

Legendre's Equation

$$\text{The D.E. } (1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + p(p+1)y = 0$$

is known as Legendre's equation. ————— (1)

This can be rewritten as

$$\frac{d^2 y}{dx^2} - \frac{2x}{(1-x^2)} \frac{dy}{dx} + \frac{p(p+1)}{1-x^2} y = 0$$

Here $x=0$ is an ordinary point. ————— (2)

We can apply power series method to solve it.

$$\text{Let } y = \sum_{n=0}^{\infty} a_n x^n \text{ be the}$$

solution of (2) ————— (A)

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting these in eqn (1) we have

$$(1-x)^p \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + p(p+1) \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2 \sum_{n=1}^{\infty} n a_n x^n + p(p+1) \sum_{n=0}^{\infty} a_n x^n = 0.$$

③

~~Now~~ Now for the term

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\text{Let } n-2 = m$$

$$\Rightarrow n = m+2$$

$$\text{when } n=2, m=0$$

So this term can be rewritten as

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

Since index summation is a dummy variable we can write $m=n$.

So the 1st term can be written as

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Now from ③

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2 \sum_{n=1}^{\infty} n a_n x^n + p(p+1) \sum_{n=0}^{\infty} a_n x^n = 0$$

Since in the 2nd term, the ~~sum~~ sum corresponding to $n=0$ & 1 vanish

So we can take the limit for $n=0$ instead of $n=2$.

Similarly, in 3rd term the sum corresponding to $n=0$ vanish

So we can summation from $n=0$ instead of $n=1$.

Hence,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n - 2 \sum_{n=0}^{\infty} n a_n x^n + p(p+1) \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - n(n-1) a_n - 2n a_n + p(p+1) a_n] x^n = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (-n^2 + n - 2n + p^2 + p) a_n] x^n = 0$$

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$$\Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (-n^2 - n + p^2 + p)a_n] x^n = 0$$

Equating the coefficients of x^n to zero

$$(n+2)(n+1)a_{n+2} + (p^2 - n^2 + p - n)a_n = 0$$

$$\Rightarrow (n+2)(n+1)a_{n+2} + [(p-n)(p+n) + p-n]a_n = 0$$

$$\Rightarrow a_{n+2} = - \frac{(p-n)(p+n+1)}{(n+2)(n+1)} a_n$$

$$\Rightarrow a_{n+2} = - \frac{(p+n+1)(p-n)}{(n+2)(n+1)} a_n$$

which is the recursion formula for Legendre's equation.

for $n=0, 1, 2, 3, \dots$ we have

$$a_2 = - \frac{p(p+1)}{2 \cdot 1} a_0$$

$$a_3 = - \frac{(p+2)(p+1)}{3 \cdot 2} a_1$$

$$a_4 = - \frac{(p+3)(p+2)}{4 \cdot 3} a_2$$

$$= - \frac{(p+3)(p+2)}{4 \cdot 3} x - \frac{p(p+1)}{2!} a_0$$

$$\Rightarrow a_4 = - \frac{p(p+1)(p+3)(p+2)}{4 \cdot 3 \cdot 2 \cdot 1} a_0$$

$$a_5 = - \frac{p(p+4)(p+3)}{5 \cdot 4} a_3$$

$$= - \frac{(p+4)(p+3)}{5 \cdot 4} x - \frac{(p+2)(p+1)}{3 \cdot 2} a_1$$

$$\Rightarrow a_5 = \frac{(p+2)(p+4)(p+3)(p+1)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} a_1$$

in general,

$$a_{2m} = \frac{(-1)^m (p+2m-1)(p+2m-3)\dots(p+1)p(p-1)\dots(p-2m+2)}{(2m)!} a_0$$

$$a_{2m+1} = \frac{(-1)^m (p+2m)(p+2m-2)\dots(p+2)p(p-1)(p-3)\dots(p-2m+1)}{(2m+1)!} a_1$$

where $m=1, 2, 3, \dots$

Substituting these values in eqn (A) we have

$$y = a_0 + a_1 x + \frac{p(p+1)}{2!} a_0 x^2 - \frac{(p+2)(p+1)}{3!} x^3 + \frac{p(p+1)(p+3)(p+2)}{4!} x^4 + \dots$$

$$\Rightarrow y = a_0 \left[1 - \frac{P(P+1)}{2!} x^2 + \frac{P(P+3)(P+1)(P-2)}{4!} x^4 - \dots \right]$$

$$+ a_1 \left[x - \frac{(P+2)(P+1)}{3!} x^3 + \frac{(P+4)(P+2)(P+1)(P-3)}{5!} x^5 - \dots \right]$$

$$\Rightarrow y = a_0 y_1 + a_1 y_2 \text{ (Say).}$$

which is the required Power series solution of Legendre's equation.

Legendre's Polynomial ($P_n(x)$)

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of the Legendre's equation is

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

The solution of the above eqⁿ in the series of descending power of x is

$$y = a_0 \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n+1)(n-2)(n-3)}{(2n-1)(2n-3) \cdot 2} x^{n-4} - \dots \right]$$

where a_0 is an arbitrary constant.

if n is a positive integer

$$\text{and } a_0 = \frac{1 \cdot 3 \cdot 5 \cdot \dots (2n-1)}{n!}$$

then the previous solution y is $P_n(x)$.

Hence
$$P_n(x) = \frac{1 \cdot 3 \cdot 5 \cdot \dots (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{(2n-1) \cdot 2} x^{n-2} + \dots \right]$$

This is a terminating series.

$P_n(x)$ is called the Legendre's function of the 1st kind.

* There is another solution of Legendre's equation when n is a positive integer

$$y = a_0 \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \dots \right]$$

if we take $a_0 = \frac{n!}{1 \cdot 3 \cdot 5 \cdot \dots (2n+1)}$

The above solution is called $Q_n(x)$.

Hence
$$Q_n(x) = \frac{n!}{1 \cdot 3 \cdot 5 \cdot \dots (2n+1)} \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \dots \right]$$

$Q_n(x)$ is the Legendre's 2nd kind.

Since $P_n(x)$ & $Q_n(x)$ are two independent solutions of Legendre's eqⁿ,

So the most general solution of Legendre's equation is

$$y = AP_n(x) + BQ_n(x)$$

where A & B are two arbitrary constants.

Sub Rodrigue's Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

is the Rodrigue's formula of Legendre's eqⁿ.

Leibnitz Theorem

$$D^n(uv) = D^n(u)v + {}^nC_1(D^n u)(Dv) + {}^nC_2(D^{n-2}u)(D^2v) + \dots + uD^n(v)$$

Proof

$$\text{Let } y = (x^2-1)^n$$

$$\Rightarrow \frac{dy}{dx} = n(x^2-1)^{n-1} \cdot 2x$$

multiplying (x^2-1) on both sides we have

$$(x^2-1) \frac{dy}{dx} = 2n(x^2-1)x$$

$$\Rightarrow (x^2-1) \frac{dy}{dx} = 2nxy$$

Now differentiating w.r.t x (n+1) times

$$D^{n+1}[(x^2-1)y] = D^{n+1}[2nxy] \quad \text{where } y_1 = \frac{dy}{dx}$$

$$\Rightarrow D^{n+1}(y_1)(x^2-1) + {}^{n+1}C_1 D^n(y_1) D(x^2-1) +$$

$${}^{n+1}C_2 D^{n-1}(y_1) D^2(x^2-1) =$$

$$2n [D^{n+1}(y) + {}^{n+1}C_1 D^n(y) D(x)]$$

$$\Rightarrow (x^2-1)y_{n+2} + (n+1)y_{n+1} (2x) +$$

$$\frac{n(n+1)}{2} y_n \cdot 2 = 2n [y_{n+1}x + (n+1)y_n]$$

$$\begin{aligned} \Rightarrow (x^2-1)y_{n+2} + 2nx y_{n+1} + 2xy_{n+1} + (n^2+n-2n^2-2n)y_n \\ = 2nx y_{n+1} + 2n^2 y_n + 2ny_n \end{aligned}$$

$$\Rightarrow (x^2-1)y_{n+2} + 2xy_{n+1} + (n^2+n-2n^2-2n)y_n$$

$$\Rightarrow (x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

$$\Rightarrow (x^2-1) \frac{d^2 y_n}{dx^2} + 2x \frac{dy_n}{dx} - n(n+1)y_n = 0$$

$$\Rightarrow (1-x^2) \frac{d^2 y_n}{dx^2} - 2x \frac{dy_n}{dx} + n(n+1)y_n = 0$$

$$\text{Let } y_n = z$$

$$\Rightarrow (1-x^2) \frac{d^2 z}{dx^2} - 2x \frac{dz}{dx} + n(n+1)z = 0 \quad \text{--- (1)}$$

This is a Legendre eqⁿ & its solution is $z = C P_n(x)$

$$\Rightarrow y_n = C P_n(x) \rightarrow$$

$$\hookrightarrow \frac{d^n y}{dx^n} = C P_n(x) \quad (1)$$

$$\Rightarrow \left[\frac{d^n y}{dx^n} \right]_{x=1} = C P_n(1) \quad (2)$$

$$\Rightarrow \left[\frac{d^n y}{dx^n} \right]_{x=1} = C \quad (3)$$

we know $P_n(1) = 1$

$$\text{we know } y = (x^2 - 1)^n$$

$$\Rightarrow y = (x+1)^n (x-1)^n$$

Differentiating w.r.t x n times
by Leibnitz Theorem,

$$D^n y = D^n [(x+1)^n (x-1)^n]$$

$$\Rightarrow y_n = D^n (x+1)^n (x-1)^n + n D^{n-1} (x+1)^{n-1} (x-1)^n$$

$$+ \dots + (x+1)^n D^n (x-1)^n$$

$$\Rightarrow \left[\frac{d^n y}{dx^n} \right]_{x=1} = n! (x+1)^n \frac{d^n}{dx^n} (x-1)^n + n(x+1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} (x-1)^n + \dots + (x+1)^n \frac{d^n}{dx^n} (x-1)^n = 0$$

$$\Rightarrow \left[\frac{d^n y}{dx^n} \right]_{x=1} = n! (x+1)^n + (x-1) [\dots]$$

$$\Rightarrow \left[\frac{d^n y}{dx^n} \right]_{x=1} = n! \cdot 2^n + 0 \quad (4)$$

by eqⁿ (3) & (4)

$$\Rightarrow \boxed{C = n! \cdot 2^n}$$

Hence by (1)

$$\frac{d^n y}{dx^n} = n! \cdot 2^n \cdot P_n(x)$$

$$\Rightarrow \boxed{P_n(x) = \frac{1}{n! \cdot 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n}$$

(Proved)

Legendre Polynomials

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n \quad (\text{Rodriguez's formula})$$

$$\text{if } n=0, P_0(x) = \frac{1}{2^0 \cdot 0!} = 1$$

$$\begin{aligned} \text{if } n=1, P_1(x) &= \frac{1}{2^1 \cdot 1!} \cdot \frac{d}{dx} (x^2-1) \\ &= \frac{1}{2} \cdot 2x = x \end{aligned}$$

$$\begin{aligned} \text{if } n=2, P_2(x) &= \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2} (x^2-1)^2 \\ &= \frac{1}{8} \frac{d}{dx} [4x(x^2-1)] \\ &= \frac{1}{8} (12x^2 - 4) \\ &= \frac{1}{2} (3x^2 - 1) \end{aligned}$$

$$\text{Similarly, } P_3(x) = \frac{1}{2} (5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x)$$

$$P_n(x) = \sum_{r=0}^n \frac{(-1)^r (2n-2r)!}{2^n r! (n-r)! (n-2r)!} x^{n-2r}$$

when $n = \frac{n}{2}$ if n is even

$n = \frac{1}{2}(n-1)$ if n is odd

Evaluation of $P_n(x)$ by using Binomial Theorem

$$\begin{aligned} (x^2-1)^n &= \sum_{r=0}^n {}^nC_r (x^2)^{n-r} (-1)^r \\ &= \sum_{r=0}^n (-1)^r \frac{n!}{r!(n-r)!} x^{2n-2r} \end{aligned}$$

$$\begin{aligned} \text{Now, } P_n(x) &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n \\ &= \frac{1}{2^n n!} \sum_{r=0}^n (-1)^r \frac{n!}{r!(n-r)!} \frac{d^n}{dx^n} x^{2n-2r} \\ &= \sum_{r=0}^n \frac{(-1)^r (2n-2r)!}{2^n (n-r)! (n-2r)!} x^{n-2r} \end{aligned}$$

Either n^0 or x^1 is the last term

$$n-2r=0 \Rightarrow r = \frac{n}{2} \quad (n \text{ is even})$$

$$n-2r=1 \Rightarrow r = \frac{1}{2}(n-1) \quad (n \text{ is odd})$$

* Express $f(x) = 4x^3 + 6x^2 + 7x + 2$ in terms of Legendre Polynomials.

$$\text{Sol}^n \quad 4x^3 + 6x^2 + 7x + 2$$

$$= aP_3(x) + bP_2(x) + cP_1(x) + dP_0(x)$$

$$= a\left(\frac{5x^3}{2} - \frac{3x}{2}\right) + b\left(\frac{3x^2-1}{2}\right) + cx + d$$

$$= \frac{5ax^3}{2} - \frac{3ax}{2} + \frac{3bx^2}{2} - \frac{b}{2} + cx + d$$

$$= \frac{5ax^3}{2} + \frac{3bx^2}{2} + \left(-\frac{3a}{2} + c\right)x - \frac{b}{2} + d$$

Equating the coefficients of like

powers of x , we have

$$\frac{5a}{2} = 4 \Rightarrow \boxed{a = \frac{8}{5}}$$

$$\frac{3b}{2} = 6 \Rightarrow \boxed{b = 4}$$

$$-\frac{39}{2} + c = 7$$

$$\Rightarrow -\frac{24}{10} + c = 7 \Rightarrow c = 7 + \frac{24}{10}$$

$$\Rightarrow \boxed{c = \frac{47}{5}}$$

$$-\frac{b}{2} + d = 2 \Rightarrow -\frac{4}{2} + d = 2$$

$$\Rightarrow \boxed{d = 4}$$

Putting the values of a, b, c & d in ① we have

$$\frac{3}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4P_0(x)$$

Generating Function of Legendre Equation is

$P_n(x)$ is the coefficients of z^n in the expansion of $(1-2xz+z^2)^{-1/2}$ in ascending powers of z .

$$\text{we } (1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)z^n \quad \text{--- ②}$$

APRIL

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$$(1-2xz+z^2)^{-1/2} = P_0(x) + zP_1(x) + z^2P_2(x) + \dots + z^nP_n(x) + \dots$$

The coefficients of z, z^2, z^3 etc in ①

are $P_1(x), P_2(x), P_3(x), \dots$

IMP
* Prove that $P_n(1) = 1$

we know that

$$(1-2xz+z^2)^{-1/2} = 1 + zP_1(x) + z^2P_2(x) + \dots + z^nP_n(x) + \dots$$

Substituting $x=1$ in the above eqⁿ.

$$(1-2z+z^2)^{-1/2} = 1 + zP_1(1) + z^2P_2(1) + \dots + z^nP_n(1) + \dots$$

$$\Rightarrow [(1-z)^2]^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(1)$$

$$\Rightarrow (1-z)^{-1} = \sum z^n P_n(1)$$

$$\Rightarrow \sum z^n P_n(1) = (1-z)^{-1} = 1 + z + z^2 + z^3 + \dots + z^n + \dots$$

Equating the coefficients of z^n

on both sides we have

$$P_n(1) = 1$$

CProved

Q Express in terms of Legendre Polynomials

$$1+x-x^2$$

sol

$$1 = P_0(x)$$
$$x = P_1(x)$$

~~Q2~~ we know $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$\Rightarrow 2P_2(x) = 3x^2 - 1$$

$$\Rightarrow 2P_2(x) + 1 = 3x^2$$

$$\Rightarrow 2P_2(x) + P_0(x) = 3x^2$$

$$\Rightarrow x^2 = \frac{2P_2(x) + P_0(x)}{3}$$

Now, $1+x-x^2$

$$= P_0(x) + P_1(x) - \frac{2}{3}P_2(x) - \frac{1}{3}P_0(x)$$

$$= -\frac{2}{3}P_2(x) + P_1(x) + \frac{2}{3}P_0(x)$$