phi_n} = \frac{F_p}{\Phi_p} = \Phi$$

$$\frac{x}{2a^2l} = \frac{y}{2b^2m} = \frac{z}{2c^2n} = \frac{1}{2p} \quad \text{--- (4)}$$

From equation (1)

$$\frac{x}{a^2 l} = \frac{y}{b^2 m} = \frac{z}{c^2 n} = \frac{1}{p}$$

$$l = \frac{px}{a^2}, \quad m = \frac{py}{b^2}, \quad n = \frac{pz}{c^2}$$

Putting the value of l, m, n in eq^s

$$\frac{px^2}{a^2} + \frac{py^2}{b^2} + \frac{pz^2}{c^2} - p = 0$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{proved}$$

⑧. Sphere of constant radius b have their centres on the fixed circle $x^2 + y^2 = a^2, z=0$.
Prove that their envelop is the surface

$$(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$$

Proof

Let the centre of sphere = $C = (h, k, 0)$

Eqⁿ of the sphere

$$(x-h)^2 + (y-k)^2 + (z-0)^2 = b^2 \quad \text{--- (1)}$$

Since $C(h, k, 0)$ lies in the circle

$$h^2 + k^2 = a^2 \quad \text{--- (2)}$$

Hence family of surface $(x-h)^2 + (y-k)^2 + z^2 = b^2$

Parameter h and k are connected by the relation

$$h^2 + k^2 - a^2 = 0$$

$$\text{Here } F = (x-h)^2 + (y-k)^2 + z^2 - b^2$$

$$\Phi = h^2 + k^2 - a^2$$

Equation of envelop can be obtained by eliminating h, k from the following eq^s.

$$F=0, (x-h)^2 + (y-k)^2 + z^2 - b^2 = 0 \quad \text{--- (3)}$$

$$\phi=0, h^2 + k^2 - a^2 = 0 \quad \text{--- (4)}$$

$$\frac{F_h}{F_n} = \frac{F_k}{F_n}$$

$$\Rightarrow \frac{-2(x-h)}{2h} = \frac{-2(y-k)}{2k}$$

$$\Rightarrow \frac{x-h}{h} = \frac{y-k}{k} \quad \text{--- (5)}$$

From eqn --- (5)

$$\frac{x-h}{h} = \frac{y-k}{k}$$

$$\Rightarrow \frac{x}{h} - 1 = \frac{y}{k} - 1$$

$$\Rightarrow \frac{x}{h} = \frac{y}{k}$$

$$\Rightarrow k = \frac{hy}{x}$$

Putting $k = \frac{hy}{x}$ in eqn --- (4)

$$h^2 + \frac{h^2 y^2}{x^2} = a^2$$

$$\Rightarrow h^2 \left(1 + \frac{y^2}{x^2}\right) = a^2$$

$$\Rightarrow h^2 (a^2 + y^2) = a^2 x^2$$

$$\Rightarrow h = \frac{ax}{\sqrt{a^2 + y^2}}$$

$$k = \frac{hy}{x} = \frac{y}{x} \cdot \frac{ax}{\sqrt{a^2 + y^2}} = \frac{ay}{\sqrt{a^2 + y^2}}$$

Putting value of k & h in eqn (3) we have

$$\left(x - \frac{ax}{\sqrt{a^2 + y^2}}\right)^2 + \left(y - \frac{ay}{\sqrt{a^2 + y^2}}\right)^2 + z^2 - b^2 = 0$$

$$\Rightarrow \left(x\sqrt{a^2 + y^2} - ax\right)^2 + \left(y\sqrt{a^2 + y^2} - ay\right)^2 + (z^2 - b^2)(a^2 + y^2) = 0$$

$$\Rightarrow x^2(a^2 + y^2) + a^2x^2 - 2a^2x\sqrt{a^2 + y^2} + y^2(a^2 + y^2) + a^2y^2 - 2a^2y\sqrt{a^2 + y^2} + z^2a^2 + z^2y^2 - b^2a^2 - b^2y^2 = 0$$

$$\begin{aligned}
 &\Rightarrow x^4 + 2xy^2 + a^2z^2 - 2aa^2\sqrt{x^2+y^2} + y^2z^2 + y^4 + 2ay^2z \\
 &\quad - 2ay^2\sqrt{x^2+y^2} + z^2a^2 - z^2y^2 - b^2x^2 - b^2y^2 = 0 \\
 &\Rightarrow x^4 + 2xy^2 + y^4 + a^2z^2 + a^2y^2 + z^2a^2 - z^2y^2 - b^2x^2 \\
 &\quad - b^2y^2 - 2aa^2\sqrt{x^2+y^2} - 2ay^2\sqrt{x^2+y^2} = 0 \\
 &\Rightarrow (x^2+y^2)^2 + a^2(x^2+y^2) - b^2(x^2+y^2) - 2aa^2\sqrt{x^2+y^2} \\
 &\quad - 2ay^2\sqrt{x^2+y^2} = 0 \\
 &\Rightarrow x^2(x^2+y^2+a^2-b^2-2a\sqrt{x^2+y^2}) + y^2(x^2+y^2+a^2-b^2-2a\sqrt{x^2+y^2}) = 0 \quad (\because z=0) \\
 &\Rightarrow (x^2+y^2+a^2-b^2-2a\sqrt{x^2+y^2}) a^2 = 0 \quad (\because x^2+y^2 = a^2) \\
 &\Rightarrow x^2+y^2+a^2-b^2 = 2a\sqrt{x^2+y^2} \\
 &\Rightarrow (x^2+y^2+a^2-b^2)^2 = 4a^2(x^2+y^2) \quad \text{Proved}
 \end{aligned}$$

⑥ Prove that any tangent plane to the surface $a(x^2+y^2) + xyz = 0$ meets it again in a conic whose projection on the xy -plane

② The envelope of a plane, the sum of the square of whose intercept on the axes is constant is a surface.

$$x^2/3 + y^2/3 + z^2/3 = \text{const.}$$

Solution:- let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{--- (1)}$$

By given condition

$$a^2 + b^2 + c^2 = k^2 \text{ where } k \text{ is a constant} \quad (i)$$

$$\text{Here } F(a, b, c) = \frac{a}{a} + \frac{b}{b} + \frac{c}{c} - 1$$

$$f(a, b, c) = a^2 + b^2 + c^2 - k^2$$

Equation of envelop can be obtained by eliminating a, b, c from the following equation.

$$F=0 \quad \frac{a}{a} + \frac{b}{b} + \frac{c}{c} - 1 = 0 \quad \text{--- (ii)}$$

$$f=0 \quad a^2 + b^2 + c^2 - k^2 = 0 \quad \text{--- (iv)}$$

$$\frac{F_a}{F_a} = \frac{F_b}{F_b} = \frac{F_c}{F_c}$$

$$\frac{-a/a^2}{2a} = \frac{-b/b^2}{2b} = \frac{-c/c^2}{2c} \quad \text{--- (v)}$$

From the equation (v) we get

$$\left(\frac{-a}{a^2}\right)/2a = \left(\frac{-b}{b^2}\right)/2b = \left(\frac{-c}{c^2}\right)/2c$$

$$\Rightarrow \frac{a}{a^3} = \frac{b}{b^3} = \frac{c}{c^3} = \lambda \text{ (say)}$$

$$\Rightarrow a^3 = \frac{a}{\lambda}, \quad b^3 = \frac{b}{\lambda}, \quad c^3 = \frac{c}{\lambda}$$

$$\Rightarrow a = \left(\frac{a}{\lambda}\right)^{1/3} \quad b = \left(\frac{b}{\lambda}\right)^{1/3} \quad c = \left(\frac{c}{\lambda}\right)^{1/3}$$

Putting the above value in (iv)

$$\frac{a^{2/3}}{\lambda^{2/3}} + \frac{b^{2/3}}{\lambda^{2/3}} + \frac{c^{2/3}}{\lambda^{2/3}} - k^2 = 0$$

$$\Rightarrow \frac{a^{2/3} + b^{2/3} + c^{2/3}}{\lambda^{2/3}} - k^2 = 0$$

$$\Rightarrow a^{2/3} + b^{2/3} + c^{2/3} = k^2 \lambda^{2/3}$$

$$\Rightarrow \boxed{a^{2/3} + b^{2/3} + c^{2/3} = \text{const.}}$$

i.e. the required equation of the envelope

(13) Prove that the envelope of a plane which forms with the co-ordinate planes a tetrahedron of constant volume is a surface $xyz = \text{constant}$.

Proof:- let the eqⁿ of family of planes is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{--- (i)}$$

By given condⁿ

Volume of the tetrahedron = constant

$$\Rightarrow \frac{1}{3} \left(\frac{abc}{2} \right) (c) = \text{constant}$$

$$\Rightarrow abc = 6 \times \text{constant}$$

$$\Rightarrow abc = k^3 \quad \text{--- (ii)} \quad \text{where } k \text{ is constant}$$

$$\text{Here } F(a, b, c) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1$$

$$f(a, b, c) = abc - k^3$$

Eqⁿ of envelope can be obtained by eliminating a, b, c from the following eq^s

$$F = 0 \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0 \quad \text{--- (iii)}$$

$$f = 0 \quad abc - k^3 = 0 \quad \text{--- (iv)}$$

$$\frac{F_a}{f_a} = \frac{F_b}{f_b} = \frac{F_c}{f_c} = \frac{0}{f_c} = \frac{\frac{-x}{a^2}}{\frac{-x}{a^2}} = \frac{\frac{-y}{b^2}}{\frac{-y}{b^2}} = \frac{\frac{-z}{c^2}}{\frac{-z}{c^2}} \quad \text{--- (v)}$$

From eqⁿ (v)

$$\frac{(-x/a^2)}{bc} = \frac{(-y/b^2)}{ac} = \frac{(-z/c^2)}{ab}$$

$$\frac{-a}{abc} = \frac{-y}{ab^2c} = \frac{-z}{abc^2}$$

$$\frac{a}{a} = \frac{y}{b} = \frac{z}{c} = \lambda \text{ (say)}$$

$$\text{let } \frac{a}{a} = \frac{y}{b} = \frac{z}{c} = \lambda \text{ (say)}$$

$$a = \frac{a}{\lambda}, b = \frac{y}{\lambda}, c = \frac{z}{\lambda}$$

Putting the value of a, b, c in eqn (iv)

$$\frac{xyz}{\lambda^3} - \lambda^3 = 0$$

$$xyz = \lambda^6$$

$$xyz = \text{const}$$

i.e. the required equation of the envelope

$$\frac{xyz}{\lambda^6} = 1, \frac{xyz}{\lambda^6} = 1, \frac{xyz}{\lambda^6} = 1$$

Therefore the required equation of the envelope is $xyz = \text{const}$

$$\frac{xyz}{\lambda^6} = 1, \frac{xyz}{\lambda^6} = 1, \frac{xyz}{\lambda^6} = 1$$

Since the two points are adjacent to each other, the distance between them is small, and the curve is nearly a straight line.

curvature co-ordinate on a surface:-

Suppose $\vec{r} = r(u, v)$ be the eqⁿ of surface where u, v are independent parameters any relation betⁿ the two parameter (surface).

any $f(u, v) = 0$ represents a curve on the surface.

we shall use u as the 1st parameter and v as the 2nd parameter and r is the surface r_1 and r_2 to denote the partial derivative w.r.t u & v respectively

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u}, \quad \vec{r}_2 = \frac{\partial \vec{r}}{\partial v}$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2}, \quad \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial v^2}$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u \partial v}, \quad \vec{r}_{21} = \frac{\partial^2 \vec{r}}{\partial v \partial u}$$

Fundamental magnitudes of 1st order:-

consider two neighbourhood point on the surface with position vectors r and $r + dr$ corresponding to the parameters u, v and $u + du, v + dv$ respectively then

$$\begin{aligned} d\vec{r} &= \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv \\ &= \vec{r}_1 du + \vec{r}_2 dv \end{aligned} \quad \begin{matrix} u \\ r \\ v \end{matrix}$$

Since the two points are adjacent point on a curve passing through them the length ds of the element of arc joining them is equals to there actual distance $|d\vec{r}|$.

$$\text{thus } ds^2 = d\vec{r}^2$$

$$= (\vec{r}_1 du + \vec{r}_2 dv)^2$$

$$= \vec{r}_1^2 du^2 + \vec{r}_2^2 dv^2 + 2\vec{r}_1 \cdot \vec{r}_2 du dv$$

$$= E du^2 + 2F du dv + G dv^2 \quad (1)$$

$$\text{where } \left. \begin{aligned} E &= \vec{r}_1^2 \\ F &= \vec{r}_1 \cdot \vec{r}_2 \\ G &= \vec{r}_2^2 \end{aligned} \right\} \quad (2)$$

The quantities denoted by E, F, G are called fundamental magnitudes of the 1st order. Relation (1) is called 1st fundamental form of the metric form of the surface.

$$\boxed{H^2 = EG - F^2}$$

Q. calculate the fundamental of 1st order for the surface $x = u \cos \phi, y = u \sin \phi, z = f(u)$

$$\text{Sol: } \vec{r} = u \cos \phi \hat{i} + u \sin \phi \hat{j} + f(u) \hat{k}$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = \cos \phi \hat{i} + \sin \phi \hat{j} + f'(u) \hat{k}$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial \phi} = -u \sin \phi \hat{i} + u \cos \phi \hat{j} + 0 \hat{k}$$

$$E = \vec{r}_1^2 = \cos^2 \phi + \sin^2 \phi + f'(u)^2 = 1 + [f'(u)]^2$$

$$G = \vec{r}_2^2 = u^2 \sin^2 \phi + u^2 \cos^2 \phi = u^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = -u \sin \phi \cos \phi + u \sin \phi \cos \phi = 0$$

$$H^2 = EG - F^2 = u^4 - 0 = u^4$$

$$H = \sqrt{u^4} = u^2$$

$$H = u \sqrt{1 + [f'(u)]^2}$$

Theorem

The necessary and sufficient condition that the parametric curves are orthogonal is $F = 0$

Proof:- condition necessary:-

Given that Parametric curves are orthogonal.

To prove $F = 0$

The parametric curves are $u = c_1$ and $v = c_2$ where c_1 & c_2 are constant.

We know that angle betⁿ two curves is the angle betⁿ the tangent at the point of intersection

$\vec{r}_1$ and $\vec{r}_2$ are tangent vectors

In the parametric curve $v = c_2$ and $u = c_1$ $\vec{r}_1 = c_2$ and $\vec{r}_2 = c_1$ respectively.

Angle betⁿ $\vec{r}_1$ and $\vec{r}_2$ is $\phi = 90^\circ$

from the dot product we know that

$$\vec{r}_1 \cdot \vec{r}_2 = |\vec{r}_1| |\vec{r}_2| \cos \phi$$

$$= |\vec{r}_1| |\vec{r}_2| \cos 90^\circ$$

$$= 0$$

$$\boxed{F = 0}$$

condition Sufficient:-

Given that $F = 0$

To prove = Parametric curves are orthogonal

Since $F = 0$

$$\vec{r}_1 \cdot \vec{r}_2 = 0$$

$$\Rightarrow |\vec{r}_1| |\vec{r}_2| \cos \phi = 0 \quad (\phi \text{ is the angle betw } \vec{r}_1 \text{ \& } \vec{r}_2)$$

$$\Rightarrow \cos \phi = 0$$

$$\Rightarrow \phi = \frac{\pi}{2}$$

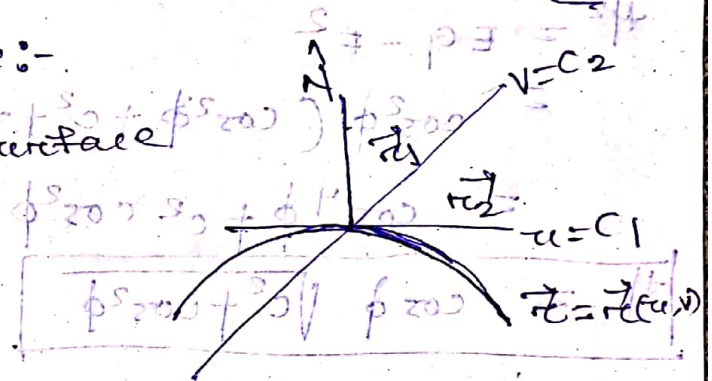
$\therefore$ angle between $\vec{r}_1$ and $\vec{r}_2$ is $\frac{\pi}{2}$

Hence the parametric curves are orthogonal
proved

Normality of a Surface:-

$\hat{n}$ = unit normal surface

$$\begin{aligned} \vec{r} &= \vec{r}(u, v) \\ \vec{r}_1 &= \frac{\partial \vec{r}}{\partial u} \\ \vec{r}_2 &= \frac{\partial \vec{r}}{\partial v} \\ \vec{n} &= \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} \end{aligned}$$



$$\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

Fundamental Magnitude of Second order:-

$$L = \vec{r}_1 \cdot \vec{n}$$

$$M = \vec{r}_1 \cdot \vec{n} \quad \text{where } \vec{n} \text{ is the unit normal}$$

$$N = \vec{r}_2 \cdot \vec{n}$$

$$T^2 = LN - M^2$$

Q calculate the fundamental magnitude for the right helicoid given by $x = u \cos \phi$, $y = u \sin \phi$, $z = c\phi$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = u \cos \phi \hat{i} + u \sin \phi \hat{j} + c\phi \hat{k}$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = \cos \phi \hat{i} + \sin \phi \hat{j} \quad \vec{r}_2 = \frac{\partial \vec{r}}{\partial \phi} = -u \sin \phi \hat{i} + u \cos \phi \hat{j} + c \hat{k}$$

$$\vec{r}_1 = \cos \phi \hat{i} + \sin \phi \hat{j} \quad \vec{r}_2 = -u \sin \phi \hat{i} + u \cos \phi \hat{j} + c \hat{k}$$

$$E = \vec{r}_1^2 = \cos^2 \phi + \sin^2 \phi = 1 \quad G = \vec{r}_2^2 = u^2 \sin^2 \phi + u^2 \cos^2 \phi + c^2 = u^2 + c^2$$

$$\vec{F} = \vec{r}_1 \cdot \vec{r}_2 = -r \sin \phi \cos \phi \hat{z}$$

$$F^2 = r^2 \sin^2 \phi \cos^2 \phi$$

$$H^2 = EG - F^2$$

$$= \cos^2 \phi (\cos^2 \phi + c^2)$$

$$L = 0, M = 0, N = -\sin \phi \hat{j}$$

$$F^2 = LN - M^2$$

$$= 0$$

$$H^2 = EG - F^2$$

$$= \cos^2 \phi (\cos^2 \phi + c^2 + r^2 \sin^2 \phi) - r^2 \sin^2 \phi \cos^2 \phi$$

$$= \cos^4 \phi + c^2 \cos^2 \phi$$

$$|H| = \cos \phi \sqrt{c^2 + \cos^2 \phi}$$

$$L = 0, M = 0$$

$$\vec{N} = \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \phi & 0 & 0 \\ -r \sin \phi & \cos \phi & c \end{vmatrix}$$

$$= \hat{i}(0 - 0) - \hat{j}(c \cos \phi) + \hat{k}(\cos^2 \phi)$$

$$= \cos^2 \phi \hat{k} - c \cos \phi \hat{j}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{\cos^4 \phi + c^2 \cos^2 \phi}$$

$$= \cos \phi \sqrt{c^2 + \cos^2 \phi}$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{-c \cos \phi \hat{j} + \cos^2 \phi \hat{k}}{\cos \phi \sqrt{c^2 + \cos^2 \phi}}$$

$$\vec{L} = \vec{r}_1 \cdot \vec{N} = 0$$

$$M = \vec{r}_{12} \cdot \vec{N}$$

$$= 0$$

$$N = \vec{r}_{22} \cdot \vec{N}$$

$$= \frac{\cos \phi \cos \phi}{\cos \phi \sqrt{c^2 + \cos^2 \phi}} = \frac{\cos \phi}{\sqrt{c^2 + \cos^2 \phi}}$$

$$\boxed{r^2 = LN - M^2}$$

$$= 0 - 0 = 0$$

Q. calculate the fundamental magnitude and unit normal to the surface (N) given in Monge's form. Sol. Eqⁿ of surface in Monge's form is given by $z = f(x, y)$ (1) $\vec{r} = (x, y, z)$ (2)

where x, y are independent parameters

Since z is a funⁿ of x & y

Therefore in eqⁿ (2) x, y can be regarded as parameters.

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = (1, 0, p)$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = (0, 1, q) = (0, 1, q)$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = (0, 0, r) = (0, 0, r)$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = (0, 0, s) = (0, 0, s)$$

$$\vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = (0, 0, t) = (0, 0, t)$$

$$E = \vec{r}_1^2 = 1 + p^2$$

$$G = \vec{r}_2^2 = 1 + q^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = pq$$

$$\begin{aligned}
 T^2 &= EG - F^2 \\
 &= (1+p^2)(1+q^2) - p^2q^2 \\
 &= 1+q^2+p^2+p^2q^2-p^2q^2
 \end{aligned}$$

$$T^2 = 1+p^2+q^2$$

$$\vec{r} = \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & p \\ 0 & 1 & q \end{vmatrix}$$

$$\begin{aligned}
 &= \hat{i}(0-p) - \hat{j}(q-0) + \hat{k}(1) \\
 &= -p\hat{i} - q\hat{j} + \hat{k} \quad \hat{n} = \frac{-p\hat{i} - q\hat{j} + \hat{k}}{\sqrt{1+p^2+q^2}}
 \end{aligned}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{p^2+q^2+1}$$

$$L = \vec{r}_{11} \cdot \hat{n} = \frac{r_1}{\sqrt{1+p^2+q^2}}$$

$$M = \vec{r}_{12} \cdot \hat{n} = \frac{s}{\sqrt{1+p^2+q^2}}$$

$$N = \vec{r}_{22} \cdot \hat{n} = \frac{t}{\sqrt{1+p^2+q^2}}$$

$$\begin{aligned}
 T^2 &= LN - M^2 \\
 &= rt - s^2
 \end{aligned}$$

$$T^2 = rt - s^2$$

$$T^2 = \frac{rt}{\sqrt{1+p^2+q^2}} - \frac{s^2}{1+p^2+q^2} = \frac{rt}{\sqrt{1+p^2+q^2}}$$

$$T^2 = \frac{rt - s^2}{1+p^2+q^2} \left(\frac{p}{\sqrt{1+p^2+q^2}}, 0, 0 \right) = \frac{rt - s^2}{\sqrt{1+p^2+q^2}}$$

$$\left(\frac{p}{\sqrt{1+p^2+q^2}}, 0, 0 \right) = \left(\frac{p}{\sqrt{1+p^2+q^2}}, 0, 0 \right) = \frac{p}{\sqrt{1+p^2+q^2}}$$

$$p+1 = \frac{s}{\sqrt{1+p^2+q^2}} \quad q+1 = \frac{s}{\sqrt{1+p^2+q^2}}$$

Q. 3. Taking x, y as parameters calculate the fundamental magnitude & normal to the surface $ax^2 + by^2 = 2z$

$$\text{Sol}^1: z = \frac{1}{2}(ax^2 + by^2)$$

$$\vec{r} = (x, y, z)$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$$= x\hat{i} + y\hat{j} + \frac{1}{2}(ax^2 + by^2)\hat{k}$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = \hat{i} + ax\hat{k}$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = \hat{j} + by\hat{k}$$

$$\vec{r}_1^2 = 1 + a^2x^2 \quad \vec{r}_2^2 = 1 + b^2y^2$$

$$\vec{r}_1 \cdot \vec{r}_2 = axby$$

$$\vec{r}_1^2 = 1 + (ax)^2 \quad \vec{r}_2^2 = 1 + (by)^2$$

$$\vec{r}_1 \cdot \vec{r}_2 = (ax)(by)$$

$$\vec{r}_1 = h\hat{k} \quad \vec{r}_{11} = a\hat{k}, \quad \vec{r}_{22} = b\hat{k}$$

$$H^2 = EG - F^2$$

$$= (1 + a^2x^2)(1 + b^2y^2) - (axby)^2$$

$$H^2 = 1 + (b^2x^2 + a^2y^2)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & ax \\ 0 & 1 & by \end{vmatrix} = \hat{i}(by) - \hat{j}(ax) + \hat{k}(1)$$

$$\hat{N} = \frac{(-a \sin y) \hat{i} - (b \sin y) \hat{j} + \hat{k}}{\sqrt{1 + (a \sin y)^2 + (b \sin y)^2}}$$

$$L = \vec{r}_{11} \cdot \hat{N} = \frac{a}{\sqrt{1 + (a \sin y)^2 + (b \sin y)^2}}$$

$$M = \frac{b}{\sqrt{1 + (a \sin y)^2 + (b \sin y)^2}}$$

$$N = \frac{1}{\sqrt{1 + (a \sin y)^2 + (b \sin y)^2}}$$

$$T^2 = L^2 + M^2 + N^2 = \frac{a^2 + b^2 + 1}{1 + (a \sin y)^2 + (b \sin y)^2}$$

Ex-2.11 calculate the fundamental magnitude for a surface of revolution.

$$r = (u \cos v, u \sin v, \pm cu)$$

$$\vec{r}_1 = (\cos v, \sin v, \pm cu)$$

$$\vec{r}_2 = (-u \sin v, u \cos v, 0)$$

$$\vec{r}_1 = (0, 0, \pm cu)$$

$$\vec{r}_2 = (-u \cos v, -u \sin v, 0)$$

$$\vec{r}_1 = (0, 0, \pm cu)$$

$$E = \vec{r}_1^2 = \cos^2 v + \sin^2 v + (\pm cu)^2 = 1 + c^2 u^2$$

$$G = \vec{r}_2^2 = u^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = -u \sin v \cos v \pm u \sin v \cos v = 0$$

$$\frac{1}{\mu^2} = \epsilon \epsilon_1 - F^2$$

$$\frac{1}{\mu^2} = \epsilon \epsilon^2 [1 + \epsilon^2(u)^2]$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & \epsilon(u) \\ -\epsilon \sin v & \epsilon \cos v & 0 \end{pmatrix} = \hat{i} (-\epsilon \sin v \cos v) + \hat{j} (\epsilon \sin^2 v) + \hat{k} (\epsilon \cos^2 v)$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{-\epsilon \sin v \cos v \hat{i} + \epsilon \sin^2 v \hat{j} + \epsilon \cos^2 v \hat{k}}{\epsilon \sqrt{1 + \epsilon^2(u)^2}}$$

$$L = \vec{r}_{11} \cdot \vec{N} = \frac{\epsilon^2(u)}{1 + \epsilon^2(u)^2}$$

$$M = \vec{r}_{12} \cdot \vec{N} = \frac{\epsilon \sin v \cos v \epsilon(u) + \epsilon \sin^2 v \cos v \epsilon(u)}{1 + \epsilon^2(u)^2}$$

$$N = \vec{r}_{22} \cdot \vec{N} = \frac{\epsilon \cos^2 v \epsilon(u) + \epsilon \sin^2 v \epsilon(u)}{1 + \epsilon^2(u)^2} = \frac{\epsilon \epsilon(u)}{1 + \epsilon^2(u)^2}$$

$$\therefore T^2 = L^2 + M^2 + N^2 = \frac{\epsilon^4(u) \epsilon^2(u)}{1 + \epsilon^2(u)^2} + \frac{4 \epsilon^2 \sin^2 v \cos^2 v \epsilon^2(u)}{1 + \epsilon^2(u)^2} + \frac{\epsilon^4(u) \epsilon^2(u)}{1 + \epsilon^2(u)^2}$$

$$\vec{N} = \frac{-\epsilon \sin v \cos v \hat{i} + \epsilon \sin^2 v \hat{j} + \epsilon \cos^2 v \hat{k}}{\sqrt{1 + \epsilon^2(u)^2}}$$

$$L = \vec{r}_{11} \cdot \vec{N} = \frac{\epsilon^2(u)}{1 + \epsilon^2(u)^2}$$

$$M = \vec{r}_{12} \cdot \vec{N} = \frac{\epsilon \sin v \cos v \epsilon(u)}{1 + \epsilon^2(u)^2}$$

$$N = \vec{r}_{22} \cdot \vec{N} = \frac{\epsilon^2(u) \cos^2 v + \epsilon^2(u) \sin^2 v}{1 + \epsilon^2(u)^2} = \frac{\epsilon^2(u)}{1 + \epsilon^2(u)^2}$$

$$J^2 = LN - M^2$$

$$T^2 = \frac{u^3 f(u) f''(u)}{f'(u)^2}$$

Ex-112:- Find the fundamental magnitude for the surface given by

$$x = a(u+v), y = b(u-v), z = uv$$

Sol:- $\vec{r}(u,v) = a(u+v)\hat{i} + b(u-v)\hat{j} + uv\hat{k}$

$$\vec{r}_1 = a\hat{i} + b\hat{j} + v\hat{k}$$

$$\vec{r}_2 = a\hat{i} - b\hat{j} + u\hat{k}$$

$$\vec{r}_{12} = \hat{k}$$

$$\vec{r}_{11} = 0$$

$$\vec{r}_{22} = 0$$

$$E = \vec{r}_1 \cdot \vec{r}_1 = a^2 + b^2 + v^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = a^2 - b^2 + uv$$

$$G = \vec{r}_2 \cdot \vec{r}_2 = a^2 + b^2 + u^2$$

$$H^2 = EG - F^2$$

$$= (a^2 + b^2 + v^2)(a^2 + b^2 + u^2) - (a^2 - b^2 + uv)^2$$

$$= a^4 + a^2b^2 + a^2u^2 + a^2v^2 + b^4 + u^2b^2 + a^2v^2 +$$

$$b^2v^2 + u^2v^2 - a^4 - b^4 - u^2v^2 + 2a^2b^2$$

$$+ 2b^2uv - 2a^2uv$$

$$= 4a^2b^2 + a^2(u^2 + v^2 - 2uv) + b^2(u^2 + v^2 + 2uv)$$

$$H^2 = 4a^2b^2 + a^2(u-v)^2 + b^2(u+v)^2$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & v \\ a & -b & u \end{vmatrix} = \hat{i}(bu + bv) - \hat{j}(au - av) + \hat{k}(-a^2 - b^2)$$

$$\hat{n} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{b(u+v)\hat{i} - a(u-v)\hat{j} - (a^2 + b^2)\hat{k}}{H}$$

$$L = \vec{r}_{11} \cdot \vec{N} = 0$$

$$M = \vec{r}_{12} \cdot \vec{N} = \frac{-2ab}{4}$$

$$N = \vec{r}_{22} \cdot \vec{N} = 0$$

$$H^2 = LN - M^2 = \left(-\frac{yazb^2}{4} \right)$$

$$H^2 = \frac{-yazb^2}{4}$$

EXERCISE - 2 (b)

1. Show that on the following surfaces, the parametric curves are orthogonal.

(i) $r = (u \cos v, u \sin v, \pm u)$ (Surface of revolution)

Proof:- $\vec{r}_1 = \cos v \hat{i} + \sin v \hat{j} + \hat{k}$

$$\vec{r}_2 = -u \sin v \hat{i} + u \cos v \hat{j} + \frac{1}{u} \hat{k}$$

$$\vec{r}_1 \cdot \vec{r}_2 = -u \sin v \cos v + u \cos v \sin v + \frac{1}{u} = 0$$

$\therefore$ parametric curves are orthogonal.

(ii) $r = a \cos u \cos v, \sin u \sin v, \cos u$ (Sphere)
a being constant.

Proof:- $\vec{r}_1 = a \cos u \cos v \hat{i} + a \sin u \cos v \hat{j} - a \sin u \hat{k}$

$$\vec{r}_2 = -a \sin u \sin v \hat{i} + a \sin u \cos v \hat{j} + a \cos u \hat{k}$$

$$\vec{r}_1 \cdot \vec{r}_2 = -a^2 \sin u \cos u \sin v \cos v + a^2 \sin u \sin v \cos v + a^2 \sin u \cos u \cos v = 0$$

$\therefore$ Parametric curves are orthogonal.

(iii) $x = u \cos v, y = u \sin v, z = u \cot \phi$ (Right circular cone)

where ϕ is the semi-vertical angle.

sol:- $\vec{r} = u \cos v \hat{i} + u \sin v \hat{j} + u \cot \phi \hat{k}$

$$\vec{r}_1 = \cos v \hat{i} + \sin v \hat{j} + \cot \phi \hat{k}$$

$$\vec{r}_1 \cdot \vec{r}_2 = -u \sin v \cos v + u \cos v \sin v = 0$$

(iv) $x = u \cos v$, $y = u \sin v$, $z = cv$ Right helicoid

$$\text{Sol: } \vec{r} = (u \cos v, u \sin v, cv)$$

$$\vec{r}_1 = (-u \sin v, u \cos v, 0)$$

$$\vec{r}_2 = (0, -u \cos v, u \sin v)$$

$$\vec{r}_1 = (\cos v, \sin v, 0)$$

$$\vec{r}_2 = (-u \sin v, u \cos v, c)$$

$$\vec{r}_1 \cdot \vec{r}_2 = -u \sin v \cos v + u \sin v \cos v + 0 = 0$$

(5) Find the metric of the paraboloid
 $\vec{r} = (u, v, u^2 - v^2)$

$$\text{Sol: } \vec{r} = (u, v, u^2 - v^2)$$

$$\vec{r}_1 = (1, 0, 2u)$$

$$\vec{r}_2 = (0, 1, -2v)$$

$$E = \vec{r}_1^2 = 1 + 4u^2 \quad F = \vec{r}_1 \cdot \vec{r}_2 = -4uv$$

$$G = \vec{r}_2^2 = 1 + 4v^2$$

$$ds^2 = E du^2 + G dv^2 + 2F du dv$$

$$= (1 + 4u^2) du^2 + (1 + 4v^2) dv^2 - 8uv du dv$$

$$= (1 + 4u^2) du^2 - 8uv du dv + (1 + 4v^2) dv^2$$

in the metric form.

4. calculate the fundamental magnitudes & the unit normal for the conoid: $x = u \cos v$, $y = u \sin v$, $z = f(v)$ with u, v are parameters.

$$\begin{aligned} \vec{r}_1 &= u \cos v \hat{i} + u \sin v \hat{j} + f(v) \hat{k} \\ \vec{r}_2 &= \cos v \hat{i} + \sin v \hat{j} + 0 \hat{k} \\ \vec{r}_3 &= -u \sin v \hat{i} + u \cos v \hat{j} + f'(v) \hat{k} \\ \vec{r}_4 &= 0 \end{aligned}$$

$$\begin{aligned} \vec{r}_5 &= -\sin v \hat{i} + \cos v \hat{j} \\ \vec{r}_6 &= -u \cos v \hat{i} - u \sin v \hat{j} + f''(v) \hat{k} \end{aligned}$$

$$E = \vec{r}_1 \cdot \vec{r}_1 = 1$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = 0$$

$$G = \vec{r}_2 \cdot \vec{r}_2 = u^2 + (f'(v))^2$$

$$\begin{aligned} H^2 &= EG - F^2 \\ &= u^2 + (f'(v))^2 \end{aligned}$$

$$\begin{aligned} \vec{r}_1 \times \vec{r}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & f'(v) \end{vmatrix} \\ &= \hat{i} (0 \sin v - f'(v)) - \hat{j} (f'(v) \cos v) + \hat{k} (-u \cos^2 v + u \sin^2 v) \\ &= 0 \sin v f'(v) \hat{i} - f'(v) \cos v \hat{j} + u \hat{k} \end{aligned}$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{0 \sin v f'(v) \hat{i} - f'(v) \cos v \hat{j} + u \hat{k}}{\sqrt{u^2 + (f'(v))^2}}$$

$$L = \vec{r}_1 \cdot \vec{N} = 0$$

$$M = \vec{r}_2 \cdot \vec{N} = \frac{-\sin^2 v f'(v) - \cos^2 v f'(v)}{\sqrt{u^2 + (f'(v))^2}} = \frac{-f'(v)}{\sqrt{u^2 + (f'(v))^2}}$$

$$N = \vec{r}_3 \cdot \vec{N} = \frac{-u f''(v)}{\sqrt{u^2 + (f'(v))^2}}$$

$$t^2 = LM - M^2$$

$$t^2 = 1 - \frac{\{f'(v)\}^2}{4}$$

⑤ calculate the fundamental magnitudes & the unit normal for the helicoid

$q = u \cos v$, $y = u \sin v$, $z = f(u) + cv$
where c is a constant & u, v are parameters.

$$\vec{r} = u \cos v \hat{i} + u \sin v \hat{j} + (f(u) + cv) \hat{k}$$

$$\vec{r}_1 = \cos v \hat{i} + \sin v \hat{j} + f'(u) \hat{k}$$

$$\vec{r}_2 = -u \sin v \hat{i} + u \cos v \hat{j} + c \hat{k}$$

$$E = \vec{r}_1^2 = \cos^2 v + \sin^2 v + \{f'(u)\}^2 = 1 + \{f'(u)\}^2$$

$$G = \vec{r}_2^2 = u^2 + c^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = c f'(u)$$

$$H^2 = EG - F^2$$

$$= (u^2 + c^2) [1 + \{f'(u)\}^2] - c^2 \{f'(u)\}^2 = u^2 + c^2 + u^2 \{f'(u)\}^2$$

$$H = u \sqrt{1 + \{f'(u)\}^2} + \frac{c^2}{2u}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & f'(u) \\ -u \sin v & u \cos v & c \end{vmatrix}$$

$$= \hat{i} (c \cos v - u f'(u) \sin v) - \hat{j} (c \sin v + u f'(u) \cos v) + \hat{k} (u \cos^2 v + u \sin^2 v) = \hat{i} (c \cos v - u f'(u) \sin v) - \hat{j} (c \sin v + u f'(u) \cos v) + \hat{k} (u)$$

$$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{\|\vec{r}_1 \times \vec{r}_2\|} = \frac{\begin{vmatrix} c \sin v & -u \cos v & u \\ c \cos v & u \sin v & 0 \end{vmatrix}}{\sqrt{(c \sin v + u \cos v)^2 + (c \cos v - u \sin v)^2}}$$

$$\vec{r}_1 = \begin{pmatrix} c \sin v \\ c \cos v \\ u \end{pmatrix}$$

$$\vec{r}_2 = \begin{pmatrix} -u \sin v \\ u \cos v \\ 0 \end{pmatrix}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ c \sin v & c \cos v & u \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$

$$= \begin{pmatrix} -u \cos v \\ -u \sin v \\ c \cos^2 v + u \sin^2 v \end{pmatrix}$$

$$\|\vec{r}_1 \times \vec{r}_2\| = \sqrt{u^2 \cos^2 v + u^2 \sin^2 v + (c \cos^2 v + u \sin^2 v)^2}$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{\|\vec{r}_1 \times \vec{r}_2\|} = \frac{\begin{pmatrix} -u \cos v \\ -u \sin v \\ c \cos^2 v + u \sin^2 v \end{pmatrix}}{\sqrt{u^2 \cos^2 v + u^2 \sin^2 v + (c \cos^2 v + u \sin^2 v)^2}}$$

$$\vec{N} = \frac{\begin{pmatrix} -u \cos v \\ -u \sin v \\ c \cos^2 v + u \sin^2 v \end{pmatrix}}{\sqrt{u^2 \cos^2 v + u^2 \sin^2 v + (c \cos^2 v + u \sin^2 v)^2}}$$

$$\vec{N} = \frac{\begin{pmatrix} -u \cos v \\ -u \sin v \\ c \cos^2 v + u \sin^2 v \end{pmatrix}}{\sqrt{u^2 \cos^2 v + u^2 \sin^2 v + (c \cos^2 v + u \sin^2 v)^2}}$$

$$\vec{N} \cdot \vec{N} = 1$$

$$\vec{N} \cdot \vec{N} = \frac{u^2 \cos^2 v + u^2 \sin^2 v + (c \cos^2 v + u \sin^2 v)^2}{u^2 \cos^2 v + u^2 \sin^2 v + (c \cos^2 v + u \sin^2 v)^2} = 1$$

$$\vec{N} \cdot \vec{N} = 1$$

⑥ calculate the fundamental magnitude including the unit normal for the surface $\vec{r} = u \cos v \hat{i} + u \sin v \hat{j} + 2u \hat{k}$

$$\vec{r} = u \cos v \hat{i} + u \sin v \hat{j} + 2u \hat{k}$$

$$\vec{r}_1 = \cos v \hat{i} + \sin v \hat{j} + 2 \hat{k}$$

$$\vec{r}_2 = -u \sin v \hat{i} + u \cos v \hat{j} + 0 \hat{k}$$

$$E = \vec{r}_1 \cdot \vec{r}_1 = 1 + 1 + 4 = 6$$

$$G = \vec{r}_2 \cdot \vec{r}_2 = u^2 \sin^2 v + u^2 \cos^2 v = u^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = -\sin v \cos v + 2 \cos v \sin v + 0 = \sin v \cos v$$

$$L^2 = EG - F^2$$

$$= 4u^4 - 4u^2 \quad u^2(4u^2 + 1) = 0$$

$$= 4u^2(1 + u^2)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$

$$= \hat{i}(-2u^2 \cos v) - \hat{j}(2u^2 \sin v) + \hat{k}(u \cos^2 v + u \sin^2 v)$$

$$= -2u^2 \cos v \hat{i} - 2u^2 \sin v \hat{j} + u \hat{k}$$

$$\hat{n} = \frac{\vec{r}_1 \times \vec{r}_2}{\|\vec{r}_1 \times \vec{r}_2\|} = \frac{-2u^2 \cos v \hat{i} - 2u^2 \sin v \hat{j} + u \hat{k}}{u \sqrt{4u^2 + 1}}$$

$$\vec{r}_1 = u \hat{i} + v \hat{j} + 2u^2 \hat{k}$$

$$\vec{r}_2 = -\sin v \hat{i} + \cos v \hat{j}$$

$$\vec{r}_{22} = -u \cos v \hat{i} - u \sin v \hat{j}$$

$$L = \vec{r}_{11} \cdot \hat{n} = \frac{2u}{\sqrt{4u^2 + 1}}$$

$$M = \vec{r}_{12} \cdot \hat{n} = \frac{2u^2 \cos v \sin v - 2u^2 \sin v \cos v}{\sqrt{4u^2 + 1}} = 0$$

$$N = \vec{r}_{22} \cdot \hat{n} = \frac{2u^3 \cos^2 v + 2u^3 \sin^2 v}{\sqrt{4u^2 + 1}} = \frac{2u^3}{\sqrt{4u^2 + 1}}$$

$$J^2 = L^2 N^2 - M^2 = \frac{4u^4}{4u^2 + 1} - 0$$

$$\frac{1}{J^2} = \frac{4u^4}{4u^2 + 1}$$

$$\begin{aligned} x &= a(u+v) \\ y &= b(u-v) \\ z &= 2uv \end{aligned}$$

7. Show that on the surface given by
 $x = a(u+v)$, $y = b(u-v)$, $z = uv$
 the parametric curves are straight lines.

Proof:- $\vec{r} = a(u+v)\hat{i} + b(u-v)\hat{j} + uv\hat{k}$

$\vec{r}_1 = a\hat{i} + b\hat{j} + v\hat{k}$

$\vec{r}_2 = a\hat{i} - b\hat{j} + u\hat{k}$

$E = \vec{r}_1 \cdot \vec{r}_1 = a^2 + b^2 + v^2$

$G = \vec{r}_2 \cdot \vec{r}_2 = a^2 + b^2 + u^2$

$F = \vec{r}_1 \cdot \vec{r}_2 = a^2 - b^2 + uv$

$H^2 = EG - F^2$

$= (a^2 + b^2 + v^2)(a^2 + b^2 + u^2) - (a^2 - b^2 + uv)^2$
 $= a^4 + a^2b^2 + a^2u^2 + a^2v^2 + b^4 + b^2u^2 + b^2v^2 + u^2v^2 - a^4 - b^4 - u^2v^2 + 2a^2b^2$
 $+ 2b^2uv - 2a^2uv$

$= 4a^2b^2 + a^2(v^2 + u^2 - 2uv) + b^2(u^2 + v^2 + 2uv)$
 $= 4a^2b^2 + a^2(u-v)^2 + b^2(u+v)^2$

$H^2 = 4a^2b^2 + a^2(u-v)^2 + b^2(u+v)^2$

Now $\vec{r}_1 \times \vec{r}_2 =$

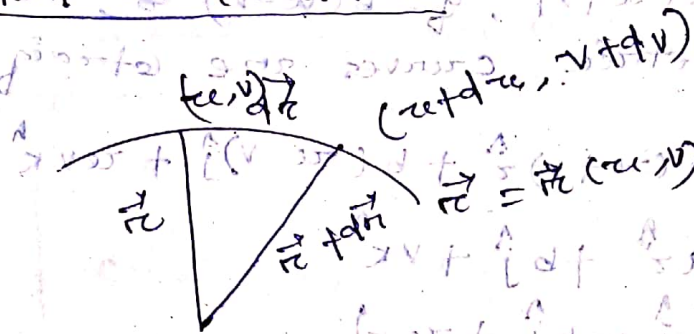
$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & v \\ a & -b & u \end{vmatrix}$
 $= \hat{i}(bv + bv) - \hat{j}(au - av) + \hat{k}(-abu)$

$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{(bv + bv)\hat{i} - (au - av)\hat{j} + 2ab\hat{k}}{H}$

$\vec{r}_{11} = 0$, $\vec{r}_{22} = 0$

$L = \vec{r}_{11} \cdot \vec{N} = 0$, $M = \vec{r}_{12} \cdot \vec{N} = 0$, $N = \vec{r}_{22} \cdot \vec{N} = 0$

$T^2 = LN - M^2 = 0 - \frac{4a^2b^2}{H^2} = -\frac{4a^2b^2}{H^2}$

Direction on a surface:-

Any direction on the surface from a given point (u, v) is determined by the increments du, dv . From a small displacement in the dirⁿ.

let ds be the length of displacement of dr corresponding to du, dv .

let δs be the length of another displacement δr corresponding to increments $\delta u, \delta v$.

$$dr = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv$$

$$\delta r = \frac{\partial \vec{r}}{\partial u} \delta u + \frac{\partial \vec{r}}{\partial v} \delta v$$

$$= \vec{r}_1 du + \vec{r}_2 dv$$

let ψ be the angle between two directions the $dr, \delta r$.

$$\cos \psi = \frac{dr \cdot \delta r}{|dr| |\delta r|}$$

$$\cos \psi = \frac{dr \cdot \delta r}{|dr| |\delta r|}$$

$$= \frac{(\vec{r}_1 du + \vec{r}_2 dv) \cdot (\vec{r}_1 \delta u + \vec{r}_2 \delta v)}{\sqrt{(\vec{r}_1 du + \vec{r}_2 dv) \cdot (\vec{r}_1 du + \vec{r}_2 dv)} \sqrt{(\vec{r}_1 \delta u + \vec{r}_2 \delta v) \cdot (\vec{r}_1 \delta u + \vec{r}_2 \delta v)}} = \frac{\vec{r}_1 \cdot \vec{r}_1 du \delta u + \vec{r}_1 \cdot \vec{r}_2 du \delta v + \vec{r}_2 \cdot \vec{r}_1 dv \delta u + \vec{r}_2 \cdot \vec{r}_2 dv \delta v}{\sqrt{(\vec{r}_1 \cdot \vec{r}_1 du^2 + 2\vec{r}_1 \cdot \vec{r}_2 du dv + \vec{r}_2 \cdot \vec{r}_2 dv^2)} \sqrt{(\vec{r}_1 \cdot \vec{r}_1 \delta u^2 + 2\vec{r}_1 \cdot \vec{r}_2 \delta u \delta v + \vec{r}_2 \cdot \vec{r}_2 \delta v^2)}}$$

$$\cos \phi = \frac{\frac{r_1^2}{r_1} du \delta v + \frac{r_2^2}{r_2} dv \delta u + 2r_1 r_2 \frac{1}{r_1 r_2} du dv}{ds^2}$$

$$\cos \phi = \frac{r_1^2 du \delta v + r_2^2 dv \delta u + 2r_1 r_2 du \delta v}{ds^2} \quad (1)$$

As two disks are $\perp$ to each other

$$0 = \frac{1}{2} E \frac{du \delta v}{\delta v} + F (du \delta v + \delta u dv) + G \delta v dv$$

$$\Rightarrow E \frac{du}{dv} \frac{\delta v}{\delta v} + F \left(\frac{du \delta v + \delta u dv}{\delta v dv} \right) + G = 0$$

$$\text{Now } |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \phi$$

$$\Rightarrow \sin \phi = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$$

$$2 = \frac{|\vec{a} \times \vec{b}|}{ds^2}$$

$$= \frac{(\vec{r}_1 du + \vec{r}_2 dv) \times (\vec{r}_1 \delta u + \vec{r}_2 \delta v)}{ds^2}$$

$$0 = \frac{(\vec{r}_1^2 du \delta v + \vec{r}_2^2 dv \delta u + 2\vec{r}_1 \cdot \vec{r}_2 du \delta v)}{ds^2}$$

$$= (\vec{r}_1 \times \vec{r}_2) du \delta v + (\vec{r}_2 \times \vec{r}_1) dv \delta u$$

$$\sin \phi = \frac{4 |du \delta v - dv \delta u|}{ds^2}$$

Particular case:-

Case-1: The angle α betⁿ the two dirⁿ du, dv and that of the vector $V = \text{constant}$ may be deducted from the result - (1) by substituting $\delta v = 0$ and $\delta s = \sqrt{E} \delta u$

$$\text{Thus } \cos \alpha = \frac{1}{\sqrt{E}} \left(E \frac{du}{ds} + F \frac{dv}{ds} \right) = 0$$

Similarly,

its inclination β to the parametric curve $u = \text{constant}$ is obtained by putting $\delta u = 0$ and $\delta s = \sqrt{G} \delta v$

$$\text{Thus } \cos \beta = \frac{1}{\sqrt{G}} \left(F \frac{du}{ds} + G \frac{dv}{ds} \right)$$

Orthogonal trajectory:-

Define orthogonal trajectory of a family of curve

Find the differential of family of curve

Find the eqⁿ of orthogonal trajectories

Find the condⁿ that to family of curve given by $P du^2 + Q du dv + R dv^2 = 0$

will be orthogonal.

Ans: let $\phi(u, v) = c_1$ (1) be

eqⁿ of family of curve lying on the

surface $\vec{r} = \vec{r}(u, v)$

If $\exists$ a family curves
 $\psi(u, v) = c_2$ — (2)

on the same surface such that at every point of the surface of the two curves one from each family are orthogonal then family of curves — (ii) is called the orthogonal trajectories of the family of curves.

To find the differential equation of orthogonal trajectories of a given family of curves.

Let the differential equation of a given family of curves be

$$P \delta u + Q \delta v = 0 \text{ — (iii)}$$

where P, Q are functions of u, v

$$\Rightarrow P \delta u = -Q \delta v$$

$$\frac{\delta u}{\delta v} = -\frac{Q}{P} \text{ — (iv)}$$

We know that two dirⁿ (du, dv) and $(\delta u, \delta v)$ are orthogonal (perpendicular) to each other

$$\Rightarrow E \frac{du}{dv} \frac{\delta u}{\delta v} + F \left(\frac{du}{dv} + \frac{\delta u}{\delta v} \right) + G = 0 \text{ — (v)}$$

Putting $\frac{\delta u}{\delta v} = -\frac{Q}{P}$ in (v), we get

$$E \frac{du}{dv} \left(-\frac{Q}{P} \right) + F \left(\frac{du}{dv} - \frac{Q}{P} \right) + G = 0$$

$$\Rightarrow -EQ \frac{du}{dv} + PF \frac{du}{dv} - QF + PG = 0$$

$$\Rightarrow (PF - EQ) \frac{du}{dv} = QF - PG$$

$$\Rightarrow (PF - EQ) du = (QF - PG) dv$$

$$\Rightarrow (QE - PF) du + (QF - PG) dv = 0 \text{ — (vi)}$$

This is the differential equation of orthogonal trajectories of a given family of curves.

Next to find the condⁿ that two family of curves given by

$P du^2 + Q du dv + R dv^2 = 0$ will be orthogonal

$$P du^2 + Q du dv + R dv^2 = 0 \quad \text{--- (vii)}$$

dividing both sides by dv^2 , we get

$$P \left(\frac{du}{dv} \right)^2 + Q \left(\frac{du}{dv} \right) + R = 0 \quad \text{--- (viii)}$$

This is the quadratic eqⁿ in $\frac{du}{dv}$, so this will be two values of $\frac{du}{dv}$

let $\frac{du}{dv}$ and $\frac{\delta u}{\delta v}$ be two roots of eqⁿ (viii)

By properties of roots of a quadratic equation we get

$$\frac{du}{dv} + \frac{\delta u}{\delta v} = -\frac{Q}{P} \quad \text{--- (ix)}$$

$$\frac{du}{dv} \cdot \frac{\delta u}{\delta v} = \frac{R}{P}$$

We know that disⁿ (du, dv) and $(\delta u, \delta v)$ are orthogonal to each other.

$$\Leftrightarrow E \frac{du}{dv} \cdot \frac{\delta u}{\delta v} + F \left(\frac{du}{dv} + \frac{\delta u}{\delta v} \right) + G = 0 \quad \text{--- (x)}$$

$$\text{Putting } \frac{du}{dv} + \frac{\delta u}{\delta v} = -\frac{Q}{P}, \quad \frac{du}{dv} \cdot \frac{\delta u}{\delta v} = \frac{R}{P}$$

in eqⁿ (ix) we get

$$E \cdot \frac{R}{P} + F \left(-\frac{Q}{P} \right) + G = 0$$

$$\boxed{ER - FQ + PG = 0}$$

This is the required condition.

Q Find the tangent to the angle between the two directions on the surface determined by the quadratic
 $P du^2 + Q du dv + R dv^2 = 0$

Ans:- $P du^2 + Q du dv + R dv^2 = 0$ (i)
 dividing both sides by dv^2 , we get
 $P \left(\frac{du}{dv}\right)^2 + Q \left(\frac{du}{dv}\right) + R = 0$ (ii)

This is a quadratic equation in $\frac{du}{dv}$
 will give two values of $\frac{du}{dv}$

let $\frac{du}{dv}$ and $\frac{su}{sv}$ are two roots and roots of eqn (ii) a quadratic eqn we get

$$\frac{du}{dv} + \frac{su}{sv} = -\frac{Q}{P}$$

$$\frac{du}{dv} \cdot \frac{su}{sv} = \frac{R}{P}$$

let ψ be the angle betⁿ the two direction
 (du, dv) and (su, sv)

$$\tan \psi = \frac{\sin \psi}{\cos \psi}$$

$$= \frac{\{ \frac{1}{ds} | du sv - su dv | \}}{\{ \frac{1}{ds} | du sv + su dv | \}}$$

$$= \frac{\{ E du su + F (du sv + su dv) + G dv sv \}}{\{ E du su + F (du sv + su dv) + G dv sv \}}$$

$$= \frac{\{ | du sv - su dv | \}}{\{ E du su + F (du sv + su dv) + G dv sv \}}$$

$$\{ \frac{1}{dv sv} | du sv - su dv | \}$$

$$= \frac{\{ \frac{1}{dv sv} | du sv - su dv | \}}{\{ E du su + F (du sv + su dv) + G dv sv \}}$$

$$= \frac{1}{dv sv} \frac{| du sv - su dv |}{E du su + F (du sv + su dv) + G dv sv}$$

$$0 = \frac{1}{dv sv} \frac{| du sv - su dv |}{E du su + F (du sv + su dv) + G dv sv}$$

$$H \left| \frac{du}{dv} - \frac{su}{sv} \right|$$

$$= \frac{E \frac{du}{dv} \frac{su}{sv} + F \left(\frac{du}{dv} + \frac{su}{sv} \right) + G}{\dots}$$

$$H \sqrt{\left(\frac{du}{dv} + \frac{su}{sv} \right)^2 - 4 \frac{du}{dv} \frac{su}{sv}}$$

$$= \frac{E \frac{R}{P} + F \left(-\frac{Q}{P} \right) + G}{\dots}$$

$$H \sqrt{\left(-\frac{Q}{P} \right)^2 - 4 \frac{R}{P}}$$

$$= \frac{\frac{ER}{P} - \frac{FQ}{P} + G}{\dots}$$

$$H \sqrt{\frac{Q^2 - 4PR}{P^2}}$$

$$= \frac{ER - FQ + GP}{P}$$

$$= \frac{H \sqrt{Q^2 - 4PR}}{ER - FQ + GP}$$

Curves on a surface:-

$$\vec{r}'' = k_n \vec{N}^1$$

$$\vec{N} \cdot \vec{r}'' = k_n$$

$$\vec{r}' = \vec{r}_1' u' + \vec{r}_2' v'$$

$$\vec{r}'' = \vec{r}_1'' u'' + u' \vec{r}_1' + \vec{r}_2'' v'' + v' \vec{r}_2'$$

$$= \vec{r}_1'' u'' + u' (\vec{r}_1' u' + \vec{r}_2' v') + \vec{r}_2'' v'' + v' \vec{r}_2'$$

$$\vec{r}_1'' u'' + \vec{r}_2'' v'' + \vec{r}_1' u'^2 + 2 \vec{r}_1' u' v' + \vec{r}_2' v'^2$$

Since $\vec{N}^1 \cdot \vec{r}_1' = 0$
 $\vec{N}^1 \cdot \vec{r}_2' = 0$

$$\vec{N} \cdot \vec{r} = 0$$

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$$K_n = N^1 \cdot \vec{r}^{11}$$

$$= N^1 \cdot (\vec{r}_1^1 u^{11} + \vec{r}_2^1 v^{11} + \vec{r}_1^2 u^{12} + \vec{r}_2^2 v^{12} + \vec{r}_2^2 v^{12})$$

$$= N^1 \cdot \vec{r}_1^1 u^{11} + N^1 \cdot \vec{r}_2^1 v^{11} + N^1 \cdot \vec{r}_2^2 v^{12}$$

$$= L u^{12} + 2M u^{11} v^{12} + N v^{12}$$

$$= \frac{L u^{12} + 2M u^{11} v^{12} + N v^{12}}{(ds)^2}$$

$$= L \left(\frac{du}{ds} \right)^2 + 2M \left(\frac{du}{ds} \right) \left(\frac{dv}{ds} \right) + N \left(\frac{dv}{ds} \right)^2$$

$$= \frac{L (du)^2 + 2M du dv + N (dv)^2}{(ds)^2}$$

$$K_n = \frac{L (du)^2 + 2M du dv + N (dv)^2}{E (du)^2 + 2F du dv + G (dv)^2}$$

The above eqⁿ gives the curvature of the normal section parallel to the direction $\frac{du}{dv}$

Find curvature of normal section of right helicoid. $x = u \cos v$, $y = u \sin v$, $z = cv$

$$\vec{r} = u \cos v \hat{i} + u \sin v \hat{j} + cv \hat{k}$$

$$\vec{r}_1 = \cos v \hat{i} + \sin v \hat{j}$$

$$\vec{r}_2 = -u \sin v \hat{i} + u \cos v \hat{j} + c \hat{k}$$

$$E = \vec{r}_1^2 = 1$$

$$G = \vec{r}_2^2 = u^2 + c^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = -u \sin v \cos v + u \sin v \cos v = 0$$

$$H^2 = EG - F^2 = u^2 + c^2$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ c \sin v & c \cos v & 0 \\ -u \sin v & u \cos v & c \end{vmatrix}$$

$$= \hat{i} (c \cos v - 0) - \hat{j} (c \cos v) + \hat{k} (c)$$

$$= c \cos v \hat{i} - c \cos v \hat{j} + c \hat{k}$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{c \cos v \hat{i} - c \cos v \hat{j} + c \hat{k}}{c}$$

$$L = \vec{r}_1 \cdot \vec{N}$$

$$= 0$$

$$M = \vec{r}_2 \cdot \vec{N}$$

$$= \frac{(-u \cos v \hat{i} + u \sin v \hat{j}) \cdot (c \cos v \hat{i} - c \cos v \hat{j} + c \hat{k})}{c}$$

$$= \frac{-u c \cos^2 v - u c \cos^2 v}{c}$$

$$= \frac{-2u c \cos^2 v}{c} = -2u \cos^2 v$$

$$N = \vec{r}_2 \cdot \vec{N}$$

$$= (-u \cos v \hat{i} - u \sin v \hat{j}) \cdot (c \cos v \hat{i} - c \cos v \hat{j} + c \hat{k})$$

$$= -u c \cos^2 v + u c \cos^2 v = 0$$

$$K_n = \frac{v \frac{d}{dt} \sqrt{a^2 + c^2}}{\sqrt{a^2 + c^2}}$$

$$du^2 + (u^2 + c^2) dv^2$$

Theorem

Métienier's Theorem :-

Let k and k_n are curvatures of oblique and normal section through the same tangent line and θ be the angle between the section then $k_n = k \cos \theta$.

Proof:- Suppose the section considered is not a normal section.

Then the principal normal to the curve is not parallel to N .

It is parallel to r'' and the unit principal normal is r''/K , where K is the curvature of the section considered.

If θ is the angle betⁿ N & the principal normal to the curve then

$$\cos \theta = N \cdot \frac{r''}{K} = \frac{1}{K} (Lr'^2 + 2Mr'v' + Nv'^2)$$

since at the given point both the curves have the same (du, dv) , the last eqⁿ can be written as

e.g. 3.2 $\cos \theta = \frac{k_n}{K} \Rightarrow k_n = K \cos \theta$

Principal curvature :-

$$H^2 K^2 - (EN - 2FM + GL)K + F^2 = 0$$

This is a quadratic eqⁿ gives two values

k_a & k_b of K as required

The first curvature of surface at any point may be defined as sum of the principal curvature we will denote by J

Thus $J = k_a + k_b = \frac{EN - 2FM + GL}{H^2}$

The second curvature / specific curvature of the surface at any point is the product of the principal curvatures.

It is also called as Gaussian curvature.

defined as

$$K = k_1 k_2 = \frac{J^2}{H^2} = \frac{M^2 - LN}{E^2 - EG}$$

case-1:- If the 1st curvature of the surface is zero at all point. Then the surface is called minimal surface.

Then a surface is minimal iff

$$EM - 2FM + GL = 0 \text{ at every point of the surface.}$$

case-2:-

If the second curvature of a surface is zero then the surface is called developable.

Then a surface is developable iff

$$M^2 - LN = 0$$

Q. Show that the surface $z = e^x \cos y$ is minimal.

Proof:- The equation of surface is given by

$$z = e^x \cos y$$

$$e^x = \frac{\cos y}{\cos z}$$

$$x = \log \cos y - \log \cos z$$

Thus the equation of the surface is in Monge's form.

$$r = (x, y, z)$$

$$r = (x, y, \log \cos y - \log \cos x)$$

$$\vec{r}_1 = (1, 0, \tan x), \quad \vec{r}_2 = (0, 1, -\tan y)$$

$$\vec{r}_{11} = (0, 0, \sec^2 x), \quad \vec{r}_{22} = (0, 0, -\sec^2 y)$$

$$\vec{r}_{12} = 0$$

$$E = \vec{r}_1^2 = 1 + \tan^2 x = \sec^2 x$$

$$G = \vec{r}_2^2 = 1 + \tan^2 y = \sec^2 y$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = -\tan x \cdot \tan y$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \tan x \\ 0 & 1 & -\tan y \end{vmatrix} = \hat{i}(-\tan x) - \hat{j}(-\tan y) + \hat{k}(1)$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{\|\vec{r}_1 \times \vec{r}_2\|} = \frac{-\tan x \hat{i} + \tan y \hat{j} + \hat{k}}{\sqrt{1 + \tan^2 x + \tan^2 y}}$$

$$L = \vec{r}_{11} \cdot \vec{N} = \sec^2 x / \sqrt{1 + \tan^2 x + \tan^2 y}, \quad M = \vec{r}_{12} \cdot \vec{N} = 0$$

$$N = \vec{r}_{22} \cdot \vec{N} = -\sec^2 y / \sqrt{1 + \tan^2 x + \tan^2 y}$$

$$\text{Now } EM - 2FM + GL$$

$$= \sec^2 x \cdot \frac{-\sec^2 y}{\sqrt{1 + \tan^2 x + \tan^2 y}} + 2 \tan x \cdot \tan y \cdot 0 + \sec^2 y \cdot \frac{\sec^2 x}{\sqrt{1 + \tan^2 x + \tan^2 y}}$$

$$= \frac{-\sec^2 x \cdot \sec^2 y}{\sqrt{1 + \tan^2 x + \tan^2 y}} + \frac{\sec^2 x \cdot \sec^2 y}{\sqrt{1 + \tan^2 x + \tan^2 y}}$$

$$= 0$$

$\therefore$ Hence the given surface is minimal.

Lines of a curvature :-

A curve drawn on a surface and possessing a property that normal to the surface at conjugate points intersect, is called lines of curvature.

Thus the direction of lines of curvature at any point is a principal direction at that point and the differential equation of the lines of curvature is given by the equation

$$(EM - FL) du^2 + (EN - GL) du dv + (FN - GM) dv^2 = 0$$

Q) Find the principal curvature and lines of curvature of right helicoids.

$$x = u \cos \phi, y = u \sin \phi, z = c\phi$$

Sol :- The fundamental magnitudes for this surface were found & are as follows.

$$E = 1, F = 0, G = u^2 + c^2, H^2 = u^2 + c^2$$

$$L = 0, M = -\frac{c}{H}, N = 0, T^2 = \frac{c^2}{H^2}$$

The principal curvature is

$$H^2 K^2 - (EN - 2FM + GL)K + T^2 = 0$$

$$\Rightarrow (u^2 + c^2) K^2 - (0 - 0 + 0)K + \frac{c^2}{H^2} = 0$$

$$\Rightarrow (u^2 + c^2) K^2 - c^2 = 0$$

$$\Rightarrow K = \pm \frac{c}{u^2 + c^2}$$

the 1st curvature

$$= EN - 2FM + GL$$

$$= 0 - 0 + 0$$

$$= 0$$

hence the surface is a minimal surface.

The second curvature is

$$K = \frac{\tau^2}{H^2} = \frac{-\frac{c}{H}}{H} \times \frac{1}{H^2} = \frac{-\frac{c}{(u^2+c^2)^2}}$$

$\therefore$ The lines of curvature

$$(EM - FL) du^2 + (EN - GL) du dv + (FN - GM) dv^2$$

$$= 0$$

$$\Rightarrow \left(\left(-\frac{c}{H} - 0 \right) du^2 + (0 - 0) du dv + H \left(0 + \frac{c}{H} \right) dv^2 \right) = 0$$

$$\Rightarrow -\frac{c}{H} du^2 + \frac{c(u^2+c^2)}{H} dv^2 = 0$$

$$\Rightarrow -c du^2 + c(u^2+c^2) dv^2 = 0$$

$$\Rightarrow -du^2 = (u^2+c^2) dv^2$$

$$\Rightarrow dv = \pm \frac{du}{\sqrt{u^2+c^2}}$$

Integrating both sides we get

$$v = \pm \sinh^{-1} \left(\frac{u}{c} \right) + \text{const}$$

$$u = c \sinh v$$

$$u^2 = c^2 \sinh^2 v$$

$$u^2 + c^2 = c^2 \cosh^2 v$$

$$\sqrt{u^2 + c^2} = c \cosh v$$

$$H = \frac{1}{c} \sqrt{u^2 + c^2} = \cosh v$$

$$K = \frac{-\frac{c}{H}}{H} = -\frac{c}{H^2} = -\frac{c}{\cosh^2 v}$$

$$K = -\frac{c}{\cosh^2 v}$$

$$K = -\frac{c}{\cosh^2 v}$$

THEOREM :-

The necessary and sufficient condition that the lines of curvature be the parametric curves are $F=0$ and $M=0$

Proof :- condition necessary :-

Let the lines of curvature be parametric curves. Since the parametric curves are represented by

$$u = \text{const.} \quad v = \text{const.}$$

their joint differential eqⁿ is of the form $du dv = 0$ (1)

The lines of curvature are given by

$$(EM - FL) du^2 + (EN - GL) du dv + (FN - GM) dv^2 = 0 \quad (2)$$

$$(EM - FL) du^2 = 0 \quad (EN - GL) \neq 0 \quad FN - GM = 0 \quad (3)$$

$$EM - FL = 0 \Rightarrow EM = FL \Rightarrow M = \frac{FL}{E}$$

$$FN - GM = FN - \frac{GL}{E} = (EN - GL) \frac{F}{E} = 0$$

$$\text{Similarly } (EN - GL) M = 0$$

Since $(EN - GL) \neq 0$ it follows that $F=0, M=0$

condition sufficient :-

If $F=0$ & $M=0$, the differential eqⁿ to the lines of curvature reduces to

$$(EN - GL) du dv = 0$$

Now $(EN - GL) \neq 0$, because $L/E = N/G$

Hence it follows that $du dv = 0$

$\therefore$ Thus the lines of curvature are parametric curves.

Euler's Theorem:

The normal curvature (k_n) of a curve in any direction at point intersects of principal curvatures k_a and k_b is given by

$$k_n = k_a \cos^2 \psi + k_b \sin^2 \psi$$

where ψ is the angle betⁿ the dirⁿ dv of the normal section of the surface and the principal direction $dv=0$

Proof:- Let the lines of curvature be taken as parametric curves so that $F=M=0$

then the normal curvature reduces to the form

$$k_n = \frac{L du^2 + N dv^2}{E du^2 + G dv^2} = \frac{L du^2 + N dv^2}{E du^2 + G dv^2} \quad (1)$$

The principal curvature k_a , being the normal curvature for the direction $dv = \text{const}$.

i.e $dv=0$ is given by

$$k_a = \frac{L}{E} \quad (2)$$

Similarly, the principal curvature k_b , being the normal curvature for the direction $du=0$ is given by

$$k_b = \frac{N}{G}$$

At the normal section of the surface in the dirⁿ (du, dv) makes an angle ψ with the principal dirⁿ $dv=0$, we have

$$\cos \psi = \sqrt{E} \frac{du}{ds} \quad (4)$$

$$\cos \left(\frac{\pi}{2} - \psi \right) = \sin \psi = \sqrt{G} \frac{dv}{ds} \quad (5)$$

Substituting in (1)

$$k_n = \frac{L du^2 + N dv^2}{E du^2 + G dv^2}$$

$$k = k_a (\sqrt{E} u)^2 + k_b (\sqrt{G} v)^2$$

$$= k_a \cos^2 \psi + k_b \sin^2 \psi$$

Savin's Theorem:-

The sum of the normal curvatures in two orthogonal directions is constant and equals to the sum of the principal curvatures

Proof:- By Euler's Theorem

$$k_{n1} = k_a \cos^2 \psi + k_b \sin^2 \psi$$

$$\text{and } k_{n2} = k_a \cos^2 \left(\frac{\pi}{2} + \psi \right) + k_b \sin^2 \left(\frac{\pi}{2} + \psi \right)$$

$$= k_a \sin^2 \psi + k_b \cos^2 \psi$$

$$\therefore k_{n1} + k_{n2} = k_a + k_b = \text{const}$$

Ex-36:-

Find the eqn for the principal curvatures the lines of curvature & the first & second curvature of the surface of revolution $x = u \cos \phi$, $y = u \sin \phi$, $z = f(u)$

Sol:- With u, ϕ as parameter & r the position vector of a current pt on the surface, we have

$$r = (u \cos \phi, u \sin \phi, f(u))$$

$$r_1 = (\cos \phi, \sin \phi, f'(u))$$

$$r_2 = (-u \sin \phi, u \cos \phi, 0)$$

$$E = 1 + f'(u)^2, F = 0, G = u^2, H = -u^2(1 + f'(u)^2)$$

The unit normal to the surface is

$$N = \frac{1}{H} (u f'(u) - f(u) u \cos \phi, -f(u) u \sin \phi, u)$$

Further $r_{u1} = (0, 0, f''(u))$

$$r_{u2} = (-\sin \phi, \cos \phi, 0), r_{v2} = (-u \cos \phi, -u \sin \phi, 0)$$

So that the second order magnitude are

$$L = \frac{u f''(u)}{H}, M = 0, N = \frac{-u^2 f'(u)}{H}, T^2 = \frac{u^3 f'(u)}{f''(u) H^2}$$

The eqⁿ for the principal curvature reduces to

$$u(1 + f'(u)^2) x^2 - \sqrt{1 + f'(u)^2} \{ u f''(u) + f'(1 + f'(u)^2) \} x + (f'(u) f''(u)) = 0$$

The roots of which are

$$k_a = \frac{f''(u)}{\{1 + f'(u)^2\}^{3/2}} \text{ and } k_b = \frac{f'(u)}{u(1 + f'(u)^2)^{1/2}}$$

The principal curvatures are given by

$$J = \frac{u f''(u) + f'(1 + f'(u)^2)}{u(1 + f'(u)^2)^{3/2}} = \frac{u f''}{u^2} + \frac{u f'}{u^2}$$

Gaussian curvature is given by

$$K = \frac{f'(u) f''(u)}{u(1 + f'(u)^2)^2}$$

$$0 = \frac{u f''}{u^2} + \frac{u f'}{u^2}$$

$$0 = \frac{u f''}{u^2} + \frac{u f'}{u^2}$$

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conjugate direction:

~~the dirⁿ (du, dv) & (su, sv) at a point~~

The dirⁿ (du, dv) & (su, sv) at a pt. P on the surface are conjugate iff

$$L \frac{su}{sv} + \frac{du}{dv} + M \left(\frac{su}{sv} + \frac{du}{dv} \right) + N = 0$$

Theorem:

To show that the two dirⁿ given by the eqⁿ $P du^2 + Q du dv + R dv^2 = 0$ are conjugate iff $LR - MQ + NP = 0$

Proof: - The given parametric eqⁿ can be written as

$$P \left(\frac{du}{dv} \right)^2 + Q \frac{du}{dv} + R = 0$$

Let the two dirⁿ given by (1) be

(du, dv) and (su, sv) then

$\frac{su}{sv}$ and $\frac{du}{dv}$ are the roots of eqⁿ (1)

$$\frac{su}{sv} + \frac{du}{dv} = -\frac{Q}{P}$$

$$\frac{su}{sv} \cdot \frac{du}{dv} = -\frac{R}{P}$$

Now the dirⁿ (du, dv) & (su, sv) are

conjugate iff $L \frac{su}{sv} + \frac{du}{dv} + M \left(\frac{su}{sv} + \frac{du}{dv} \right) + N = 0$

$$\frac{LR}{P} - \frac{MQ}{P} + N = 0$$

$$\boxed{LR - MQ + NP = 0}$$

Theorem:-

The necessary & sufficient condⁿ that a parametric curve are conjugate dirⁿ is $M=0$

Condⁿ Necessary:-

the parametric curves are conjugate direction

$$du dv = 0$$

We know two dirⁿ given by eqⁿ

$$P du^2 + Q du dv + R dv^2 = 0$$

are conjugate

$$\text{I.B.B } LR - MQ + NP = 0$$

we have $P=R=0, Q \neq 0$

Since the two dirⁿ are conjugate

$$LR - MQ + NP = 0$$

$$\Rightarrow -MQ = 0$$

$$\Rightarrow MQ = 0$$

$$\Rightarrow M = 0 \quad (\because Q \neq 0)$$

Condⁿ Sufficient:-

Given $M=0$ is satisfied when the

lines of curvature are taken as parametric curves.

Hence the principal dirⁿ at a pt. on the

Surface are conjugate direction.

Q. Show that the parametric curves are conjugate on the following surfaces.

(i) Surface of revolution

$$x = u \cos v, \quad y = u \sin v, \quad z = f(u)$$

(ii) The Surface $z = f(x) + g(y)$ where x, y are parameters

Prob (i) :-

$$\vec{r} = x \cos v \hat{i} + x \sin v \hat{j} + f(x) \hat{k}$$

$$\vec{r}_1 = \cos v \hat{i} + \sin v \hat{j} + f'(x) \hat{k}$$

$$\vec{r}_2 = -x \sin v \hat{i} + x \cos v \hat{j}$$

$$\vec{r}_{12} = -\sin v \hat{i} + \cos v \hat{j}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & f'(x) \\ -x \sin v & x \cos v & 0 \end{vmatrix}$$

$$= \hat{i} (-x \cos v \cdot x \cos v) - \hat{j} (x \sin v \cdot x \sin v) + \hat{k} (x \cos^2 v + x \sin^2 v)$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{-x^2 \cos v \sin v \hat{i} - x^2 \sin v \cos v \hat{j} + x \hat{k}}{H}$$

$$\therefore M = \vec{r}_2 \cdot \vec{N}$$

$$= \frac{-x^2 \cos v \sin v \cos v \sin v - x^2 \sin v \cos v \sin v \cos v + x \sin v \cos v}{H}$$

$$= 0$$

(ii) :-

$$\vec{r} = x \hat{i} + y \hat{j} + \{f(x) + f(y)\} \hat{k}$$

$$\vec{r}_1 = \hat{i} + f'(x) \hat{k}$$

$$\vec{r}_2 = \hat{j} + f'(y) \hat{k}$$

$$\vec{r}_{12} = 0$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f'(x) \\ 0 & 1 & f'(y) \end{vmatrix} = \hat{i} (-f'(x)) - \hat{j} (f'(y)) + \hat{k} (1)$$

$$M = \vec{r}_2 \cdot \vec{N} = 0$$

Asymptotic lines:-

* direction at a pt on the surface is called asymptotic if it is self conjugate and a asymptotic line is a curve whose dirⁿ at every pt. is self conjugate.

The differential eqⁿ of asymptotic line is

$$L du^2 + 2M du dv + N dv^2 = 0$$

Theorem:- The necessary & sufficient condⁿ that the parametric curves are asymptotic lines are $L=0$, $N=0$, $M \neq 0$

Proof:- condⁿ necessary:-

The diff. eqⁿ of the parametric curve is $du \cdot dv = 0$ (1)

and the diff. eqⁿ of the asymptotic line is

$$L du^2 + 2M du dv + N dv^2 = 0$$

Suppose the parametric curves are asymptotic lines then the above two eqⁿ must be identical.

Comparing the co-efficient we get

$$L=0, M \neq 0, N=0$$

condⁿ sufficient:-

Given that $L=0, N=0, M \neq 0$

The diff. eqⁿ of the asymptotic line is

$$L du^2 + 2M du dv + N dv^2 = 0$$

$$\Rightarrow 0 \cdot du^2 + 2M du dv + 0 \cdot dv^2 = 0$$

$$\Rightarrow 2M du dv = 0$$

$$\Rightarrow M du dv = 0$$

$$\Rightarrow du dv = 0 \quad (\because M \neq 0)$$

i.e the asymptotic lines are parametric curves

Q Find the asymptotic line of conoid

$$x = u \cos v, y = u \sin v, z = f(v)$$

$$\vec{r} = u \cos v \hat{i} + u \sin v \hat{j} + f(v) \hat{k}$$

$$\vec{r}_1 = -\sin v \hat{i} + \cos v \hat{j} + f'(v) \hat{k}$$

$$\vec{r}_2 = -u \sin v \hat{i} + u \cos v \hat{j}$$

$$\vec{r}_1 = -\sin v \hat{i} + \cos v \hat{j} + f'(v) \hat{k}$$

$$\vec{r}_2 = -u \sin v \hat{i} + u \cos v \hat{j}$$

$$\vec{r}_{22} = -u \cos v \hat{i} - u \sin v \hat{j}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & f'(v) \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$

$$= \hat{i} (-u f'(v) \cos v) - \hat{j} (-u f'(v) \sin v) + \hat{k} (u)$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{u^2 f'(v)^2 \cos^2 v + u^2 f'(v)^2 \sin^2 v + u^2}$$

$$= \sqrt{u^2 f'(v)^2 + u^2}$$

$$= u \sqrt{1 + f'(v)^2}$$

$$L = \frac{u \sqrt{1 + f'(v)^2} \sin v \cos v - u f'(v) \sin v \cos v}{u \sqrt{1 + f'(v)^2}}$$

$$= \frac{f''(v)}{\sqrt{1 + f'(v)^2}}$$

$$M = \tau_{12} \cdot N^1$$

$$= \frac{u \sin v \cos v \mp' (u) - (u \mp' (v) \sin v \cos v)}{u \sqrt{1 + \mp' (u)^2}}$$

$$= 0$$

$$N = \tau_{22} \cdot N^2$$

$$= \frac{u^2 \cos^2 v \mp' (u) + u^2 \mp' (u) \sin^2 v}{u \sqrt{1 + \mp' (u)^2}}$$

$$= \frac{u \mp' (u)}{\sqrt{1 + \mp' (u)^2}}$$

The eqⁿ of asymptotic line
 $L du^2 + M du dv + N dv^2 = 0$

$$\Rightarrow \frac{\mp' (u)}{\sqrt{1 + \mp' (u)^2}} du^2 + \frac{u \mp' (u)}{\sqrt{1 + \mp' (u)^2}} dv^2 = 0$$

$$\Rightarrow \mp' (u) du^2 + u \mp' (u) dv^2 = 0$$

$$\frac{1}{\mp' (u) + u} \cdot \frac{du}{dv} = \frac{1}{\mp' (u) + u}$$

Find the curve of revolution

$$f(u, v) = z$$

$$(u, v, z)$$

$$(x, y, z) = \begin{pmatrix} u \cos v \\ u \sin v \\ z \end{pmatrix}$$

$$(x, y, z) = \begin{pmatrix} 1 \cdot \cos 0 \\ 1 \cdot \sin 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(x, y, z) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(x, y, z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(x, y, z) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise - 3(a)

- ① Find the normal curvature for the surface of revolution
 $x = u \cos v, y = u \sin v, z = f(u)$

Sol:- The fundamental magnitudes are

$$E = 1 + f_1^2(u), F = 0, G = u^2, H = \frac{1}{2} \sqrt{4f_1^2 + u^2}$$

$$L = \frac{u f_{11}}{H}, M = 0, N = \frac{u^2 f_{11}}{H^2}$$

∴ Normal curvature $k_n = \frac{E du^2 + 2F du dv + G dv^2}{E du^2 + 2F du dv + G dv^2}$

$$= \frac{\frac{u f_{11}}{H} du^2 + \frac{u^2 f_{11}}{H^2} dv^2}{(1 + f_1^2) du^2 + u^2 dv^2}$$

$$k_n = \frac{u f_{11} du^2 + u^2 f_{11} dv^2}{H (1 + f_1^2) (du^2 + dv^2)}$$

$$k_n = \frac{u f_{11} du^2 + u^2 f_{11} dv^2}{(1 + f_1^2) du^2 + u^2 dv^2} \cdot \frac{1}{u \sqrt{1 + f_1^2}}$$

- ② Find the normal curvature for the disk (du, dv) of the surface of revolution

$$z = x^2 + y^2$$

Sol:- $\vec{r} = (x, y, z)$
 $= (x, y, x^2 + y^2)$

$$\vec{r}_1 = (1, 0, 2x)$$

$$\vec{r}_2 = (0, 1, 2y)$$

$$\vec{r}_{11} = (0, 0, 2)$$

$$\vec{r}_{22} = (0, 0, 2)$$

$$\vec{r}_{12} = (0, 0, 0)$$

$$z = f(x, y)$$

$$(x, y, z)$$

Fundamental magnitudes of 1st order are

$$E = \sqrt{1+4x^2}$$

$$E = \vec{r}_1 \cdot \vec{r}_1 = 1+4x^2, \quad F = 4xy, \quad G = 1+4y^2$$

$$\begin{aligned} H^2 &= EG - F^2 \\ &= (1+4x^2)(1+4y^2) - 16x^2y^2 \\ &= 1+4x^2+4y^2 \end{aligned}$$

$$H = \sqrt{1+4x^2+4y^2}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \hat{i}(-2x) - \hat{j}(2y) + \hat{k}(1) = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$\vec{N} = \frac{-2x\hat{i} - 2y\hat{j} + \hat{k}}{\sqrt{1+4x^2+4y^2}}$$

$$\text{say } L = \vec{r}_{11} \cdot \vec{N} = \frac{2}{\sqrt{1+4x^2+4y^2}}$$

$$M = \vec{r}_{12} \cdot \vec{N} = 0$$

$$N = \vec{r}_{22} \cdot \vec{N} = \frac{2}{\sqrt{1+4x^2+4y^2}}$$

$\therefore$ normal curvatures

$$k_n = \frac{\frac{2}{\sqrt{1+4x^2+4y^2}} du^2 + \frac{2}{\sqrt{1+4x^2+4y^2}} dv^2}{(1+4x^2)du^2 + 8xy du dv + (1+4y^2)dv^2}$$

$$\frac{2}{\sqrt{1+4x^2+4y^2}} \cdot \frac{du^2 + dv^2}{(1+4x^2)du^2 + 8xy du dv + (1+4y^2)dv^2}$$

③. What are the normal curvatures for the directions along the parametric curves.

Sol: The parametric curves are $u = \text{const.}$ and $v = \text{const.}$

The normal curvature for $u = \text{const.}$ i.e. $du = 0$

$$k_n = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2} = \frac{N}{G}$$

The normal curvature for parametric curve

$$v = \text{const} \therefore \dot{v} \cdot e \, dv = 0$$

$$k_n = \frac{L}{E}$$

(4). Find the principal dirⁿ & the principal curvature on the surface

$$x = a(u+v), \quad y = b(u-v), \quad z = uv$$

$$\vec{r} = (au + av, bu - bv, uv)$$

$$\vec{r}_1 = (a, b, v)$$

$$\vec{r}_2 = (a, -b, u)$$

$$\vec{r}_{11} = (0, 0, 0)$$

$$\vec{r}_{12} = (0, 0, 1)$$

$$\vec{r}_{22} = (0, 0, 0)$$

$$E = \vec{r}_1^2 = (a^2 + b^2 + v^2), \quad G = (a^2 + b^2 + u^2)$$

$$F = a^2 - b^2 + uv$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & v \\ a & -b & u \end{vmatrix} = \hat{i}(bu + bv) - \hat{j}(au - av) + \hat{k}(-2ab)$$

$$\vec{n} = \frac{b(u+v)\hat{i} - a(u-v)\hat{j} - 2ab\hat{k}}{\sqrt{b^2(u+v)^2 + a^2(u-v)^2 + 4a^2b^2}}$$

$$L = \vec{r}_{11} \cdot \vec{n} = 0, \quad M = \frac{-2ab}{H}, \quad N = 0$$

$$\text{Principal Curvature} = \frac{LN - M^2}{H^2} = \frac{-4a^2b^2}{H^2}$$

$$H^2 k^2 - (EN - 2FM + GL)k + F^2 = 0$$

$$\Rightarrow H^2 k^2 + 2(a^2 - b^2 + uv) \cdot \frac{-2ab}{H} - \frac{4a^2b^2}{H^2} = 0$$

$$\Rightarrow H^2 k^2 - \frac{4ab}{H} (a^2 - b^2 + uv) - \frac{4a^2b^2}{H^2} = 0$$

$$\Rightarrow H^4 k^2 - 4abH(a^2 - b^2 + uv) - 4a^2b^2 = 0$$

where $J^2 = b^2(u+v)^2 + a^2(u-v)^2 + 4a^2b^2$

The 1st curvature

$$J = \frac{EN - 2FM + GL}{J^2}$$

$$= \frac{-2(a^2 - b^2 + uv)}{J^2} \cdot \frac{-2ab}{J} \cdot \frac{1}{J^2}$$

$$= \frac{1}{J^3} \frac{4ab}{J} (a^2 - b^2 + uv)$$

The second curvature

$$K = \frac{GL - 2FM + EN}{J^2} = \frac{-4a^2b^2}{J^4}$$

(iii) The surface $2x = ax^2 + 2hxy + by^2$
 $z = \frac{1}{2}ax^2 + hxy + \frac{b}{2}y^2$

$$\vec{r} = (x, y, \frac{1}{2}ax^2 + hxy + \frac{b}{2}y^2)$$

$$\vec{r}_1 = (1, 0, ax + hy)$$

$$\vec{r}_2 = (0, 1, hx + by)$$

$$\vec{r}_{11} = (0, 0, a)$$

$$\vec{r}_{22} = (0, 0, b)$$

$$\vec{r}_{12} = (0, 0, h)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & ax+hy \\ 0 & 1 & hx+by \end{vmatrix} = \hat{i}(-ax-hy) - \hat{j}(bx+ay) + \hat{k}(1-0)$$

$$\vec{N} = \frac{-(ax+by)^2 - (bx+ay)^2 + 1}{\sqrt{1 + (ax+by)^2 + (bx+ay)^2}}$$

$$E = \vec{r}_1^2 = 1 + (ax+by)^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = (ax+by)(bx+ay) = abx^2 + abxy + b^2ay + b^2by^2$$

$$G = \vec{r}_2^2 = 1 + (bx+ay)^2$$

$$L = \vec{r}_1 \cdot \vec{N} = \frac{a}{1}, M = \frac{b}{1}, N = \frac{b}{1}, T^2 = \frac{ab - b^2}{1^2}$$

lines of curvature

$$(EM - FL) du^2 + (EN - GL) du dv + (FN - GM) dv^2 = 0$$

$$\Rightarrow \frac{b(1 + (ax+by)^2)}{1} - \frac{a(ax+by)(bx+ay)}{1} = 0$$

$$\Rightarrow (EM - FL) da^2 + (EN - GL) da dy + (FN - GM) dy^2 = 0$$

Principal radii

$$H^2 K^2 - (EN - 2FM + GL) K + T^2 = 0$$

$$\Rightarrow H^2 - (EN - 2FM + GL) J + T^2 g^2 = 0$$

(iv) Surface of revolution, $x = u \cos v$, $y = u \sin v$, $z = u^2$

$$\vec{r} = (u \cos v, u \sin v, u^2)$$

$$\vec{r}_1 = (\cos v, \sin v, 2u)$$

$$\vec{r}_2 = (-u \sin v, u \cos v, 0)$$

$$\vec{r}_3 = (0, 0, 2)$$

$$\vec{r}_{22} = (-u \cos v, -u \sin v, 0)$$

$$\vec{r}_{12} = (-\sin v, \cos v, 0)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$

$$= \hat{i} (-2u^2 \cos v) - \hat{j} (2u^2 \sin v) + \hat{k} (u)$$

$$\hat{n} = \frac{-2u^2 \cos v \hat{i} - 2u^2 \sin v \hat{j} + u \hat{k}}{\sqrt{u^2 + 4u^4 \cos^2 v + 4u^4 \sin^2 v}}$$

$$= \frac{-2u^2 \cos v \hat{i} - 2u^2 \sin v \hat{j} + u \hat{k}}{\sqrt{u^2 + 4u^4}}$$

$$E = \vec{r}_u \cdot \vec{r}_u = 1 + 4u^2, \quad F = 0, \quad G = u^2$$

$$L = \vec{r}_u \cdot \hat{n} = \frac{2u}{\sqrt{u^2 + 4u^4}}, \quad M = \frac{2u^2 \sin v \cos v - 2u^2 \sin v \cos v}{\sqrt{u^2 + 4u^4}} = 0$$

$$N = \frac{2u^3}{\sqrt{u^2 + 4u^4}}, \quad T^2 = \left(\frac{4u^4}{u^2 + 4u^4} \right) = \frac{4u^2}{1 + 4u^2}$$

The lines of curvature are parametric.

Principal radii

$$\Rightarrow H^2 - (EN - 2FM + GL) + T^2 = 0$$

$$\Rightarrow H^2 - \left(\frac{2u^3(1+4u^2)}{\sqrt{u^2+4u^4}} + \frac{2u^3}{\sqrt{u^2+4u^4}} \right) + \frac{4u^4}{H^2} = 0$$

$$\Rightarrow H^2 - \frac{2u^3(1+4u^2) + 2u^3}{\sqrt{u^2+4u^4}} + \frac{4u^4}{H^2} = 0$$

$$\Rightarrow H^2 - \frac{(2u^3 + 8u^5 + 2u^3)}{H} + \frac{4u^4}{H^2} = 0$$

$$\Rightarrow H^2 - \frac{4u^3(1+2u^2)}{H} + \frac{4u^4}{H^2} = 0$$

The 1st curvature $\frac{1}{R_1} = \frac{EN - 2FM + GL}{H^3}$

$$= \frac{1}{H^3} \left(\frac{2u^3(1+4u^2)}{\sqrt{u^2+4u^4}} + \frac{2u^3}{\sqrt{u^2+4u^4}} \right) = \frac{4u^4}{H^3}$$

Second curvature

$$= \frac{1}{H^3} \left(\frac{2u^3(1+4u^2)}{\sqrt{u^2+4u^4}} + \frac{2u^3}{\sqrt{u^2+4u^4}} \right) = \frac{4u^4}{H^3}$$

$$= \frac{1}{H^3} \left(\frac{2u^3(1+4u^2)}{\sqrt{u^2+4u^4}} + \frac{2u^3}{\sqrt{u^2+4u^4}} \right) = \frac{4u^4}{H^3}$$

$$= \frac{1}{H^3} \left(\frac{2u^3(1+4u^2)}{\sqrt{u^2+4u^4}} + \frac{2u^3}{\sqrt{u^2+4u^4}} \right) = \frac{4u^4}{H^3}$$

⑥ Find the eqⁿ for the principal curvatures and the differential equation of the lines of curvature, for the surface

$$(i) \quad z = \frac{x^2}{a} + \frac{y^2}{b}$$

$$\vec{r} = (x, y, \frac{x^2}{2a} + \frac{y^2}{2b})$$

$$\vec{r}_1 = (1, 0, \frac{x}{a})$$

$$\vec{r}_2 = (0, 1, \frac{y}{b})$$

$$\vec{r}_{11} = (0, 0, \frac{1}{a})$$

$$\vec{r}_{12} = (0, 0, 0)$$

$$\vec{r}_{22} = (0, 0, \frac{1}{b})$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{x}{a} \\ 0 & 1 & \frac{y}{b} \end{vmatrix} = \hat{i} \left(-\frac{y}{b} \right) - \hat{j} \left(\frac{x}{a} \right) + \hat{k} (1-0)$$

$$\hat{n} = \frac{-\frac{y}{b}\hat{i} - \frac{x}{a}\hat{j} + \hat{k}}{\sqrt{\frac{y^2}{b^2} + \frac{x^2}{a^2} + 1}}$$

$$E = 1 + \frac{x^2}{a^2}, \quad F = \frac{xy}{ab}, \quad G = 1 + \frac{y^2}{b^2}$$

$$L = \vec{r}_{11} \cdot \hat{n} = \frac{1}{a\sqrt{H}}, \quad M = 0, \quad N = \frac{1}{b\sqrt{H}}$$

Principal curvatures

$$H^2 K^2 + 2(EN - 2FM + GL)K + T^2 = 0$$

$$H^2 K^2 \left(\frac{1}{b^2} \left(1 + \frac{x^2}{a^2} \right) - 0 + \frac{1}{a^2} \left(1 + \frac{y^2}{b^2} \right) \right) K + \frac{1}{ab} = 0$$

$$H^4 K^2 - \left[\frac{1}{b^2} \left(1 + \frac{x^2}{a^2} \right) + \frac{1}{a^2} \left(1 + \frac{y^2}{b^2} \right) \right] K + \frac{1}{ab} = 0$$

$$H^4 K^2 - \left[\frac{b(a^2 + x^2) + a(b^2 + y^2)}{a^2 b^2} \right] K + \frac{1}{ab} = 0$$

$$2abH^4 K^2 - [b(a^2 + x^2) + a(b^2 + y^2)]HK + ab = 0$$

Lines of curvature

$$\Rightarrow (EM - FL) dx^2 + (EN - GL) dx dy + (FN - GM) dy^2 = 0$$

$$\Rightarrow -\frac{xy}{a^2b+1} dx^2 + \frac{1}{b+1} \left(1 + \frac{x^2}{a^2}\right) - \left(1 + \frac{y^2}{b^2}\right) \frac{1}{a+1} \frac{dx dy}{b} + \frac{xy}{ab^2+1} dy^2 = 0$$

$$\Rightarrow -\frac{xy}{a^2b+1} dx^2 + \left[\frac{(a^2+x^2)}{a^2b+1} - \frac{(b^2+y^2)}{ab^2+1} \right] dx dy + \frac{xy}{ab^2+1} dy^2 = 0$$

$$\Rightarrow \frac{-bxy dx^2 + [(a^2+x^2)b - (b^2+y^2)a] dx dy + axy dy^2}{a^2b^2+1} = 0$$

$$\Rightarrow -bxy dx^2 + \{b(a^2+x^2) - a(b^2+y^2)\} dx dy + axy dy^2 = 0$$

(ii) $z = ax^3 + by^3$

$$x = \frac{ax^3}{3} + \frac{by^3}{3}$$

$$\vec{r} = (x, y, \frac{ax^3}{3} + \frac{by^3}{3})$$

$$\vec{r}_1 = (1, 0, ax^2)$$

$$\vec{r}_{11} = (0, 0, 2ax)$$

$$\vec{r}_2 = (0, 1, by^2)$$

$$\vec{r}_{22} = (0, 0, 2by)$$

$$\vec{r}_{12} = (0, 0, 0)$$

$$E = \vec{r}_1^2 = 1 + a^2x^4, F = abx^2y^2, G = \vec{r}_2^2 = 1 + b^2y^4$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & ax^2 \\ 0 & 1 & by^2 \end{vmatrix} = \hat{i}(-ax^2) - \hat{j}(by^2) + \hat{k}(1)$$

$$\vec{N} = \frac{-ax^2\hat{i} - by^2\hat{j} + \hat{k}}{\sqrt{a^2x^4 + b^2y^4 + 1}}$$

$$L = \vec{r}_{11} \cdot \vec{N} = \frac{2ax}{1}, M = 0, N = \vec{r}_{22} \cdot \vec{N} = \frac{2by}{1}$$

Principal curvature

$$I^2 = LN - M^2 = \frac{4abxy}{1^2}$$

$$H^2K^2 - (EN - 2FM + GL)K + I^2 = 0$$

$$\Rightarrow H^2K^2 - \left[\frac{2by(1+a^2x^4)}{1} - 0 + \frac{2ax(1+b^2y^4)}{1} \right] K + I^2 = 0$$

$$\Rightarrow \frac{1}{H^2} \left[\frac{2by(1+a^2x^4) + 2ax(1+b^2y^4)}{H} \right] K + \frac{4abxy}{H^2} = 0$$

$$\Rightarrow H^4 K^2 - [2by(1+a^2x^4) + 2ax(1+b^2y^4)] HK + 4abxy = 0$$

Lines of curvature

$$(EN - FL) dx^2 + (EN - GL) dx dy + (FN - GM) dy^2 = 0$$

$$\Rightarrow \frac{-abx^2y^2 \cdot 2ax}{H} dx^2 + \frac{(1+a^2x^4)2by}{H} - \frac{2ax(1+b^2y^4)}{H} dx dy + \left(\frac{abx^2y^2 \cdot 2by}{H} - 0 \right) dy^2 = 0$$

$$\Rightarrow -2a^3bx^3y^2 dx^2 + [2by(1+a^2x^4) - 2ax(1+b^2y^4)] dx dy + 2ab^2x^2y^3 dy^2 = 0$$

$$\Rightarrow -abx^3y^2 dx^2 + [2by(1+a^2x^4) - 2ax(1+b^2y^4)] dx dy + ab^2x^2y^3 dy^2 = 0$$

⑦ calculate the 1st and 2nd curvatures of the helicoid $x = u \cos v$, $y = u \sin v$, $z = f(u) + cv$ and show that the latter is const along a helix ($u = \text{const.}$)

$$\vec{r} = (u \cos v, u \sin v, f(u) + cv)$$

$$\vec{r}_1 = (\cos v, \sin v, f'(u))$$

$$\vec{r}_2 = (-u \sin v, u \cos v, c)$$

$$\vec{r}_{11} = (0, 0, f''(u))$$

$$\vec{r}_{12} = (-\sin v, \cos v, 0)$$

$$\vec{r}_{22} = (-u \cos v, -u \sin v, 0)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & f'(u) \\ -u \sin v & u \cos v & c \end{vmatrix} = \hat{i} (c \sin v - u \cos v f'(u)) - \hat{j} (c \cos v + f'(u) u \sin v) + \hat{k} (u)$$

$$L = \vec{r}_{11} \cdot \vec{N} = \frac{u f''(u)}{H}$$

$$E = 1 + f'(u)^2$$

$$M = \vec{r}_{12} \cdot \vec{N} = \frac{-c}{H}$$

$$F = c f'(u)$$

$$N = \vec{r}_{22} \cdot \vec{N} = \frac{-u f'(u)}{H}$$

$$G = u^2 + c^2$$

1st curvature

$$\frac{EN - 2FM + GL}{H^2} = \frac{1}{H^2} \left\{ \frac{u f'(u) (1 + f'(u)^2)}{H} + \frac{2c^2 f'(u)}{H} \right.$$

$$\left. + \frac{(u^2 + c^2) u f''(u)}{H} \right\}$$

$$= \frac{u f'(u) (1 + f'(u)^2) + 2c^2 f'(u) + u f''(u) (u^2 + c^2)}{H^3}$$

Second curvature

$$\frac{L^2}{H^2}$$

$$= \frac{u^2 f'(u)^2 f''(u)^2 - c^2}{H^4}$$

~~$\mu = \text{constant} = a c i e t$~~

Q10 Find the principal curvatures at the origin of the paraboloid.

$$2z = 5x^2 + 4xy + 2y^2$$

$$z = \frac{5}{2}x^2 + 2xy + y^2$$

Sol:- $\vec{r} = (x, y, \frac{5}{2}x^2 + 2xy + y^2)$

$$\vec{r}_1 = (1, 0, 5x + 2y)$$

$$\vec{r}_2 = (0, 1, 2x + 2y)$$

$$\vec{r}_{11} = (0, 0, 5)$$

$$\vec{r}_{12} = (0, 0, 2)$$

$$\vec{r}_{22} = (0, 0, 2)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 5x+2y \\ 0 & 1 & 2x+2y \end{vmatrix} = \hat{i}(-5x-2y) - \hat{j}(2x+2y) + \hat{k}(1-0)$$

$$L = \vec{r}_{11} \cdot \vec{N}^1 = \frac{5}{H}$$

$$M = \vec{r}_{12} \cdot \vec{N}^1 = \frac{2}{H}$$

$$N = \vec{r}_{22} \cdot \vec{N}^1 = \frac{2}{H}$$

$$E = 1 + (5x+2y)^2$$

$$F = (5x+2y)(2x+2y)$$

$$G = 1 + (2x+2y)^2$$

$$T^2 = \frac{10}{H^2} - \frac{4}{H^2} = \frac{6}{H^2}$$

1st curvature $J = \frac{EN - 2FM + GL}{H^3}$

$$= \frac{2(1 + (5x+2y)^2) - 4(5x+2y)(2x+2y) + 1(1 + (2x+2y)^2)}{H^3}$$

at origin $J = \frac{2+1}{H^3} = \frac{3}{H^3} = 7$

$$H^2 K^2 - (EN - 2FM + GL)K + T^2 = 0$$

$$\Rightarrow K^2 - 7K + 6 = 0$$

$$\Rightarrow K^2 - 6K - K + 6 = 0$$

$$\Rightarrow K = 1, 6$$

$$\Rightarrow K(K-6) - 1(K-6) = 0$$

$$K_1 = 1, K_2 = 6$$

(1) Show that on the following surfaces, the parametric curves are asymptotic lines.

(i) Right helicoid $x = u \cos \phi$, $y = u \sin \phi$, $z = c\phi$

Proof:- $\vec{r} = (u \cos \phi, u \sin \phi, c\phi)$

$\vec{r}_1 = (\cos \phi, \sin \phi, 0)$

$\vec{r}_2 = (-u \sin \phi, u \cos \phi, c)$

$\vec{r}_{11} = (0, 0, 0)$

$\vec{r}_{12} = (-\sin \phi, \cos \phi, 0)$

$\vec{r}_{22} = (-u \cos \phi, -u \sin \phi, 0)$

$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \phi & \sin \phi & 0 \\ -u \sin \phi & u \cos \phi & c \end{vmatrix} = \hat{i}(c \sin \phi) - \hat{j}(c \cos \phi) + \hat{k}(u)$

$N = \frac{c \sin \phi \hat{i} - c \cos \phi \hat{j} + u \hat{k}}{\sqrt{c^2 + u^2}}$

$L = \vec{r}_{11} \cdot N = 0$

$M = \frac{-c \sin 2\phi - c \cos 2\phi}{\sqrt{c^2 + u^2}} = \frac{-c}{\sqrt{c^2 + u^2}}$

$N = \frac{-u c \sin \phi \cos \phi + u c \sin \phi \cos \phi}{\sqrt{c^2 + u^2}} = 0$

Here $L=0$ and $M=0$ & $N \neq 0$

Hence the parametric curves are asymptotic line.

(ii) Surface $x = a(u+v)$, $y = b(u-v)$, $z = uv$

$\vec{r} = (a(u+v), b(u-v), uv)$

$\vec{r}_1 = (a, b, v)$

$\vec{r}_2 = (a, -b, u)$

$$\vec{r}_{11} = (0, 0, 0)$$

$$\vec{r}_{12} = (0, 0, 1)$$

$$\vec{r}_{22} = (0, 0, 0)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & v \\ a & -b & -u \end{vmatrix} = \hat{i}(b(-u+v) - (-a(-u-v))) + \hat{j}(a(-u-v) - (-a(-u+v))) + \hat{k}(a(-b) - (-a(-b)))$$

$$= b(-u+v)\hat{i} - a(-u-v)\hat{j} - 2ab\hat{k}$$

$$\vec{N} = \frac{b(-u+v)\hat{i} - a(-u-v)\hat{j} - 2ab\hat{k}}{\sqrt{b^2(-u+v)^2 + a^2(-u-v)^2 + 4a^2b^2}}$$

$$L = \vec{r}_{11} \cdot \vec{N} = 0, \quad M = \frac{-2ab}{\sqrt{b^2(-u+v)^2 + a^2(-u-v)^2 + 4a^2b^2}}, \quad N = 0$$

∴ Hence the parametric curves are asymptotic lines.

5) Show that the asymptotic lines of the hyperboloid are given by $\phi \pm \psi = \text{constant}$

$$r = a \cos \alpha \sec \psi \hat{i} + b \sin \alpha \sec \psi \hat{j} + c \tan \psi \hat{k}$$

are given by $\phi \pm \psi = \text{constant}$

$$\vec{r}_1 = a \cos \alpha \sec \psi \hat{i} + b \sin \alpha \sec \psi \hat{j} + c \tan \psi \hat{k}$$

$$\vec{r}_2 = -a \sin \alpha \sec \psi \hat{i} + b \cos \alpha \sec \psi \hat{j} + c \sec^2 \psi \hat{k}$$

$$\vec{r}_3 = a \cos \alpha \sec \psi \tan \psi \hat{i} + b \sin \alpha \sec \psi \tan \psi \hat{j} + c \sec^2 \psi \hat{k}$$

$$\vec{r}_{11} = a \cos \alpha \sec \psi \hat{i} - b \sin \alpha \sec \psi \hat{j}$$

$$\vec{r}_{12} = -a \sin \alpha \sec \psi \tan \psi \hat{i} + b \cos \alpha \sec \psi \tan \psi \hat{j}$$

$$\vec{r}_{22} = a \cos \alpha (\sec^3 \psi + \sec \psi \tan^2 \psi) \hat{i} + b \sin \alpha (\sec^3 \psi + \sec \psi \tan^2 \psi) \hat{j} + 2c \sec^2 \psi \tan \psi \hat{k}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos \alpha \sec \psi & b \sin \alpha \sec \psi & c \tan \psi \\ -a \sin \alpha \sec \psi \tan \psi & b \cos \alpha \sec \psi \tan \psi & c \sec^2 \psi \end{vmatrix}$$

$$= \frac{1}{2} (bc \cos \theta \sec^3 \psi) - \frac{1}{2} (-ac \sin \theta \sec^3 \psi) + \dots$$

$$+ (-abc \sin^2 \theta \sec^2 \psi \tan \psi - ab \cos^2 \theta \sec^2 \psi \tan \psi)$$

$$= \frac{1}{2} (bc \cos \theta \sec^3 \psi) + \frac{1}{2} (ac \sin \theta \sec^3 \psi) - abc \sec^2 \psi \tan \psi$$

$$L = \vec{r}_{11} \cdot \vec{N}^1$$

$$= \frac{-abc \cos^2 \theta \sec^4 \psi - abc \sin^2 \theta \sec^4 \psi}{\sec^4 \psi}$$

$$= -abc \sec^4 \psi$$

$$M = \vec{r}_{12} \cdot \vec{N}^1$$

$$= \frac{-abc \sec^4 \psi \tan \psi \sin \theta \cos \theta + abc \sin \theta \cos \theta \sec^4 \psi \tan \psi}{\sec^4 \psi \tan \psi}$$

$$= 0$$

$$N = \vec{r}_{22} \cdot \vec{N}^1$$

$$= abc \cos^2 \theta \sec^3 \psi (\sec^3 \psi + \sec \psi \tan^2 \psi)$$

$$+ abc \sin^2 \theta \sec^3 \psi (\sec^3 \psi + \sec \psi \tan^2 \psi)$$

$$- 2abc \sec^4 \psi \tan^2 \psi$$

$$= abc \sec^6 \psi + abc \sec^4 \psi \tan^2 \psi + abc \sec^6 \psi + abc \sec^4 \psi \tan^2 \psi - 2abc \sec^4 \psi \tan^2 \psi$$

$$= 2abc \sec^6 \psi$$

$$= 2abc (\sec^6 \psi)$$

The asymptotic lines are

$$L d\theta^2 + 2M d\theta d\psi + N d\psi^2 = 0$$

$$\Rightarrow \frac{-abc \sec^4 \psi}{2abc \sec^6 \psi} d\theta^2 + d\psi^2 = 0$$

$$\Rightarrow 2abc \sec^4 \phi \, d\phi^2 = abc \sec^4 \phi \, d\phi^2$$

$$\Rightarrow 2 \sec^2 \phi \, d\phi^2 = d\phi^2$$

$$\Rightarrow \sqrt{2} \sec \phi \, d\phi = d\phi$$

⑥ Show that the asymptotic lines on the general helicoid $x = u \cos v$, $y = u \sin v$, $z = \phi(u) + cv$ are given by $u \phi_{11} du^2 - 2c du dv + u^2 \phi_{11} dv^2 = 0$

$$\text{Sol: } \vec{r} = (u \cos v, u \sin v, \phi(u) + cv)$$

$$\vec{r}_1 = (\cos v, \sin v, \phi_1(u))$$

$$\vec{r}_2 = (-u \sin v, u \cos v, c)$$

$$\vec{r}_{11} = (0, 0, \phi_{11}(u))$$

$$\vec{r}_{12} = (-\sin v, \cos v, 0)$$

$$\vec{r}_{22} = (-u \cos v, -u \sin v, 0)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & \phi_1(u) \\ -u \sin v & u \cos v & c \end{vmatrix} = \hat{i} (c \sin v - \phi_1(u) \cdot u \cos v) - \hat{j} (c \cos v + u \phi_1(u) \sin v) + \hat{k} (u)$$

$$L = \vec{r}_{11} \cdot \hat{N} = \frac{u \phi_{11}(u)}{+1}, M = \vec{r}_{12} \cdot \hat{N} = \frac{-c}{+1}, N = \vec{r}_{22} \cdot \hat{N} = \frac{-u \phi_1(u)}{+1}$$

The asymptotic lines are

$$L du^2 + 2M du dv + N dv^2 = 0$$

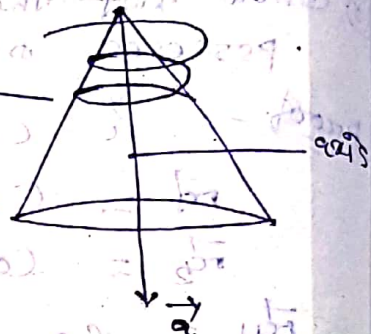
$$\Rightarrow \frac{u \phi_{11}(u)}{+1} du^2 - \frac{2c}{+1} du dv + \frac{-u \phi_1(u)}{+1} dv^2 = 0$$

$$\Rightarrow u \phi_{11}(u) du^2 - 2c du dv - u \phi_1(u) dv^2 = 0 \quad \text{Proved}$$

Q. Given that a curve is drawn on a right circular cone every where inclined at the same angle α to the axis

To prove that $k = z \tan \alpha$

Proof: let $\vec{a}$ be a constant vector along the axis. Angle betⁿ $\vec{a}$ and $\hat{t}$ is α



$$\text{So } \vec{a} \cdot \hat{t} = |\vec{a}| |\hat{t}| \cos \alpha$$

$$\vec{a} \cdot \hat{t} = |\vec{a}| \cos \alpha$$

$$\Rightarrow \frac{d}{ds} (\vec{a} \cdot \hat{t}) = \frac{d}{ds} |\vec{a}| \cos \alpha$$

$$\Rightarrow \left(\frac{d\vec{a}}{ds} \cdot \hat{t} + \vec{a} \cdot \frac{d\hat{t}}{ds} \right) = 0$$

$$\Rightarrow 0 + \vec{a} \cdot \frac{d\hat{t}}{ds} = 0$$

$$\Rightarrow \vec{a} \cdot \kappa \hat{n} = 0$$

$$\Rightarrow \kappa (\vec{a} \cdot \hat{n}) = 0$$

$$\Rightarrow \vec{a} \cdot \hat{n} = 0 \quad (\because \kappa \neq 0)$$

$$\Rightarrow \vec{a} \cdot (\hat{b} \times \hat{t}) = 0$$

$$[\vec{a} \ \hat{b} \ \hat{t}] = 0$$

$\vec{a}$, $\hat{b}$, $\hat{t}$ are coplanar

$$\vec{a} = \hat{t} \cos \alpha + \hat{b} \sin \alpha$$

$$\frac{d\vec{a}}{ds} = \frac{d}{ds} (\hat{t} \cos \alpha + \hat{b} \sin \alpha)$$

$$= \cos \alpha \frac{d\hat{t}}{ds} + \sin \alpha \frac{d\hat{b}}{ds}$$

$$\Rightarrow 0 = \kappa \hat{n} \cos \alpha + \sin \alpha (-\hat{t})$$

$$\Rightarrow 0 = (\kappa \cos \alpha - \tau \sin \alpha) \hat{n}$$

$$\Rightarrow \kappa \cos \alpha = \tau \sin \alpha$$

$$\Rightarrow \boxed{\kappa = \tau \tan \alpha}$$

Q If k is zero at all points, the curve is a straightline. If z is zero at all points, the curve is plane. The necessary and sufficient condition.

$$[r' \ r'' \ r'''] = 0$$

Proof :- Given that $k=0$ at all points.
To prove that the curve is a straightline
 $k=0$ at all points

$$k\theta' = 0\theta'$$

$$\frac{dt}{ds} = 0$$

$$t' = \text{const. vector}$$

The curve is a straightline.

Given that $z=0$ at all points.

To prove that the curves lie on a plane.

$$z=0 \text{ at all points}$$

$$\Rightarrow z\theta' = 0\theta' \text{ at all pts.}$$

$$\Rightarrow z\theta' = 0$$

$$-z\theta' = 0$$

$$\frac{db'}{ds} = 0$$

$$b' = \text{const. vector}$$

If the curve is given by $\vec{r}(s)$

$$\text{then } \frac{d}{ds} (\vec{r} \cdot \vec{b}') = \frac{d\vec{r}}{ds} \cdot \vec{b}' + \vec{r} \cdot \frac{d\vec{b}'}{ds}$$

$$= t' \cdot \vec{b}' + \vec{r} \cdot 0$$

$$= 0 + 0$$

$$= 0$$

$$\vec{r} \cdot \vec{b}' = \text{constant}$$

The vector from the origin to the various pts of the curve are orthogonal to the fixed direction $\vec{b}'$.

Hence the curve will be a plane.

To prove that

The curve is a plane curve

$$\Rightarrow [\vec{r} \quad \vec{r}' \quad \vec{r}''] = 0$$

$$\left[\frac{d\vec{r}}{ds} \quad \frac{d^2\vec{r}}{ds^2} \quad \frac{d^3\vec{r}}{ds^3} \right] = 0$$

$$\Rightarrow [t^1, \kappa n^1, \kappa' n^1 + \kappa \tau b^1 - \kappa^2 t^1] = 0$$

$$\Rightarrow [t^1, \kappa n^1, \kappa' n^1 + \kappa \tau b^1 - \kappa^2 t^1] = 0$$

$$\Rightarrow (t^1 \times \kappa n^1) \cdot (\kappa' n^1 + \kappa \tau b^1 - \kappa^2 t^1) = 0$$

$$\Rightarrow (t^1 \times n^1) \kappa \cdot (\kappa' n^1 + \kappa \tau b^1 - \kappa^2 t^1) = 0$$

$$\kappa \cdot (\kappa' n^1 + \kappa \tau b^1 - \kappa^2 t^1) = 0$$

$$\kappa^2 \tau = 0$$

$$\tau = 0$$

The curve is a plane curve.

Examples

Unit-IV

Prove that the differential equations of the curves which bisect the angle between the parametric curves are $\sqrt{E} du + \sqrt{G} dv = 0$

Solution:

The parametric curves are $u = \text{constant}$ and $v = \text{const.}$

If α_1 is the angle betⁿ the directions du, dv and that of the curve $v = \text{const.}$

$$\cos \alpha_1 = \frac{1}{\sqrt{E}} \left(E \frac{du}{ds} + F \frac{dv}{ds} \right)$$

Similarly, if α_2 is the angle betⁿ the dirⁿs du, dv and that of the curve $u = \text{const.}$

$$\cos \theta_2 = \frac{1}{\sqrt{G}} \left(F \frac{du}{ds} + G \frac{dv}{ds} \right)$$

Since the dirⁿ (du, dv) bisects the angle betⁿ the parametric curves

therefore $\cos \theta_1 = \cos \theta_2$

$$\frac{1}{\sqrt{E}} \left(E \frac{du}{ds} + F \frac{dv}{ds} \right) = \frac{1}{\sqrt{G}} \left(F \frac{du}{ds} + G \frac{dv}{ds} \right)$$

$$\Rightarrow \sqrt{G} (E du + F dv) = \sqrt{E} (F du + G dv)$$

$$\Rightarrow G (E du + F dv)^2 = E (F du + G dv)^2$$

$$\Rightarrow (EG - F^2) (E du^2 - G dv^2) = 0$$

$$\Rightarrow E du^2 - G dv^2 = 0 \quad \therefore EG - F^2 = 0$$

$$\Rightarrow \sqrt{E} du - \sqrt{G} dv = 0 \quad \text{and} \quad \sqrt{E} du + \sqrt{G} dv = 0$$

Example - Given the curve $uv = \text{const.}$ on the surface

$\vec{r} = u\hat{i} + v\hat{j}$ find the orthogonal trajectories

$$\vec{r} = u\hat{i} + v\hat{j} \quad \vec{r}_1 = \hat{i}, \quad \vec{r}_2 = \hat{j}$$

$$E = \vec{r}_1^2 = 1, \quad F = \vec{r}_1 \cdot \vec{r}_2 = 0, \quad G = \vec{r}_2^2 = 1$$

The given curve $uv = \text{const.}$

$$v \delta u + u \delta v = 0$$

i.e of the form $P \delta u + Q \delta v = 0$

$$\text{where } P = v, \quad Q = u$$

The diff. eqⁿ for the orthogonal trajectories is

$$(EQ - FP) du + (FQ - GP) dv = 0$$

$$u du - v dv = 0$$

$$\frac{1}{2} u^2 - \frac{1}{2} v^2 = \frac{1}{2} c^2$$

$$u^2 - v^2 = c^2$$

are orthogonal trajectories.