

Limit exist at a point $x=a$ if it satisfies the condⁿ.

(i) LHL exist

(ii) RHL exist

Working rule:-

$$\text{LHL} = \lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} f(a-h)$$

$$\text{RHL} = \lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} f(a+h)$$

$$\boxed{\lim_{x \rightarrow a} f(x) = l.}$$

* $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = \frac{2^2-4}{2-2} = \left(\frac{0}{0}\right)$ (wrong) (form)

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2-4}{x-2} &= \lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{(x-2)} \\ &= \lim_{x \rightarrow 2} (x+2) \\ &= 2+2 = 4 \end{aligned}$$

Continuous function:- A function is continuous at a point $x=a$ if it satisfies the following conditions i.e.

(i) LHL exist and finite also

(ii) RHL exist and finite also

(iii) LHL = RHL

(iv) $\lim_{x \rightarrow a} f(x) = f(a)$

this is also represent by i.e.

$\boxed{\text{limiting value} = \text{functional value}}$

Ex $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

2- $\lim_{x \rightarrow 0} f(x)$

$$= \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h)$$

$$= \lim_{h \rightarrow 0} -h \sin(-\frac{1}{h})$$

$$= \lim_{h \rightarrow 0} h \sin(\frac{1}{h})$$

$$= 0 \times \sin(\frac{1}{0})$$

$$= 0 \times \sin \infty$$

$$= 0 \text{ oscillating in between } [-1, 1]$$

$$f(0) = 0$$

→ Derivative :-

1st principle method or ab-initio method or by definition of

derivative :-

st-1 let $y = f(x)$ ——— ①

$y + \delta y = f(x + \delta x)$ ——— ②

st-2 subtracting eq(1) from eq(2) then we have

$$y + \delta y - y = f(x + \delta x) - f(x)$$

$$\Rightarrow f(x + \delta x) - f(x) = \delta y$$

st-3 dividing both the sides by δx , we get :-

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

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St-4 Taking limits in both the sides,

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

here define $\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

ab-initio

increment + ration = intiam

ab = first letters.

* Remark $\rightarrow h \gg 0$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \quad h \gg 0.$$

$$* y = x^n \quad \text{--- (1)}$$

$$y + \delta y = (x + \delta x)^n \quad \text{--- (2)}$$

subtracting eq (2) from eq (1)

$$y + \delta y - y = (x + \delta x)^n - x^n$$

$$\boxed{(x + \delta x)^n - x^n = \delta y}$$

dividing δx on the both sides,

$$\frac{\delta y}{\delta x} = \frac{(x + \delta x)^n - x^n}{\delta x}$$

taking limit on both the sides,

$$\lim_{x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{x \rightarrow 0} \frac{(x + \delta x)^n - x^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{(x + \delta x)^n - x^n}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{(x + \delta x)^n - x^n}{(x + \delta x) - x} \quad (\text{formulation})$$

$$= nx^{n-1}$$

By definition:-

$$y = x^n$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^n$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{x+h-x}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{(x+h) - x}$$

$$= nx^{n-1} \quad (\text{formulation})$$

$$* y = \sqrt{ax^2 + bx + c}$$

$$y = (ax^2 + bx + c)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(ax^2 + bx + c)^{-1/2} (2ax + b)$$

$$* y = \frac{1 - \sin x}{1 + \sin x}$$

$$y = \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2\sin \frac{x}{2} \cdot \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cdot \cos \frac{x}{2}}$$

$$y = \frac{(\cos \frac{x}{2} - \sin \frac{x}{2})^2}{(\sin \frac{x}{2} + \cos \frac{x}{2})^2} \Rightarrow y = \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}}$$

5 ~~5~~ dividing $\cos x/2$ in ~~numerator~~ ^{numerator} and denominator we get,

$$y = \frac{1 - \tan x/2}{1 + \tan x/2}$$

$$y = \frac{\tan \frac{\pi}{4} - \tan x/2}{1 + \tan \frac{\pi}{4} \cdot \tan x/2}$$

$$y = \tan \left(\frac{\pi}{4} - \frac{x}{2} \right)$$

$$\frac{dy}{dx} = \sec^2 \left(\frac{\pi}{4} - \frac{x}{2} \right) \left(-\frac{1}{2} \right)$$

Integration :-
antiderivative

$$\rightarrow \frac{d}{dx} \left(\frac{x^{n+1}}{n+1} \right) = x^n$$

$$\rightarrow y = \sin x$$

$$\frac{d}{dx} \left(\frac{x^{n+1}}{n+1} + C \right) = x^n.$$

Integration :-

(i) indefinite integral (ii) definite integral.

$$\ast \int \log x \, dx$$

$$= \log x \int dx - \int \left(f(x) \cdot \frac{d}{dx} \log x \right) dx.$$

$$= \log x \cdot x - \int \frac{1}{x} \cdot x \, dx.$$

$$= x \log x - x + C.$$

Integration by parts :-

$$\int (u \cdot v) \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \times \int v \, dx \right) dx.$$

⑥ $\int e^x [f(x) + f'(x)] dx = e^x \cdot f(x) + C.$

2. $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C.$

* $\int e^{ax} \cdot \cos bx$

let $I = \int \frac{e^{ax}}{u} \cdot \frac{\cos bx}{1}$

$= \cos bx \int e^{ax} dx - \int \left(\frac{d}{dx} \cos bx \cdot \int e^{ax} dx \right) dx$

$= \cos bx \cdot \frac{e^{ax}}{a} - \int (-\sin bx) \cdot b \frac{e^{ax}}{a} dx.$

$= \cos bx \cdot \frac{e^{ax}}{a} + \frac{b}{a} \int \frac{e^{ax}}{u} \cdot \frac{\sin bx}{1} dx.$

$= \cos bx \cdot \frac{e^{ax}}{a} + \frac{b}{a} \left[\sin bx \int e^{ax} dx - \int \frac{d}{dx} \sin bx \int e^{ax} dx \right]$

$= \cos bx \cdot \frac{e^{ax}}{a} + \frac{b}{a} \left[\sin bx \cdot \frac{e^{ax}}{a} - \int \cos bx \cdot b \cdot \frac{e^{ax}}{a} \right]$

$= \cos bx \cdot \frac{e^{ax}}{a} + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx dx.$

$\therefore I = \frac{e^x}{a} \cos bx - \frac{b^2}{a^2} \int e^{ax} \cos bx dx + \frac{b}{a^2} e^{ax} \sin bx$

$I = \frac{e^{ax}}{a} \cos bx - \frac{b^2}{a^2} I + \frac{b}{a^2} e^{ax} \sin bx + C$

$I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx + C.$

$I \left(1 + \frac{b^2}{a^2} \right) = \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx + C.$

$I \left(\frac{a^2 + b^2}{a^2} \right) = \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx + C$

$I = \frac{a^2}{a^2 + b^2} \cdot \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} \frac{a^2}{a^2 + b^2} \sin bx + \frac{C}{a^2 + b^2}$

(7)

Sol

$$= \frac{a}{a^2+b^2} e^{ax} \cos bx + \frac{b}{a^2+b^2} + \frac{b}{a^2+b^2} \sin bx + \frac{c}{a^2+b^2}$$

$$= \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx] + k$$

Partial fraction :-

1. Whenever factor of denominator are linear and no. repeated.

$$\frac{px+q}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

2. If factors are linear and repeated

$$\frac{px+q}{(x+a)^2(x+b)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{x+b}$$

3. factors are quadratic and non-repeated

$$\frac{px+q}{(x^2+a)(x^2+b)} = \frac{Ax+B}{x^2+a} + \frac{Cx+D}{x^2+b}$$

4. factors are quadratic and repeated.

$$\frac{px+q}{(x^2+a)^2(x^2+b)^3} = \frac{Ax+B}{x^2+a} + \frac{Cx+D}{(x^2+a)^2} + \frac{Ex+F}{x^2+b} + \frac{Gx+H}{(x^2+b)^2} + \frac{Ix+J}{(x^2+b)^3}$$

Q. $\int \frac{2x+5}{(x-5)(x+3)} dx.$

$$\Rightarrow \frac{2x+5}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

$$\Rightarrow \frac{2x+5}{(x-5)(x+3)} = \frac{A(x+3) + B(x-5)}{(x-5)(x+3)}$$

$$\Rightarrow 2x+5 = A(x+3) + B(x-5)$$

put $x = 3$

$$2(3) + 5 = B(3 - 5)$$

$$-8B = -6 + 5 \Rightarrow 8B = 1$$

$$\Rightarrow \boxed{B = \frac{1}{8}}$$

put $x = 5$

$$2(5) + 5 = 8A$$

$$\Rightarrow 15 = 8A \Rightarrow \boxed{A = \frac{15}{8}}$$

$$\int \frac{2x+5}{(x-5)(x+3)} dx = \int \frac{A}{x-5} dx + \int \frac{B}{x+3} dx$$

$$= \frac{15}{8} \int \frac{dx}{x-5} + \frac{1}{8} \int \frac{dx}{x+3}$$

$$= \frac{15}{8} \ln|x-5| + \frac{1}{8} \ln|x+3| + C$$

Q. $\int \frac{2x+5}{(3x-9)(5x+8)} dx$

$$\frac{2x+5}{(3x-9)(5x+8)} = \frac{A}{3x-9} + \frac{B}{5x+8}$$

$$\Rightarrow \frac{2x+5}{(3x-9)(5x+8)} = \frac{A(5x+8) + B(3x-9)}{(3x-9)(5x+8)}$$

$$\Rightarrow 2x+5 = A(5x+8) + B(3x-9)$$

put $x = -8/5$

$$2(-8/5) + 5 = A(\cancel{5}x^{-8/5} + 8) + B(3x^{-8/5} - 9)$$

$$\Rightarrow -\frac{16}{5} + 5 = B(-\frac{24}{5} - 9) \Rightarrow \boxed{B = -\frac{9}{69}}$$

$$\Rightarrow \frac{-16 + 25}{5} = B(-\frac{24 - 45}{5}) \Rightarrow \boxed{B = -\frac{3}{23}}$$

$$\Rightarrow B(-\frac{29}{5}) = \frac{9}{5}$$

put $x = 9/3$

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$$2\left(\frac{9}{3}\right) + 5 = A(5 \times \frac{9}{3} + 8) + B(3 \times \frac{9}{3} - 9)$$

$$\Rightarrow A\left(\frac{45}{3} + 8\right) = 6 + 5$$

$$\Rightarrow 11 = A\left(\frac{45 + 24}{3}\right)$$

$$\Rightarrow 11 = A\left(\frac{69}{3}\right) \Rightarrow 11 = A(23)$$

$$\Rightarrow \boxed{A = \frac{11}{23}}$$

$$\int \frac{2x+5}{(3x-9)(5x+8)} dx = \int \frac{A}{(3x-9)} dx + \int \frac{B}{5x+8} dx$$

$$= \frac{11}{23} \int \frac{dx}{3x-9} - \frac{3}{23} \int \frac{dx}{5x+8}$$

$$= \frac{11}{23} \times \frac{1}{3} \int \frac{3}{3x-9} dx - \frac{3}{23} \times \frac{1}{5} \int \frac{5dx}{5x+8}$$

$$= \frac{11}{69} \int \frac{3}{3x-9} dx - \frac{3}{115} \int \frac{5}{5x+8} dx$$

$$= \frac{11}{69} \ln|3x-9| - \frac{3}{115} \ln|5x+8| + C$$

Exponential function :-

$$\boxed{2^5 = 32}$$

$\downarrow$ base $\nearrow$ power $\rightarrow$ value

$$\boxed{\log_{\text{base}}(\text{value}) = \text{power}}$$

$$\boxed{e \approx 2.7182183}$$

Logarithm :-

(i) common logarithm ($\log_a x$)

(ii) natural logarithm ($\log_e x$) = $\log x = \ln x$

Ex 10

$$1. \log_a (AB) = \log_a A + \log_a B$$

$$2. \log_a \left(\frac{A}{B}\right) = \log_a A - \log_a B$$

$$3. \log_a x^m = m \log_a x$$

$$4. \log_a x \times \log_x a = 1.$$

$$5. \log_e e = 1, \log_a a = 1.$$

$$6. \log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a} \rightarrow \text{base change formula.}$$

Differential equation :-

* $\frac{d}{dt} \rightarrow$ only co-efficient of derivative

$$\text{so } \frac{d}{dt} \left(\frac{ds}{dt} \right) \neq \frac{d^2 s}{(dt)^2}$$

$$* \frac{d^n y}{dx^n} = n^{\text{th}} \text{ order derivation} = \frac{d}{dx} \left(\frac{d^{n-1}}{dx^{n-1}} y \right)$$

Introduction :-

After the discovery of calculus newton and leibnitz studied differential eq. in connection with problems in physics especially in the theory of bending beam oscillations of mechanical system and of electrical currents and conductions of heat etc. In case of chemistry we come across velocity of chemical reaction and diffusions of solvent etc. These days there being used in economics also John

Bernoulli and Euler made their contribution to the

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theory of different eq. but the name of making is very much associated with them.

Definition:- A differential eq. is an eq. involving independent variable, dependent variable or variables and different derivatives of the independent variable with respect to the independent variable or variables.

① * $f(x) = y = ax + b \rightarrow$ singular variables

* $Z = Z(x, y) = 2x^2y + 3xy - 5x + 9 \rightarrow$ two variables

$\frac{\delta y}{\delta x} = yx$

If 'x' is an independent variable and 'y' is a dependent variable then any eq. involving $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ is called as differential eq.

Ex ① $\frac{dy}{dx} + x^2 = 2$

② $(\frac{dy}{dx})^2 + 3y^2 = 5x$

③ $(\frac{d^2y}{dx^2})^2 - 2\frac{dy}{dx} + 1 = 0$

④ $\frac{dy}{dx} = 3x$

⑤ $\frac{d^3y}{dx^3} = 0$

⑥ $\frac{d^2y}{dx^2} = k \left[1 + (\frac{dy}{dx})^2 \right]^{3/2}$

⑦ $x^2 dy + y^2 dx = 0$
 $\Rightarrow x^2 \frac{dy}{dx} + y^2 = 0$

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$$\textcircled{12} \textcircled{8} \quad x \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \left. \begin{array}{l} \text{partial differential} \\ \text{eq.} \end{array} \right\}$$

$$\textcircled{9} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Differential eq.:-

- ① Ordinary Differential Equation (ODE)
- ② Partial Differential Equation (PDE)

① Ordinary Differential Equation:-

$$\rightarrow y = 5x^2 - 3x + 1$$

$$\frac{dy}{dx} = 10x - 3$$

$$\rightarrow z = 2x^2y - 5xy^3 + 3x^2y^2 - 3xy + 9 \text{ (implicit function)}$$

$$7x^2y + 9 = 0 \text{ (implicit function)}$$

Differentiation on both the sides w.r.t. x.

$$7 \left[x^2 \frac{dy}{dx} + y \frac{dx^2}{dx} \right] + \frac{d}{dx}(9) = 0$$

$$7 \left[x^2 \frac{dy}{dx} + y \cdot 2x \right] = 0$$

$$7x^2 \frac{dy}{dx} + 14xy = 0$$

$$7x^2 \frac{dy}{dx} = -14xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{-14xy}{7x^2}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{2y}{x}}$$

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② Partial Differential equation:-

$$7x^2y + 9 = 0$$

Differentiation on both sides w.r.t x , & y is constant here.

order & degree of differential equation:-

order:- The order of differential eq. is the order of the highest derivative present in it.

Degree:- The degree of differential eq. is the degree of the derivative of highest order present in it after expressing the eq. in a form free from radicals and fractions in it derivative.

$$\text{ex } \left(\frac{dy}{dx} + \frac{1}{2} \right)^3 + 5x \left(\frac{dy}{dx} \right)^2 - 7 \frac{dy}{dx} + 11 = 0$$

$$\text{order} = 2$$

$$\text{degree} = 3$$

14) Differential equation (D.E) order degree.

① $x dy + y dx = 0$

1

1

② $\frac{dy}{dx} + 3y = x$

1

1

③ $\left(\frac{d^2y}{dx^2}\right) = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$

2

2

$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

2

2

④ $\frac{dy}{dx} + \frac{2}{x}y = x^2$

1

1

⑤ $y = x \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

1

2

$\Rightarrow y - x \frac{dy}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$\Rightarrow \left(y - x \frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

Formation of Differential equation:-

① $y = 4 \sin x$ ——— ①

$\frac{dy}{dx} = 4 \cos x$ ——— ②

dividing eq (2) by ① -

$\frac{\frac{dy}{dx}}{y} = \frac{4 \cos x}{4 \sin x} \Rightarrow \frac{dy/dx}{y} = \cot x$

$\Rightarrow \boxed{\frac{dy}{dx} = y \cot x} \Rightarrow \boxed{\frac{dy}{dx} - y \cot x = 0}$

This is the required D.E.

② $y = Ae^x + Be^{-x}$ form D.E.

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$$\frac{dy}{dx} = Ae^x - Be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} - y = 0} \text{ which is the required D.E.}$$

③ Find the D.E. of the system of circle in xy-plane with centre at (1, 2).

~~Ans~~ we know the general eqⁿ of the circle is -

$$(x-a)^2 + (y-b)^2 = r^2 \text{ where Centre is } (a, b) \text{ \& } r$$

is radius of the circle.

Here given centre of a circle 'O' is (1, 2), therefore,

$$\boxed{(x-1)^2 + (y-2)^2 = r^2}$$

Differentiating on the both the sides w.r.t x.

$$2(x-1) + 2(y-2)\frac{dy}{dx} = 0$$

$$\Rightarrow 2(y-2)\frac{dy}{dx} = -2(x-1)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1-x}{y-2}} \text{ which the required D.E.}$$

Solution of a D.E:

The solⁿ. betⁿ a variable is called a solⁿ. of a D.E if it reduces the eqⁿ. to an identity when substitution in it.

① General solⁿ.

② Particular solⁿ.

(16) (i) General solⁿ :-

The solution which contains as many as arbitrary constant as the order of D.E is called the general solⁿ of a D.E.

ex $y = A \cos x + B \sin x$ is the general solution of

$$\boxed{\frac{d^2 y}{dx^2} + y = 0}$$

(ii) Particular solⁿ :-

Solution obtained by given particular value to the arbitrary constant in the general solⁿ of a D.E. is called particular solⁿ.

ex $y = 3 \cos x + 2 \sin x$ is the particular solⁿ of

$$\boxed{\frac{d^2 y}{dx^2} + y = 0}$$

D.E of 1st order and 1st degree :-

It involves x, y and $\frac{dy}{dx}$ then it can be put in any one of the following form.

$$\frac{dy}{dx} = f(x, y)$$

$$\text{or } f(x, y, \frac{dy}{dx}) = 0$$

Solution of 1st order and 1st degree :-

(1) variable separable form,

(2) linear differential eqⁿ,

(3) homogeneous D.E.

$$x dy + y dx = 0$$

$$\Rightarrow x dy = -y dx$$

$$\Rightarrow \frac{dy}{y} = \frac{-dx}{x} \Rightarrow \int \frac{dy}{y} = - \int \frac{dx}{x}$$

$$\Rightarrow \ln y = -\ln x + \ln c$$

$$\Rightarrow \ln y = \ln c - \ln x$$

$$\Rightarrow \ln y = \ln\left(\frac{c}{x}\right)$$

$$\Rightarrow \boxed{y = \frac{c}{x}} \text{ which is the required D.E.}$$

Solution of D.E of 1st order and 1st degree:

Type-1: equation of the type $\boxed{\frac{dy}{dx} = f(x)}$

using $dy = f(x) dx$.

taking integration in both sides, we have

$$\Rightarrow \int dy = \int f(x) dx.$$

$$\Rightarrow y = \phi(x) + c.$$

$$\text{Ex } \frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$$

$$\frac{dy}{dx} = \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}$$

$$\frac{dy}{dx} = \tan^2 \frac{x}{2}$$

$$dy = \tan^2 \frac{x}{2} dx$$

integrating on both the sides,

$$\int dy = \int \tan^2 \frac{x}{2} dx.$$

Ques

(18) $y = \int (\sec^2 \frac{x}{2} - 1) dx.$

$$y = \int \sec^2 \frac{x}{2} dx - \int dx.$$

$$\boxed{y = 2 \tan \frac{x}{2} - x + C}$$

This is the required solution of the given d.e.

Trigonometric formula :-

$$\rightarrow \sin^2 x + \cos^2 x = 1$$

$$1 - \sin^2 x = \cos^2 x$$

$$1 - \cos^2 x = \sin^2 x.$$

$$\rightarrow \sec^2 x - \tan^2 x = 1.$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$\rightarrow \operatorname{cosec}^2 x - \cot^2 x = 1$$

$$\operatorname{cosec}^2 x = 1 + \cot^2 x$$

$$\cot^2 x = \operatorname{cosec}^2 x - 1$$

$$\rightarrow \sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$$

$$\rightarrow \sin(x-y) = \sin x \cdot \cos y - \cos x \cdot \sin y$$

$$\rightarrow \cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

$$\rightarrow \cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y.$$

$$\rightarrow \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$$

$$\rightarrow \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \cdot \tan y}$$

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sol

$$\rightarrow 2 \sin m x \cdot \cos n x = \sin(m+n)x + \sin(m-n)x.$$

$$\sin m x \cdot \cos n x = \frac{1}{2} [\sin(m+n)x + \sin(m-n)x].$$

$$\rightarrow \cos m x \cdot \cos n x = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]$$

$$\rightarrow \sin m x \cdot \sin n x = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x]$$

$$\rightarrow \sin C + \sin D = 2 \sin \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$\rightarrow \sin C - \sin D = 2 \cos \frac{C+D}{2} \cdot \sin \frac{C-D}{2}$$

$$\rightarrow \cos C + \cos D = 2 \cos \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$\rightarrow \cos C - \cos D = 2 \sin \frac{C+D}{2} \cdot \sin \frac{D-C}{2}$$

$$\rightarrow \sin 2x = 2 \sin x \cdot \cos x.$$

$$\rightarrow \sin 2x = \frac{2 \tan x}{1 + \tan^2 x}$$

$$\rightarrow \cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$$

$$\rightarrow \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

$$\rightarrow \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\rightarrow \sin x = 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}$$

$$\rightarrow \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\rightarrow \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$$

(20)

$$\rightarrow \cos x = 2\cos^2 \frac{x}{2} - 1$$

$$2\cos^2 \frac{x}{2} = 1 + \cos x$$

$$\rightarrow \cos x = 1 - 2\sin^2 \frac{x}{2}$$

$$2\sin^2 \frac{x}{2} = 1 - \cos x$$

$$\rightarrow \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\rightarrow \tan x = \frac{2\tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$$

$$\rightarrow \sin 3x = 3\sin x - 4\sin^3 x$$

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

$$\rightarrow \cos 3x = 4\cos^3 x - 3\cos x$$

$$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$

D.E of type - 2: $\boxed{\frac{dy}{dx} = f(y)}$

$$\frac{dy}{dx} = f(y)$$

$$\Rightarrow \frac{dy}{f(y)} = dx$$

Integrating on both the sides,

$$\int \frac{dy}{f(y)} = \int dx$$

$$\Rightarrow \ln |f(y)| = x + C$$

$$\Rightarrow e^{x+C} = f(y)$$

21) Ques

$$\Rightarrow f(y) = e^x \cdot e^c.$$

$$\Rightarrow f(y) = e^x \cdot k \text{ where 'k' is constant } k = e^c.$$

Q2 $\frac{dy}{dx} = y+2.$

$$\frac{dy}{y+2} = dx \quad [\text{Integrating on both the sides}]$$

$$\int \frac{dy}{y+2} = \int dx \quad \Rightarrow \ln|y+2| = x+C$$

$$e^{x+C} = y+2$$

$$e^x \cdot e^C = y+2$$

$$\Rightarrow y+2 = e^x \cdot k \text{ where } \boxed{k = e^C}$$

Q3 $dx + \cot x dt = 0$

$$\cot x dt = -dx.$$

$$dt = \frac{-dx}{\cot x}$$

$$\Rightarrow dt = -\tan x dx. \quad (\text{Integrating on both the sides})$$

$$\Rightarrow \int dt = -\int \tan x \cdot dx.$$

$$\Rightarrow t = -\ln|\sec x| + C.$$

$$\Rightarrow \ln|\sec x| = C - t$$

$$\Rightarrow e^{C-t} = \sec x$$

$$\Rightarrow \sec x = e^C \cdot e^{-t}$$

$$\Rightarrow \sec x = \frac{k}{e^t}, \text{ where } e^C = k$$

22) or $t = -(-\ln |\cos x|) + c$

$$\Rightarrow t = \ln |\cos x| + c$$

$$\Rightarrow \ln |\cos x| = t - c$$

$$\Rightarrow e^{t-c} = \cos x$$

$$\Rightarrow \cos x = \frac{e^t}{e^c}$$

$$\Rightarrow \cos x = \frac{e^t}{e^k} \quad \text{where } e^c = k$$

Qn

$$\frac{dy}{dx} = y^2 + 2y$$

$$\frac{dy}{y^2 + 2y} = dx \quad (\text{integrating on both the sides})$$

$$\int \frac{dy}{y^2 + 2y} = \int dx \Rightarrow \int \frac{dy}{y(y+2)} = x + c$$

$$\Rightarrow \frac{1}{2} \int \frac{2}{y(y+2)} dy = x + c$$

$$\Rightarrow \frac{1}{2} \int \frac{y+2-y}{y(y+2)} dy = x + c$$

$$\Rightarrow \frac{1}{2} \int \frac{(y+2)}{y(y+2)} dy - \frac{1}{2} \int \frac{y}{y(y+2)} dy = x + c$$

$$\Rightarrow \frac{1}{2} \ln |y| = \frac{1}{2} \ln |y+2| = x + c$$

Type-3 (equation with variable separable)

If a given D.E. of being expressed in the form of

$$\boxed{f(x) dy + f(y) dx = 0}$$

21) Ques

$$\Rightarrow f(y) = e^x \cdot e^c$$

$$\Rightarrow f(y) = e^x \cdot k \text{ where 'k' is constant } k = e^c$$

Q2 $\frac{dy}{dx} = y+2$

$$\frac{dy}{y+2} = dx \quad [\text{Integrating on both the sides}]$$

$$\int \frac{dy}{y+2} = \int dx \Rightarrow \ln|y+2| = x+c$$

$$e^{x+c} = y+2$$

$$e^x \cdot e^c = y+2$$

$$\Rightarrow y+2 = e^x \cdot k \text{ where } \boxed{k = e^c}$$

Q3 $dx + \cot x dt = 0$

$$\cot x dt = -dx$$

$$dt = \frac{-dx}{\cot x}$$

$$\Rightarrow dt = -\tan x dx \quad (\text{Integrating on both the sides})$$

$$\Rightarrow \int dt = -\int \tan x \cdot dx$$

$$\Rightarrow t = -\ln|\sec x| + c$$

$$\Rightarrow \ln|\sec x| = c - t$$

$$\Rightarrow e^{c-t} = \sec x$$

$$\Rightarrow \sec x = e^c \cdot e^{-t}$$

$$\Rightarrow \sec x = \frac{k}{e^t}, \text{ where } e^c = k$$

(23) see

$$\Rightarrow f(x) dy = -f(y) dx.$$

$$\Rightarrow \frac{dy}{f(y)} = -\frac{dx}{f(x)} \quad (\text{which on integration we have})$$

$$\Rightarrow \int \frac{dy}{f(y)} = -\int \frac{dx}{f(x)} \Rightarrow \ln|f(y)| = -\ln|f(x)| + \ln C.$$

$$\Rightarrow \ln(f(y)) + \ln(f(x)) = \ln C.$$

$$\Rightarrow \ln(f(y) \cdot f(x)) = \ln C$$

$$\Rightarrow \boxed{f(x) \cdot f(y) = C}$$

where $f(x)$ & $f(y)$ are respectively function of y & x is called variable & separable type of equation.

ex $\frac{dy}{dx} = e^{x+1} \cdot y$

$$\frac{dy}{y} = e^{x+1} \cdot dx \quad (\text{integrating on both the sides})$$

$$\Rightarrow \int \frac{dy}{y} = \int e^{x+1} dx.$$

$$\Rightarrow \int \frac{dy}{y} = \int e^x \cdot e dx.$$

$$\Rightarrow \ln y = e \cdot e^x + C.$$

$$\Rightarrow \boxed{\ln y = e^{x+1} + C}$$

Qn $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

$$\frac{dy}{dx} = e^x \cdot e^y + e^y \cdot x^2$$

$$\Rightarrow \frac{dy}{dx} = e^y (e^x + x^2)$$

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$$\frac{dy}{e^y} = (e^x + x^2) dx$$

$$\Rightarrow \int \frac{dy}{e^y} = \int e^x dx + \int x^2 dx$$

$$\Rightarrow \boxed{-e^{-y} = e^x + \frac{x^3}{3} + C} \quad (ans)$$

Qn $x^2 (y-1) dx + y^2 (x-1) dy = 0$

$$x^2 (y-1) dx = (1-x) y^2 dy$$

$$\Rightarrow \frac{x^2}{1-x} = y^2 \frac{dy}{y-1} \quad \text{integrating on both the sides}$$

$$\int \frac{x^2}{1-x} dx = \int \frac{y^2}{y-1} dy$$

$$\Rightarrow \int \frac{x^2+1-1}{1-x} dx = \int \frac{y^2+1-1}{y-1} dy$$

$$\Rightarrow \int \frac{(y-1)(y+1)+1}{y-1} dy = \int \frac{(x-1)(x+1)+1}{1-x} dx$$

$$\Rightarrow \int \frac{(y-1)(y+1)}{(y-1)} dy + \int \frac{dy}{y-1} = - \int \frac{(x-1)(x+1)}{(x-1)} dx + \int \frac{dx}{1-x}$$

$$\Rightarrow \int y dy + \int \frac{dy}{y-1} = - \int x dx - \int \frac{dx}{1-x}$$

$$\Rightarrow \frac{y^2}{2} + y + \ln|y-1| = -\frac{x^2}{2} - x + \ln|1-x| + C$$

$$\Rightarrow \boxed{\ln|y-1| - \ln|1-x| = \frac{y^2}{2} - \frac{x^2}{2} + y - x + C}$$

Qn

$$(1+x)(1+y^2) dx + (1+y)(1+x^2) dy = 0$$

$$(1+x)(1+y^2) dx = -(1+y)(1+x^2) dy$$

$$\frac{(1+x)}{(1+x^2)} dx = - \frac{dy(1+y)}{1+y^2}$$

(25) Ques

$$\Rightarrow \frac{1+y}{1+y^2} dy = \frac{-(1+x)}{1+x^2} dx \text{ (integrating on both the sides)}$$

$$\Rightarrow \int \frac{dy}{1+y^2} + \int \frac{y}{1+y^2} dy = - \int \frac{dx}{1+x^2} - \int \frac{x}{1+x^2} dx.$$

$$\tan^{-1} y + \frac{1}{2} \int \frac{2y}{1+y^2} dy = - \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx.$$

$$\Rightarrow \tan^{-1} y + \tan^{-1} x = -\frac{1}{2} \ln |1+y^2| - \frac{1}{2} \ln |1+x^2|$$

$$\Rightarrow \tan^{-1} x + \tan^{-1} y = -\frac{1}{2} [\ln(1+y^2) + \ln(1+x^2)]$$

$$\Rightarrow -2 (\tan^{-1} x + \tan^{-1} y) = \ln(1+y^2) + \ln(1+x^2).$$

$$\Rightarrow \boxed{\ln[(1+y^2) \cdot (1+x^2)] = -2 (\tan^{-1} x + \tan^{-1} y)} \text{ (Ans)}$$

Ques $(1+x^2) dy + (1+y^2) dx = 0$

$$\Rightarrow (1+x^2) dy = -(1+y^2) dx.$$

$$\Rightarrow \frac{dy}{1+y^2} = \frac{-dx}{1+x^2} \text{ (integrating on both the sides)}$$

$$\Rightarrow \int \frac{dy}{1+y^2} = - \int \frac{dx}{1+x^2}$$

$$\Rightarrow \tan^{-1} y = -\tan^{-1} x + \tan^{-1} c$$

$$\Rightarrow \tan^{-1} y + \tan^{-1} x = \tan^{-1} c$$

$$\Rightarrow \tan^{-1} \left(\frac{x+y}{1-xy} \right) = \tan^{-1} c$$

$$\Rightarrow \frac{x+y}{1-xy} = c.$$

$$\Rightarrow x+y = c - cxy$$

$$\Rightarrow \boxed{y = c - cxy - x} \text{ (Ans)}$$

26 Solving D.E of second order :

Type-1 :

$$\boxed{\frac{d^2 y}{dx^2} = f(x)}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d(z)}{dx} = \frac{dz}{dx}$$

where $z = \frac{dy}{dx}$

$$\frac{dz}{dx} = f(x)$$

$$dz = f(x) dx$$

$$\Rightarrow \int dz = \int f(x) dx$$

$$\Rightarrow z = F(x) + C_1$$

$$\Rightarrow \frac{dy}{dx} = F(x) + C_1$$

$$\Rightarrow dy = (F(x) + C_1) dx$$

$$\Rightarrow \int dy = \int F(x) dx + C_1 \int dx$$

$$\Rightarrow \boxed{y = (x) + Gx + C_2}$$

Solve it

$$\textcircled{1} \frac{d^2 y}{dx^2} = \sec^2 x + \cos^2 x$$

Ans $\frac{d^2 y}{dx^2} = \sec^2 x + \cos^2 x$

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \sec^2 x + \cos^2 x$$

let $z = \frac{dy}{dx}$

$$\frac{dz}{dx} = \sec^2 x + \cos^2 x$$

$$\Rightarrow dz = (\sec^2 x + \cos^2 x) dx$$

(27)

Sol

$$\Rightarrow \int dz = \int \sec^2 x dx + \int \cos^2 x dx.$$

$$\Rightarrow z = \tan x + \frac{1}{2} \int 2 \cos^2 x dx.$$

$$\Rightarrow z = \tan x + \frac{1}{2} \int (1 + \cos 2x) dx$$

$$\Rightarrow z = \tan x + \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x dx.$$

$$\Rightarrow z = \tan x + \frac{x}{2} + \frac{1}{2} \cdot \frac{1}{2} \sin 2x.$$

$$\Rightarrow z = \tan x + \frac{x}{2} + \frac{\sin 2x}{4} + C_1$$

$$\Rightarrow \frac{dy}{dx} = \tan x + \frac{x}{2} + \frac{\sin 2x}{4} + C_1$$

$$\Rightarrow dy = \left(\tan x + \frac{x}{2} + \frac{\sin 2x}{4} + C_1 \right) dx.$$

$$\Rightarrow dy = \tan x dx + \frac{x}{2} dx + \frac{\sin 2x}{4} dx + C_1 dx.$$

$$\Rightarrow \int dy = \int \tan x dx + \int \frac{x}{2} dx + \frac{1}{4} \int \sin 2x dx + \int C_1 dx.$$

$$\Rightarrow y = \ln |\sec x| + \frac{1}{2} \cdot \frac{x^2}{2} - \frac{1}{4} \cdot \frac{1}{2} \cos 2x + C_1 x + C_2$$

$$y = \ln |\sec x| + \frac{x^2}{4} - \frac{1}{8} \cos 2x + C_1 x + C_2$$

which is the required of the given D.E.

$$(2) \frac{dy}{dx} = \frac{1+y^2}{1+x^2} \text{ given } y = \sqrt{3} \text{ when } x = 1$$

Ans $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + C = \tan^{-1} \left(\frac{x+C}{1-xC} \right)$$

Solⁿ

Q.8 $\Rightarrow y = \frac{x+c}{1-xc}$ (1)

Given $y = \sqrt{3}$ at $x=1$

So eq (1) becomes,

$$\sqrt{3} = \frac{1+c}{1-c}$$

$$\Rightarrow \sqrt{3} - \sqrt{3}c = 1+c$$

$$\Rightarrow -c - \sqrt{3}c = 1 - \sqrt{3}$$

$$\Rightarrow -c(1+\sqrt{3}) = 1 - \sqrt{3}$$

$$\Rightarrow c(1+\sqrt{3}) = \sqrt{3} - 1$$

$$\begin{aligned} \Rightarrow c &= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} \\ &= \frac{(\sqrt{3})^2 + (1)^2 - 2\sqrt{3}}{(\sqrt{3})^2 - (1)^2} \\ &= \frac{3+1-2\sqrt{3}}{3-1} = \frac{4-2\sqrt{3}}{2} \\ &= \frac{2(2-\sqrt{3})}{2} = 2-\sqrt{3} \end{aligned}$$

putting the value of 'c' in eq (1) we have,

$$y = \frac{x + (2-\sqrt{3})}{1 - (2-\sqrt{3})} \quad \text{const}$$

③ $\frac{d^2y}{dx^2} = 6x$ given by $y=1$ and $\frac{dy}{dx} = 2, x=0$

$$\frac{d^2y}{dx^2} = 6x$$

(29)

Sol:

$$\text{Let } \frac{dy}{dx} = z$$

$$\Rightarrow \frac{d}{dx} \left(\frac{dy}{dx} \right) = 6x$$

$$\Rightarrow \frac{dz}{dx} = 6x$$

$$\Rightarrow dz = 6x dx$$

$$\Rightarrow \int dz = \int 6x dx$$

$$\Rightarrow z = 6 \cdot \frac{x^2}{2} + c_1$$

$$\Rightarrow z = 3x^2 + c_1$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + c_1 \quad \dots \dots \textcircled{1}$$

$$\Rightarrow dy = (3x^2 + c_1) dx$$

$$\Rightarrow \int dy = \int 3x^2 dx + \int c_1 dx$$

$$\Rightarrow y = \frac{3x^3}{3} + c_1 x + c_2$$

$$\Rightarrow \boxed{y = x^3 + c_1 x + c_2} \quad \dots \dots \textcircled{11}$$

Putting the value of $\frac{dy}{dx}$ in eqⁿ ① we get,

$$2 = 3(0)^2 + c_1$$

$$\Rightarrow \boxed{2 = c_1}$$

$$\Rightarrow 1 = 0^3 + 2(0) + c_2$$

$$\Rightarrow \boxed{c_2 = 1}$$

Putting the value of c_1 & c_2 ,

$$y = x^3 + 2x + 1$$

(30)

$$1) \frac{dy}{dx} = \sec(uty)$$

$$\text{Ans } \frac{dy}{dx} = \sec(uty)$$

$$\text{put } uty = t$$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$1) \frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$2) \frac{dt}{dx} - 1 = \sec t$$

$$3) \frac{dt}{dx} = \sec t + 1$$

$$4) \int \frac{dt}{\sec t + 1} = \int dx$$

$$5) \int \frac{dt}{1 + \left(\frac{1 + \tan^2 t/2}{1 - \tan^2 t/2} \right)} = \int dx$$

$$6) \int \frac{dt}{\frac{1 - \tan^2 t/2 + 1 + \tan^2 t/2}{1 - \tan^2 t/2}} = \int dx$$

$$7) \int \frac{1 - \tan^2 t/2}{2} = \int dx$$

$$8) \frac{1}{2} \int dt - \frac{1}{2} \int \tan^2 t/2 dt = \int dx$$

$$9) \frac{t}{2} - \frac{1}{2} \int (\sec^2 t/2 - 1) dt = \int dx$$

$$10) \frac{t}{2} - \frac{1}{2} \times 2 \tan \frac{t}{2} + \frac{1}{2} t = x + c$$

Sol

(31)

$$\Rightarrow \frac{t}{2} - \tan \frac{t}{2} + \frac{t}{2} = x + c.$$

$$\Rightarrow t - \tan \frac{t}{2} = x + c.$$

$$\Rightarrow x + y - \tan \frac{(x+y)}{2} = x + c.$$

Homogeneous Equation / Function:-

Defⁿ:-1 A function $f(x, y)$ in x & y is called a homogeneous function / eqⁿ of degree 'n'. If the degree of each term is 'n'.

Defⁿ:-2 A function $f(x, y)$ in x & y is said to be a homogeneous function of degree 'n', if it satisfies the following condition.

$$\text{i.e. } \boxed{f(tx, ty) = t^n \cdot f(x, y)}$$

e.g of defⁿ:-1

$$\text{If } f(x, y) = x^2 + y^2 - 7xy$$

here degree of each term is 2.

Therefore, it is a homogeneous function of degree 2

e.g of defⁿ:-2

$$\text{Given } f(x, y) = x^2 + y^2 - 7xy$$

$$\begin{aligned} f(tx, ty) &= t^2 x^2 + t^2 y^2 - 7tx \cdot ty \\ &= t^2 x^2 + t^2 y^2 - 7t^2 xy \\ &= t^2 (x^2 + y^2 - 7xy) \end{aligned}$$

$$\Rightarrow \boxed{f(tx, ty) = t^2 \cdot f(x, y)}$$

32) which is a homogeneous function of degree 2.

$$f(u, v) = u^3 + v^3 - 3uv$$

from defⁿ 1 $\rightarrow$ Here degree of each term is not same, so it is not a homogeneous function.

Ques defⁿ 2 $\rightarrow$

$$f(u, v) = u^3 + v^3 - 3uv$$

$$\begin{aligned} f(tu, tv) &= t^3 u^3 + t^3 v^3 - 3tu \cdot tv \\ &= t^3 u^3 + t^3 v^3 - 3t^2 uv \\ &= t^2 (tu^3 + tv^3 - 3uv) \end{aligned}$$

which is not a homogeneous function.

Homogeneous Differential eqⁿ :-

A differential equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$

where $f(x, y)$ and $g(x, y)$ are homogeneous functions of same degree in x & y is called a homogeneous differential eqⁿ.

Method of solving Homogeneous Differential Equation:-

St-1 At first convert D.E to

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} \text{ form} \quad \text{--- (1)}$$

St-2 by putting $y = vx$ and $\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$ (2)

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

sol

(33) $\Rightarrow \boxed{\frac{dy}{dx} = v + x \frac{dv}{dx}} \dots \dots \dots (3)$

st-3 then putting eqⁿ (2) & (3) in eqⁿ (1) we have,

$$v + x \frac{dv}{dx} = \frac{f(x, vx)}{g(x, vx)} = f(v)$$

$$\Rightarrow \boxed{x \frac{dv}{dx} = f(v) - v}$$

st-4 then using variable separable method and taking integration in both sides we have,

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{f(v) - v} = \log|x| + C$$

st-5 now replace 'v' by $\frac{y}{x}$ to obtain the required sol

Qn solve $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

Given $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \dots \dots \dots (1)$

$\therefore$ It is a homogeneous function.

by taking $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$.

$$v = \frac{y}{x}$$

putting $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$ in eqⁿ $\dots \dots (1)$

so we have,

$$v + x \frac{dv}{dx} = \frac{v^2 x^2 - x^2}{2x \cdot vx} = \frac{x^2(v^2 - 1)}{2x^2 v}$$

Sol (34)

$$= \frac{v^2-1}{2v} = f(v)$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{v^2-1-2v^2}{2v} = \frac{-v^2-1}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = -\frac{(v^2+1)}{2v}$$

by using variable separable method & taking integral

in both sides.

$$\int \frac{2v}{v^2+1} dv = - \int \frac{dx}{x}$$

$$\Rightarrow \ln|v^2+1| = -\ln x + \ln c$$

$$\Rightarrow \ln|1+v^2| + \ln|x| = \ln c$$

$$\Rightarrow \ln\left(1 + \frac{y^2}{x^2}\right) + \ln x = \ln c$$

$$\Rightarrow \ln\left(\frac{x^2+y^2}{x^2} \cdot x\right) = \ln c$$

$$\Rightarrow \ln\left(\frac{x^2+y^2}{x}\right) = c$$

$$\Rightarrow \ln(x^2+y^2) = cx$$

$$\Rightarrow \boxed{y^2 = cx - x}$$

which is the required solⁿ,

Ques Solve $x^2 dy + y(x+y) dx = 0$

Ans $x^2 dy = -y(x+y) dx$

Ques

(35)

$$\frac{dy}{dx} = \frac{-y(x+y)}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-xy - y^2}{x^2} \dots \dots \dots (1)$$

which is a homogeneous functⁿ.

by putting $y = vx$ & $\frac{dy}{dx} = v + x \frac{dv}{dx}$ in eqⁿ (1)

$$\Rightarrow v + x \frac{dv}{dx} = \frac{-x \cdot vx - v^2 x^2}{x^2}$$

$$= \frac{-vx^2 - v^2 x^2}{x^2}$$

$$= \frac{-x^2(v + v^2)}{x^2}$$

$$\Rightarrow v + x \frac{dv}{dx} = -v - v^2$$

$$x \cdot \frac{dv}{dx} = -v - v^2 - v$$

$$\Rightarrow x \frac{dv}{dx} = -v^2 - 2v$$

using variable separable method,

$$\frac{1}{-v^2 - 2v} dv = \frac{1}{x} dx$$

$$\frac{dv}{v^2 + 2v} = -\frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{v^2 + 2v} = -\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{v(v+2)} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \int \frac{v+2-v}{v(v+2)} dv = -\int \frac{dx}{x}$$

Sect

(36)

$$\Rightarrow \frac{1}{2} \int \frac{(v+2)}{v(v+2)} dv - \int \frac{v}{v(v+2)} dv = - \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \ln|v| - \ln|v+2| = -\ln x + \ln C$$

$$\Rightarrow \frac{1}{2} \ln \left| \frac{y}{x} \right| - \ln \left| \frac{y}{x} + 2 \right| = -\ln x + \ln C$$

$$\Rightarrow \frac{1}{2} \ln \left| \frac{y}{x} \right| - \ln \left| \frac{y+2x}{x} \right| = -\ln x + \ln C$$

$$\Rightarrow \ln \sqrt{\frac{y}{x}} - \ln \left| \frac{y+2x}{x} \right| = \ln \left(\frac{C}{x} \right)$$

$$\Rightarrow \ln \left(\frac{\sqrt{y}}{\sqrt{x}} \cdot \frac{\sqrt{x}}{y+2x} \right) = \ln \left(\frac{C}{x} \right)$$

$$\Rightarrow \frac{\sqrt{xy}}{y+2x} = \frac{C}{x}$$

Qn Solve $\frac{dy}{dx} = \frac{y-x}{x+y}$

Let y be a homogeneous function

$$\text{let } y = vx \Rightarrow v = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \frac{vx-x}{x+vx} \\ &= \frac{x(v-1)}{x(v+1)} \end{aligned}$$

$$v + x \frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v-1}{v+1} \Rightarrow \frac{v+1}{v-1} dv = \frac{dx}{x}$$

sol

$$x \frac{dv}{dx} = \frac{-v^2 - 1}{v + 1}$$

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$$\Rightarrow \frac{v+1}{-v^2-1} dv = \frac{1}{x} dx \Rightarrow \frac{v+1}{v^2+1} dv = -\frac{dx}{x}$$

taking integration on both the sides,

$$\int \frac{v+1}{v^2+1} dv = -\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{v}{v^2+1} dv + \int \frac{dv}{v^2+1} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \int \frac{2v}{v^2+1} dv + \int \frac{dv}{v^2+1} = -\ln x + C$$

$$\Rightarrow \frac{1}{2} \ln |v^2+1| + \tan^{-1} v = -\ln x + C$$

$$\Rightarrow \ln \sqrt{v^2+1} + \tan^{-1} v = -\ln x + C$$

$$\Rightarrow \ln \sqrt{\frac{y^2}{x^2}+1} + \tan^{-1}\left(\frac{y}{x}\right) = -\ln x + C$$

$$\Rightarrow \ln \sqrt{\frac{y^2+x^2}{x^2}} + \ln x = C - \tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \ln \left(\frac{\sqrt{y^2+x^2}}{x} \times x \right) = C - \tan^{-1}\left(\frac{y}{x}\right) + C$$

$$\Rightarrow \ln \sqrt{y^2+x^2} = -\tan^{-1}\left(\frac{y}{x}\right) + C$$

Equation reducible to homogeneous function

Equation of the type $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$... (1) can be reduced to homogeneous form by substitution.

$$x = x+h$$

$$y = y+k$$

where h & k are constants hence the given eqn.

$$x = x + h$$

$$dx = dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dx}$$

$$y = y + k$$

$$dy = dy$$

hence the given D.E reduces to..

$$\frac{dy}{dx} = \frac{a(x+h) + b(y+k) + c}{A(x+h) + B(y+k) + c}$$

$$\Rightarrow \frac{dy}{dx} = \frac{ax + ah + by + bk + c}{Ax + Ah + By + Bk + c}$$

$$\Rightarrow \frac{dy}{dx} = \frac{ax + by + (ah + bk + c)}{Ax + By + (Ah + Bk + c)}$$

by taking $ah + bk + c = 0$ and $Ah + Bk + c = 0$ & solve these two eqⁿ and find out the value of h and k .

$$\frac{dy}{dx} = \frac{ax + by}{Ax + By}$$

it is the homogeneous form.

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + c}$$

$$f(x, y) = ax + by + c$$

$$f(tx, ty) = atx + bty + c = t(ax + by) + c$$

Case 1

① $\frac{a}{A} \neq \frac{b}{B}$ use in the previous process

$$x = x + h \quad y = y + k$$

Ques

(39)

Case-2

From the above the value of h will be not be finite.

$$\text{if } ax - by = 0$$

$$\Rightarrow ax = by$$

$$\Rightarrow \boxed{\frac{a}{A} = \frac{b}{B}}$$

Conclusion: The case of failure will arise when the two lines are parallel.

$$ax + by + c = 0$$

$$Ax + By + C = 0$$

Lines are parallel and have no common point of intersection

$\Rightarrow$ In this case, the D.E will take the form,

$$\frac{dy}{dx} = \frac{ax + by + c}{m(ax + by) + c}$$

$$\left[\begin{array}{l} \frac{a}{A} = \frac{b}{B} = \frac{1}{m} \\ \Rightarrow A = am \\ \Rightarrow B = bm \end{array} \right]$$

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C} = \frac{ax + by + c}{max + mby + c}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{ax + by + c}{m(ax + by) + c}}$$

$\rightarrow$ case by taking $z = ax + by$

$$\Rightarrow \frac{dy}{dx} = \frac{z + c}{mz + c}$$

Ques Solve $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$ --- ①

Ans It is not a homogeneous functⁿ. we know that by taking,

Sol

$$x = x+h \Rightarrow x = x-h$$

$$y = y+k \Rightarrow y = y-k$$

(40)

$$\frac{dy}{dx} = \frac{x+h+2(y+k)-3}{2(x+h)+y+k-3} = \frac{x+h+2y+2k-3}{2x+2h+y+k-3}$$

$$= \frac{x+2y+(h+2k-3)}{2x+y+(2h+k-3)}$$

$$\text{let } (h+2k-3=0) \quad 2$$

$$2h+k-3=0$$

$$-2h+4k-6=0$$

$$\frac{2h+k-3=0}{(-) \quad (-) \quad (+)}$$

$$3k-3=0$$

$$\Rightarrow 3k=3$$

$$\Rightarrow \boxed{k=1}$$

$$2h+k-3=0$$

$$\Rightarrow 2h+1-3=0$$

$$\Rightarrow 2h-2=0$$

$$\Rightarrow \boxed{h=1}$$

$$\frac{dy}{dx} = \frac{x+2y}{2x+y}$$

$$\text{let } y = vx \Rightarrow v = \frac{y}{x}$$

$$\begin{aligned} \frac{dx}{dx} &= \frac{x+2(vx)}{2x+vx} = \frac{x+2vx}{2x+vx} \\ &= \frac{x(2v+1)}{x(v+2)} \end{aligned}$$

Sol

$$\frac{dy}{dx} = \frac{2v+1}{v+2}$$

(4)

$$\frac{dy}{dx} = v+1 \times \frac{dv}{dx}$$

$$v+1 \frac{dv}{dx} \times x = \frac{2v+1}{v+2}$$

$$\begin{aligned} x \frac{dv}{dx} &= \frac{2v+1}{v+2} - v \\ &= \frac{2v+1 - v^2 - 2v}{v+2} \end{aligned}$$

$$x \frac{dv}{dx} = \frac{1-v^2}{v+2}$$

$$\Rightarrow \frac{v+2}{1-v^2} dv = \frac{1}{x} dx$$

$$\Rightarrow \frac{v+2}{-(v^2-1)} dv = \frac{1}{x} dx$$

$$\Rightarrow \frac{v+2}{v^2-1} dv = -\frac{dx}{x}$$

$$\Rightarrow \int \frac{v+2}{v^2-1} dv = -\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{v+2}{v^2-1} dv = -\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{v}{v^2-1} dv + 2 \int \frac{dv}{v^2-1} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \ln |v^2-1| + \ln \left| \frac{v-1}{v+1} \right| = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \ln \frac{2v}{v^2}$$

$$\Rightarrow \ln \sqrt{v^2-1} + \ln \left| \frac{v-1}{v+1} \right| = \ln \left| \frac{c}{x} \right|$$

$$\Rightarrow \ln \sqrt{\frac{y^2}{x^2}-1} + \ln \left| \frac{\frac{y}{x}-1}{\frac{y}{x}+1} \right| = \ln \left| \frac{c}{x-h} \right|$$

sol

(42)

$$\Rightarrow \ln \sqrt{\frac{y^2 - x^2}{x^2}} + \ln \left| \frac{y-x}{y+x} \right| = \ln \left| \frac{c}{x-1} \right|$$

$$\Rightarrow \ln \left(\frac{\sqrt{y^2 - x^2}}{x} \times \frac{y-x}{y+x} \right) = \ln \left| \frac{c}{x-1} \right|$$

$$\Rightarrow \ln \left(\frac{\sqrt{(y-k)^2 - (x-h)^2}}{(x-h)} \cdot \frac{y-k-(x-h)}{y-k+(x-h)} \right) = \ln \left(\frac{c}{x-1} \right)$$

$$\Rightarrow \ln \left(\frac{\sqrt{y^2 + x - 2y - (x^2 + 1 - 2x)}}{x-1} \cdot \frac{y-1-x+1}{y-1+x-1} \right) = \ln \left(\frac{c}{x-1} \right)$$

$$\Rightarrow \ln \left(\frac{\sqrt{y^2 - x^2 - 2y + 2x}}{x-1} \times \frac{y-x}{y+x-2} \right) = \ln \left(\frac{c}{x-1} \right)$$

$$\Rightarrow \frac{\sqrt{y^2 - x^2 + 2x - 2y}}{(x-1)} \cdot \frac{y-x}{y+x-2} = \frac{c}{(x-1)}$$

$$\Rightarrow \frac{(y-x) \sqrt{y^2 - x^2 + 2x - 2y}}{y+x-2} = c$$

solue $\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5}$

$\frac{dy}{dx} = \frac{2x+3y+4}{2(2x+3y)+5}$

put $2x+3y = z$

$\frac{dy}{dx} = \frac{z+4}{2z+5}$

$2 + 3 \frac{dy}{dx} = \frac{dz}{dx}$

$3 \frac{dy}{dx} = \frac{dz}{dx} - 2$

$\frac{dy}{dx} = \frac{1}{3} \left(\frac{dz}{dx} - 2 \right)$

$\frac{1}{3} \left(\frac{dz}{dx} - 2 \right) = \frac{z+4}{2z+5}$

$\frac{dz}{dx} - 2 = \frac{3z+12}{2z+5}$

$\frac{dz}{dx} = \frac{3z+12}{2z+5} + 2$

$= \frac{3z+12+4z+10}{2z+5}$

$\frac{dz}{dx} = \frac{7z+22}{2z+5}$

$\frac{2z+5}{7z+22} dz = dx$

$\frac{2}{7} \cdot \frac{7z+22}{7z+22} dz = dx$

Ques

(44)

$$\Rightarrow \frac{2}{7} \int dz - \frac{9}{7} \int \frac{dz}{7z+22} = \int dx$$

$$\Rightarrow \frac{2}{7} \cdot z - \frac{9}{7} \times \frac{1}{7} \int \frac{7}{7z+22} dz = \int dx$$

$$\Rightarrow \frac{2}{7} z - \frac{9}{49} \ln |7z+22| = x + c$$

$$\Rightarrow \frac{2}{7} (2x+3y) - \frac{9}{49} \ln |7(2x+3y)+22| = x + c$$

$$\Rightarrow \frac{2}{7} (2x+3y) - \frac{9}{49} \ln |14x+21y+22| = x + c.$$

Ques

$$\int \frac{2z+5}{7z+22} dz = \int dx.$$

$$\Rightarrow \frac{2}{7} \int \frac{\frac{7}{2}(2z+5)}{7z+22} dz = \int dx.$$

$$\Rightarrow \frac{2}{7} \int \frac{\frac{7}{2} \times 2z + \frac{7}{2} \times 5}{7z+22} dz = \int dx$$

$$\Rightarrow \frac{2}{7} \int \frac{7z + \frac{35}{2}}{7z+22} dz = \int dx.$$

$$\Rightarrow \frac{2}{7} \int \frac{7z+22 + \frac{35}{2} - 22}{7z+22} dz = \int dx$$

$$\Rightarrow \frac{2}{7} \int \frac{7z+22 + \frac{35-44}{2}}{7z+22} dz = \int dx.$$

Q.1

(45)

$$\Rightarrow \frac{2}{7} \int \frac{7z+22 - \frac{9}{2}}{7z+22} dz = \int dx$$

$$\Rightarrow \frac{2}{7} \int \frac{7z+22}{7z+22} dz - \frac{2}{7} \times \frac{9}{2} \int \frac{dz}{7z+22} = \int dx$$

$$\Rightarrow \frac{2}{7} z - \frac{9}{7} \times \frac{1}{7} \int \frac{7}{7z+22} dz = \int dx$$

$$\Rightarrow \frac{2}{7} z - \frac{9}{49} \ln|7z+22| = x + c$$

$$\Rightarrow \frac{2}{7} (2x+3y) - \frac{9}{49} \ln|7(2x+3y)+22| - x = c$$

(Answer)

Linear Differential Equations:-

A linear differential eq. in which said to be linear D.E in which ~~the~~ dependant variable and all its derivative arrives in the 1st derivative only & are not multiplied together is called as linear D.E.

Q.1

$$(1) \frac{dy}{dx} + 5y = x^2$$

$$(2) x \frac{d^2 y}{dy^2} + 5 \frac{dy}{dx} = 8$$

$$(3) x \left(\frac{dy}{dx} \right)^2 - y^2 \frac{dy}{dx} + 3 = 0 \text{ (not a D.E.)}$$

Type - 1 of eqⁿ of the form,

St - 1 $\rightarrow \boxed{\frac{dy}{dx} + Py = Q}$

where $P = P(x)$

$Q = Q(x)$

St - 2 I. F = Integrating factor = $e^{\int P dx}$

St - 3 Let's solution,

$$\boxed{y(I.F) = \int Q(I.F) dx}$$

Qn solve $4 \frac{dy}{dx} + 8y = 5 \cdot e^{-3x}$

sol. dividing 4 in both sides,

$$\frac{dy}{dx} + 2y = \frac{5}{4} e^{-3x}$$

$$P = 2, \quad Q = \frac{5}{4} e^{-3x}$$

$$I.F = e^{\int P dx}$$

$$= e^{2 \int dx} = e^{2x}$$

$$y \cdot e^{2x} = \frac{5}{4} \int e^{-3x} \cdot e^{2x} dx$$

$$y \cdot e^{2x} = \frac{5}{4} \int e^{-x} dx$$

$$\boxed{y \cdot e^{2x} = -\frac{5}{4} e^{-x} + C}$$

$$y = -\frac{5}{4} \frac{e^{-x}}{e^{2x}} + \frac{C}{e^{2x}}$$

Sol

$$y = -\frac{5}{4} e^{-3x} + C \cdot e^{-2x}$$

(47)

Type-2

$$\frac{dx}{dy} + P x = Q$$

$$\frac{dx}{dy} + P(y) \cdot x = Q(y)$$

$$\frac{dx}{dy} + P x = Q$$

$$I.F = e^{\int P \cdot dy}$$

$$\Rightarrow x(I.F) = \int Q(I.F) dy$$

H.W

$$y \log y \frac{dx}{dy} + x - \log y = 0$$

$$y \log y \frac{dx}{dy} + x = \log y$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$

$$I.F = e^{\int P dx}$$

$$= e^{\int \frac{1}{y \log y} dy}$$

$$= e^{\int \frac{dz}{z}} = e^{\ln z}$$

$$= e^{\ln(\log y)} = \log y$$

$$\left| \begin{array}{l} \log y = z \\ \Rightarrow \frac{1}{y} = \frac{dz}{dy} \\ \Rightarrow \frac{1}{y} dy = dz \end{array} \right.$$

$$x \log y = \int \frac{1}{y} \cdot \log y dy$$

$$\left| \begin{array}{l} \log y = z \\ \frac{1}{y} dy = dz \end{array} \right.$$

$$x \log y = \int z dz$$

$$x \log y = \frac{z^2}{2} \Rightarrow$$

$$x \log y = \frac{(\log y)^2}{2} + C$$

soln

$$(1+y^2)dx = (\tan^{-1}y - x)dy$$

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} - \frac{\tan^{-1}y}{1+y^2} + \frac{x}{1+y^2} = 0$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F = e^{\int P \cdot dy}$$

$$= e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$

$$x \times I.F = \int (Q \times I.F) dy$$

$$x \times e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \cdot e^{\tan^{-1}y} dy$$

$$x \times e^{\tan^{-1}y} = \int t \cdot e^t dt \quad \left| \quad \begin{array}{l} \tan^{-1}y = t \\ \frac{1}{1+y^2} dy = dt \end{array} \right.$$

$$\Rightarrow x \times e^{\tan^{-1}y} = t \int e^t dt - \int \left(\frac{d}{dt}(t) \cdot \int e^t dt \right) dt$$

$$\Rightarrow x e^{\tan^{-1}y} = t e^t - e^t$$

$$\Rightarrow x e^{\tan^{-1}y} = \tan^{-1}y \cdot e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$$

Ques

(49)

$$x \cdot e^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + C. \text{ const.}$$

Equation Reducible to linear form (Bernoulli eq):
Type-III eqⁿ of the form,

$$\boxed{\frac{dy}{dx} + Py = Q \cdot y^n} \quad \dots \dots \dots (1)$$

where P & Q are constants or functions of x alone and 'n' is a constant other than '0' or unity can be reduced to linear form by dividing by (y^n) and substituting $\boxed{\frac{1}{y^{n-1}} = Z}$

$$\Rightarrow \frac{dy}{dx} + Py = Q \cdot y^n$$

dividing y^n in both sides,

$$\frac{1}{y^n} \cdot \frac{dy}{dx} + P \cdot \frac{y}{y^n} = Q \cdot \frac{y^n}{y^n}$$

$$\Rightarrow \boxed{\frac{1}{y^n} \frac{dy}{dx} + P \frac{1}{y^n} = Q} \quad \dots \dots \dots (1)$$

$$\Rightarrow Z = \frac{1}{y^{n-1}}$$

$$\Rightarrow Z = y^{-(n-1)} = y^{1-n}$$

$$Z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n) y^{1-n-1} \frac{dy}{dx}$$

$$\Rightarrow \frac{dz}{dx} = (1-n) y^{-n} \frac{dy}{dx}$$

$$\Rightarrow \frac{dz}{dx} = (1-n) \frac{1}{y^n} \cdot \frac{dy}{dx}$$

$$\boxed{\frac{1}{1-n} \frac{dz}{dx} = \frac{1}{y^n} \frac{dy}{dx}} \quad \dots \dots \dots (2)$$

putting eqⁿ (2) & $z = \frac{1}{y^{n-1}}$ in eqⁿ (1) we have,

⇒ $\boxed{\frac{1}{1-n} \cdot \frac{dz}{dx} + zp = Q}$ which is a linear eqⁿ.

⇒ use the method of solution type-1

Ex solve $\frac{dy}{dx} + xy = xy^3$

$$\frac{dy}{dx} + xy = xy^3$$

divide y^3 on both the sides,

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{xy}{y^3} = \frac{xy^3}{y^3}$$

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{x}{y^2} = x$$

put $\frac{1}{y^2} = z$

$$y^{-2} = z$$

$$\frac{dz}{dx} = y^{-3} (-2) \frac{dy}{dx}$$

$$-\frac{2}{y^3} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\boxed{\frac{1}{y^3} \frac{dy}{dx} = -\frac{1}{2} \frac{dz}{dx}}$$

putting the above and $z = \frac{1}{y^2}$ in eqⁿ (1)

$$⇒ -\frac{1}{2} \frac{dz}{dx} + 2z = x$$

$$⇒ -\frac{1}{2} \times 2 \cdot \frac{dz}{dx} - 2zx = -2x$$

$$⇒ \frac{dz}{dx} + 2zx = -2x$$