

• Matrix :- A Matrix is an ordered Rectangular Array of numbers or functions and enclose the elements by paratheses i.e $[]$ or $()$.

✓ Array :- Arrangement of numbers or functions in rows and columnwise.

→ A matrix is defined or denoted by capital English Alphabet and Elements of the matrix are represented by small English alphabets as like $(a, b, c \text{ etc})$.

→ A matrix is generally represented by :-

$$A = [a_{ij}] \begin{matrix} \nearrow \text{row number} \\ \searrow \text{column number} \end{matrix} \begin{matrix} \xrightarrow{\text{number of rows}} m \times n \\ \xrightarrow{\text{no of columns}} \end{matrix}$$

• order of Matrix :-

→ $m \times n$ → number of rows by number of column is known as the order of the Matrix.

$\downarrow$
 $\underline{m \text{ by } n}$
or
 $\underline{m \text{ cross } n}$

$$A = [a_{ij}] \begin{matrix} \nearrow \text{row} \\ \searrow \text{column} \end{matrix} m \times n \rightarrow \text{order}$$

Where, $i = 1, 2, 3, \dots, m$
 $j = 1, 2, 3, \dots, n$

• Expansion of this Matrix is :-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{matrix} \xrightarrow{\text{row}} \\ \xleftarrow{\text{column}} \end{matrix} m \times n$$

• Types of Matrix :-

(i) Row Matrix :- A matrix is said to be Row Matrix if it has one row and numbers of columns.

OR

Its order is $1 \times n$

EX :- $A = [a \ b \ c \ d]_{1 \times 4}$

$B = [0 \ 0 \ 0]_{1 \times 3}$

$C = [1 \ 2]_{1 \times 2}$

(ii) column Matrix :- A matrix is said to be column Matrix if it has 1 column but numbers of rows.

OR

Its order is $m \times 1$

EX :- $A = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}_{4 \times 1}$

①
②
③
④

(iii) Null Matrix / Zero Matrix (0) :- A matrix is said to be null if all elements or entries are all 0's having any order.

EX :- $A = [0]_{1 \times 1}$

$B = [0 \ 0 \ 0 \ 0]_{1 \times 4}$

$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2 \times 3}$

$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$

(4) Square Matrix:-

→ A matrix is said to be square if No of Rows = No of column.

EX:- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3 \times 3}$

(5) Diagonal Matrix:-

→ A square Matrix is said to be diagonal matrix if

$a_{ij} = 0$ for $i \neq j$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

EX:-

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

→ Leading diagonal elements.

off diagonal elements.

(6) Scalar Matrix:-

→ A diagonal Matrix is said to be a scalar Matrix if all diagonal elements are same.

EX:- $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3 \times 3}$

(7) unit Matrix or Identity Matrix:-

→ A scalar / diagonal Matrix is said to be an identity Matrix if all diagonal elements are 1.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

(8) Lower Triangular Matrix :-

→ A square Matrix is said to be Lower Triangular Matrix if
 $a_{ij} = 0$ where $(i < j)$

EX :-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

EX :-

$$B = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 1 & 7 \end{bmatrix}_{3 \times 3}$$

(9) Upper Triangular Matrix :-

→ A square Matrix is said to be an upper Triangular matrix if
 $a_{ij} = 0$ where $(i > j)$

EX :-

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}_{3 \times 3} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ 0 & p_{22} & p_{23} \\ 0 & 0 & p_{33} \end{bmatrix}$$

Properties :-

(1) Addition / Subtraction of Matrices :-

→ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2}$

$B = \begin{bmatrix} x & y \\ z & p \end{bmatrix}_{2 \times 2}$

$$A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} x & y \\ z & p \end{bmatrix} = \begin{bmatrix} a+x & b+y \\ c+z & d+p \end{bmatrix}_{2 \times 2}$$

Subtraction :-

(2)

Multiplication of Matrices :-

→ If $A_{m \times n}$ and $B_{n \times p}$ Then product of $(AB)_{m \times p}$.

$$\Rightarrow A_{m \times n} \cdot B_{n \times p} = (AB)_{m \times p}$$

$$A_{2 \times 3} \cdot B_{3 \times 7} = (AB)_{2 \times 7}$$

Ex :- $A = \begin{bmatrix} -1 & 3 & 2 \\ 4 & 0 & 1 \\ 2 & 3 & -2 \end{bmatrix}_{3 \times 3}$

$$B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

formula

$$\begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix}_{3 \times 3}$$

$$AB = \begin{bmatrix} -1 & 3 & 2 \\ 4 & 0 & 1 \\ 2 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1 \times 2) + (3 \times 1) + (2 \times -1) & (-1 \times 3) + (3 \times 2) + (2 \times 0) & (-1 \times 4) + (3 \times 3) + (2 \times 1) \\ (4 \times 2) + (0 \times 1) + (1 \times -1) & (4 \times 3) + (0 \times 2) + (1 \times 0) & (4 \times 4) + (0 \times 3) + (1 \times 1) \\ (2 \times 2) + (3 \times 1) + (-2 \times -1) & (2 \times 3) + (3 \times 2) + (-2 \times 0) & (2 \times 4) + (3 \times 3) + (-2 \times 1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 + 3 - 2 & -3 + 6 + 0 & -4 + 9 + 2 \\ 8 + 0 - 1 & 12 + 0 + 0 & 16 + 0 + 1 \\ 4 + 3 + 2 & 6 + 6 + 0 & 8 + 9 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 7 \\ 7 & 12 & 17 \\ 9 & 12 & 15 \end{bmatrix}$$

EX :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3 \times 3}$

$$B = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}_{3 \times 1}$$

Ans

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1 \times -1) + (2 \times -2) + (3 \times -3) \\ (4 \times -1) + (5 \times -2) + (6 \times -3) \\ (7 \times -1) + (8 \times -2) + (9 \times -3) \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & -4 & -9 \\ -4 & -10 & -18 \\ -7 & -16 & -27 \end{bmatrix}$$

$$AB = \begin{bmatrix} -14 \\ -32 \\ -50 \end{bmatrix}$$

Determinant :-

→ Determinant of a matrix A is denoted by $|A|$ or $\det A$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$|A| = a_{11}$$

minor and cofactor

$$c_{ij} = \text{cofactor of } a_{ij} = (-1)^{i+j} \cdot M_{ij}$$

$$a_{11} = (-1)^{1+1} \cdot m_{11}$$

$$a_{12} = (-1)^{1+2} = (-1)^3 = -1$$

Minor (M_{ij})

$$M_{11} = \text{minor of } m_{11}$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$m_{12} = \text{minor of } m_{12}$$

$$= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \text{minor of } M_{13}$$

$$= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

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$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

→ To find out determinant, we have to use square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We can expand it as :-

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13}$$

Where M_{ij} is the minor of a_{ij}

$$= a_{11}M_{11} + a_{12}(-M_{12}) + a_{13} \cdot M_{13}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\boxed{|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}}$$

OR

$$\boxed{|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}}$$

Expanding formula.

• Minor of $a_{ij} = M_{ij} :-$

$$\rightarrow \text{let } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$= a_{22} \cdot a_{33} - a_{23} \cdot a_{32}$$

$$M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= a_{21} \cdot a_{33} - a_{23} \cdot a_{31}$$

$$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21} \cdot a_{32} - a_{22} \cdot a_{31}$$

$$M_{21} = \text{minor of } a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$= a_{12} \cdot a_{33} - a_{13} \cdot a_{32}$$

$$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= a_{11} \cdot a_{33} - a_{13} \cdot a_{31}$$

$$M_{23} = \text{minor of } a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} \cdot a_{32} - a_{12} \cdot a_{31}$$

$$M_{31} = \text{minor of } a_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{12} \cdot a_{23} - a_{13} \cdot a_{22}$$

$$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$= a_{11} \cdot a_{23} - a_{13} \cdot a_{21}$$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

• cofactor of a_{ij} :-

→ $C_{ij} \text{ or } A_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$C_{11} = \text{cofactor of } a_{11} = (-1)^{1+1} \cdot M_{11}$$

$$= (-1)^2 \cdot M_{11}$$

$$= 1 \cdot M_{11}$$

$$= M_{11}$$

$$C_{12} = (-1)^{1+2} \cdot M_{12}$$

$$= (-1)^3 \cdot M_{12}$$

$$= -1 \cdot M_{12}$$

$$= -M_{12}$$

Hence

+	-	+
-	+	-
+	-	+

$$C_{13} = M_{13}$$

$$C_{21} = -M_{21}$$

$$C_{22} = M_{22}$$

$$C_{23} = -M_{23}$$

$$C_{31} = M_{31}$$

$$C_{32} = -M_{32}$$

$$C_{33} = M_{33}$$

$$= \begin{vmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{vmatrix}$$

Adjoint of a Matrix :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Definition :- The transpose of co-factors Matrix.

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Transpose of a Matrix :- Interchanging rows and columns of a Matrix is known as its transpose.

$$\boxed{A' \text{ or } A^T}$$

EX :- let $A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & -1 & -2 \\ -3 & 1 & 4 \end{bmatrix}$

$R_i \leftrightarrow C_i$

Find out its $\text{Adj } A$.

$\text{Adj } A = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$\text{Adj } A = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$\text{Adj } A = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$\text{Adj } A = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$\text{Adj } A = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$\text{Adj } A = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

Example :-

let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

find out $\text{Adj } A$

$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 2(0-2) - 1(4-2) = -4 - 2 = -6$

$[0-2] \times 2 + [2-4] \times 1 = -4 + (-2) = -6$

$(2-2) \times 1 + (2-2) \times 2 = 0 + 0 = 0$

$2 \times 2 - 1 \times 2 = 4 - 2 = 2$

$2 \times 2 - 1 \times 2 = 4 - 2 = 2$

$\text{Adj } A = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

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• Inverse of a Matrix :-

→ let A be a square Matrix.

$$\text{Then } A^{-1} = \frac{\text{Adj} A}{|A|} \text{ whenever } |A| \neq 0.$$

or

A is nonsingular matrix.

• Remark :-

• Singular Matrix :- A square Matrix is said to be singular if $|A| = 0$

• Non-singular Matrix :- A square Matrix is said to be Non-singular Matrix if $|A| \neq 0$

• Example :-

$$\text{let } A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix} \rightarrow \text{non singular.}$$

find out $A^{-1} = ?$

$$\Rightarrow |A| = \begin{vmatrix} (-1) & (-) & (+) \\ -1 & 2 & 3 \\ 4 & 0 & 2 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= (-1) [0 + 4] - 2 [4 - 6] + 3 [-8 - 0]$$

$$= (-1)(4) - 2(-2) + 3(-8)$$

$$= -4 + 4 - 24$$

$$= -24$$

$$|A| \neq 0.$$

Adjoint

$$C_{11} = (-1)^{1+1} \cdot M_{11}$$

$$= (-1)^2 \cdot \begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix}$$

$$= 1 [0 + 4]$$

$$= 4$$

$$C_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= -1 [4 - 6]$$

$$= -1 [-2]$$

$$= 2$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = (-1)^4 \begin{vmatrix} 4 & 0 \\ 3 & -2 \end{vmatrix}$$

$$= 1 [-8 - 0]$$

$$= 1 [-8]$$

$$= -8$$

$$C_{21} = (-1)^{2+1} \cdot M_{21} = (-1)^3 \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= -1 [2 + 6]$$

$$= -1 [8]$$

$$= -8$$

$$C_{22} = (-1)^{2+2} \cdot M_{22} = (-1)^4 \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 1 [-1 - 9]$$

$$= 1 [-10]$$

$$= -10$$

$$C_{23} = (-1)^{2+3} \cdot M_{23}$$

$$= (-1)^5 \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix}$$

$$= -1 [1 \cdot (-2) - 6]$$

$$= -1 [-8]$$

$$= 8$$

$$C_{31} = (-1)^{3+1} \cdot M_{31}$$

$$= (-1)^4 \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix}$$

$$= 1 [4 - 0]$$

$$= 4$$

$$= (-1)^{6} \cdot M_{33}$$

$$C_{33} = (-1)^6 \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix}$$

$$= 1 [0 - 8]$$

$$= -8$$

$$C_{32} = (-1)^5 \cdot M_{32}$$

$$= -1 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix}$$

$$= (-1) (-2 - 12)$$

$$= -1 (-14)$$

$$= 14$$

$$\text{Adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 4 & -8 & 4 \\ 2 & -10 & 14 \\ -8 & 4 & -8 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj} A}{|A|} = \frac{\begin{bmatrix} 4 & -8 & 4 \\ 2 & -10 & 14 \\ -8 & 4 & -8 \end{bmatrix}}{-24}$$

$$= \frac{\begin{bmatrix} 4 & -8 & 4 \\ 2 & -10 & 14 \\ -8 & 4 & -8 \end{bmatrix}}{-24}$$

$$= \begin{bmatrix} \frac{+4}{-24} & \frac{-8}{-24} & \frac{+4}{-24} \\ \frac{+2}{-24} & \frac{-10}{-24} & \frac{+14}{-24} \\ \frac{-8}{-24} & \frac{+4}{-24} & \frac{-8}{-24} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{12} & \frac{5}{12} & -\frac{7}{12} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

Elementary Transformation of Matrix :-

- (1) Gauss-Elimination Method
- (2) Gauss-Jordan Method.

Elementary Row operation :-

- (1) $R_i \leftrightarrow R_j$
- (2) $R_i \rightarrow kR_i$
- (3) $R_i \rightarrow R_i + mR_j + nR_k$

Elementary column operation :-

- (1) $C_i \leftrightarrow C_j$
- (2) $C_i \rightarrow kC_i$
- (3) $C_i \rightarrow C_i + mC_j + nC_k$

(4) By the Help of Elementary Row operation, Reduce the matrix to upper triangle matrix.

EX :- $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Ans

$$A = \underline{I} A \text{ or } A \underline{I}$$

$$\underline{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+2+0 & 0+0+5 \\ 2+0+0 & 0+3+0 & 0+0+1 \\ -1+0+0 & 0+1+0 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$A = I A$$

$$A A^{-1} = I$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} = A \quad \therefore X I$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = I$$

$$I A = A I = A$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

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(17)

(1)

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 3 & 6 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + (-2R_1)$$

$$R_3 \rightarrow R_3 + R_1$$

$$2 - 2 \times 1 = 0$$

$$3 - 2 \times 2 = -1$$

$$1 - 2 \times 5 = -9$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 3 & 6 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$3 + (3 \times -1) = 0$$

$$6 + (3 \times -9) = 6 - 27 = -21$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 0 & -21 \end{bmatrix}$$

$\therefore$ This is the required upper triangle Method.

(2) By using Gauss-Jordan Method:-

To find A^{-1} ?

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Ans

We know that A is a Matrix:-

$$A = AI = IA$$

Where I = Identity Matrix of same order as A

$$\text{Here, } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = IA$$

OR

$$I = AA^{-1}$$

By Gauss-Jordan Method,

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \quad \text{or} \quad \left[A \mid I \right]$$

$$\text{OR} \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$2 - 2 \times 1 = 0$$

$$3 - 2 \times 2 = -1$$

$$1 - 2 \times 5 = -9$$

$$0 - 2 \times 1 = -2$$

$$0 - 2 \times 0 = 0$$

$$1 - 2 \times 0 = 1$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_3 \rightarrow R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} A$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{5}{21} & -\frac{3}{21} & -\frac{1}{21} \end{bmatrix} A$$

$$R_3 \rightarrow R_3 + 4R_1$$

$$0 + 4 \times 1 = 4$$

$$0 + 4 \times 2 = 8$$

$$-21 + 4 \times 5 = -1$$

$$-5 + 4 \times 1 = -1$$

$$3 + 4 \times 0 = 3$$

$$1 + 4 \times 0 = 1$$

} This process is not applicable

We have to apply formula:-

$$R_3 \rightarrow \frac{1}{-21} R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ \frac{5}{21} & -\frac{3}{21} & -\frac{1}{21} \end{bmatrix} A$$

$$R_2 \rightarrow (-1) R_2$$

$$R_2 \rightarrow R_2 - 9 \times R_3$$

$$0 - 9 \times 0 = 0$$

$$1 - 9 \times 0 = 1$$

$$9 - 9 \times 1 = 0$$

$$2 - 9 \times \frac{5}{21} = \frac{42 - 45}{21} = -\frac{1}{7}$$

$$-1 - 9 \times \left(-\frac{1}{21} \right) = -1 + \frac{9}{21} = -\frac{2}{7}$$

$$0 - \cancel{9} \times \left(-\frac{1}{21} \right) = \frac{3}{7}$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$1 - 2 \times 0 = 1$$

$$2 - 2 \times 1 = 0$$

$$5 - 2 \times 0 = 5$$

$$1 - 2 \times -\frac{1}{7} = 1 + \frac{2}{7} = \frac{9}{7}$$

$$0 - 2 \times \frac{2}{7} = -\frac{4}{7}$$

$$0 - 2 \times \frac{3}{7} = -\frac{6}{7}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{7} & -\frac{4}{7} & -\frac{6}{7} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix} A$$

$$R_1 \rightarrow R_1 - 5R_3$$

$$1 - 5 \times 0 = 1$$

$$0 - 5 \times 0 = 0$$

$$5 - 5 \times 1 = 0$$

$$\frac{9}{7} - 5 \times \frac{5}{21} = \frac{2}{21}$$

$$-\frac{4}{7} - 5 \times \left(-\frac{1}{7}\right) = -\frac{4}{7} + \frac{5}{7} = \frac{1}{7}$$

$$-\frac{6}{7} - 5 \times \frac{-1}{21} = -\frac{6}{7} + \frac{5}{21} = \frac{-13}{21}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix} A$$

$$\sim I = A^{-1} A$$

$$A^{-1} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix}$$

Ans

Gauss elimination Method :-

(Rank)

Solve $x + 2y - z = 3$

$3x - y + 2z = 1$

$2x - 2y + 3z = 2$

Ans

A = coefficient Matrix

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$$

$$\Rightarrow \boxed{AX = B}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Augmented Matrix form = K.

$$K = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \end{array} \right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ \textcircled{3} & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$3 - 3 \times 1 = 0$$

$$-1 - 3 \times 2 = -7$$

$$2 - 3 \times -1 = 5$$

$$1 - 3 \times 3 = -8$$

Next,

$$R_3 \rightarrow R_3 - 2R_1$$

$$2 - 2 \times 1 = 0$$

$$-2 - 2 \times 2 = -6$$

$$3 - 2 \times -1 = 5$$

$$2 - 2 \times 3 = -4$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \end{array} \right]$$

$$R_3 \rightarrow -\frac{1}{6} R_3$$

OR

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 1 & -4 & -4 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 1 & 0 & 4 \end{array} \right]$$

$$c_2 \leftrightarrow c_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 5 & -7 & -8 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

use Backward substitution Method,

$$\Rightarrow x - y + 2z = 3$$

$$5y - 7z = -8$$

$$z = 4$$

$$z = 4 \text{ put in } 5y - 7z = -8$$

$$\Rightarrow 5y - 7 \times 4 = -8$$

$$\Rightarrow 5y - 28 = -8$$

$$\Rightarrow 5y = -8 + 28$$

$$\Rightarrow 5y = 20$$

$$\Rightarrow y = 4$$

putting $y = 4, z = 4$ in $x - y + 2z = 3$

$$\Rightarrow x - 4 + 8 = 3$$

$$\Rightarrow x + 4 = 3$$

$$\Rightarrow x = -1$$

Rank = 3 because in this matrix non zero rows are 3

Q) Solve the system of equations,

$$2x - 3y + z = 1$$

$$x + 2y - 3z = 4$$

$$4x - y - 2z = 8 \text{ by Gauss elimination Method.}$$

Ans

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

Where $A = \text{co-efficient Matrix}$

$$= \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

By Gauss Elimination Method;

$K = [A|B]$ = Augmented Matrix

$$\sim \left[\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 1 & 2 & -3 & 4 \\ 4 & -1 & -2 & 8 \end{array} \right]$$

$$R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 2 & -3 & 1 & 1 \\ 4 & -1 & -2 & 8 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 7 & -7 \\ 4 & -1 & -2 & 8 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 4R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 7 & -7 \\ 0 & -9 & 10 & -8 \end{array} \right]$$

$$R_3 \rightarrow \left(-\frac{1}{9}\right)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 7 & -7 \\ 0 & 1 & -\frac{10}{9} & \frac{8}{9} \end{array} \right]$$

$$R_2 \rightarrow \left(-\frac{1}{7}\right)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -\frac{10}{9} & \frac{8}{9} \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1/9 & -1/9 \end{array} \right]$$

$$R_3 \rightarrow (-9)R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\boxed{z=1}$$

$$y-z=1$$

$$\Rightarrow \boxed{y=2}$$

$$x+2y-3z=4$$

$$\Rightarrow x + \cancel{4} - 3 = \cancel{4}$$

$$\Rightarrow \boxed{x=3}$$

$$\boxed{\text{RANK}=3}$$

• Rank of a Matrix :-

- A number r is said to be a rank of a non-zero $m \times n$ matrix A if (i) there is atleast one $(r \times r)$ sub matrix of A whose determinant is not equal to zero. or A is a non-singular Matrix.
- (ii) The determinant of every $(r+1)$ rowed square submatrix of A is zero.

OR In other word, the rank of matrix A is the greatest possible, positive integer r such that A has atleast one non zero minor of order r .

OR Rank of a Matrix which can be obtained by eliminating largest order of non vanishing minor of the Matrix.

→ The rank of the matrix is denoted by $r(A)$ or $\rho(A)$

• Remember :-

→ $r(A) \leq \min(\text{no of rows, no of columns})$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3 \times 2}$$

$$r(A) \leq \min(3, 2)$$

$$r(A) = 2$$

• Remark :-

- (1) If all the elements of a Matrix A are zero then rank of A taken as '0'.

$$r(A) = 0$$

i.e the rank of a null matrix is '0'

- (2) If a given matrix is transformed to its Echelon form, then the number of non zero rows gives the rank of the Matrix.

(Q) Find Rank of Matrix?

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Ans let us take,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}_{3 \times 3}$$

$$\rho(A) \leq \min(3, 3)$$

$$\rho(A) \leq 3$$

$$\{\rho(A) = 1, 2, 3\}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 1 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ &= 1(8-9) - 2(10-12) + 3(8-9) \\ &= -1 + 4 - 3 \\ &= 0 \end{aligned}$$

Here A is a singular square Matrix in which there is at least one (2×2) sub matrix.

for Example :-

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$|A_1| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 1 \times 3 - 2 \times 2 = 3 - 4 = -1 \neq 0$$

Whose determinant is not equal to zero. Hence, the rank of A is 2.

$$A = \rho + 1 = 3$$

$$\boxed{\rho = 2}$$

so the rank of the matrix = 2 which is less than the order 3 of the singular square matrix.

(Q) find the rank of the Matrix ?

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Ans let us take $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

$$\rho(A) \leq \min(4, 4)$$

$$\rho(A) \leq 4$$

$$\{\rho(A) = 1, 2, 3, 4\}$$

$$|A| = \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{vmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$R_4 \rightarrow R_4 - R_3$$

$$= \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 6 & 1 & 3 & 8 \\ 6 & 1 & 3 & 8 \end{vmatrix}$$

$$= 0 \text{ (since } R_1, R_3 \text{ and } R_4 \text{ are identical)}$$

(Q) there are two identical rows then determinant will be zero

$$|A| = 0 \text{ For } (4 \times 4) \text{ Matrix}$$

$$r+1 = 4$$

$$\Rightarrow r = 3$$

The rank of the given matrix is not 4. let us consider the minor of order 3 (i.e. submatrix of the original Matrix) which are -

$$|A_1| = \begin{vmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 10 & 3 & 9 \end{vmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \begin{vmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 1 & 3 \end{vmatrix}$$

$$= 0$$

$$|A_2| = \begin{vmatrix} 1 & 3 & 8 \\ 2 & 6 & -1 \\ 3 & 9 & 7 \end{vmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \begin{vmatrix} 1 & 3 & 8 \\ 2 & 6 & -1 \\ 1 & 3 & 8 \end{vmatrix}$$

$$= 0$$

$$|A_3| = \begin{vmatrix} 4 & 2 & 6 \\ 10 & 3 & 9 \\ 16 & 4 & 12 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 3 & 9 \\ 4 & 12 \end{vmatrix} - 2 \begin{vmatrix} 10 & 9 \\ 16 & 12 \end{vmatrix} + 6 \begin{vmatrix} 10 & 3 \\ 16 & 4 \end{vmatrix}$$

$$= 4(36-36) - 2(126-144) + 6(40-48)$$

$$= 0 - 2(-24) + 6(-8)$$

$$= 48 - 48$$

$$= 0$$

$$|A_4| = \begin{vmatrix} 2 & 6 & -1 \\ 3 & 9 & 7 \\ 4 & 12 & 15 \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - R_2$$

$$R_3 \rightarrow R_3 - R_2$$

$$R_2 \rightarrow R_3 - R_1$$

$$= \begin{vmatrix} 2 & 6 & -1 \\ 3 & 9 & 7 \\ 1 & 3 & 8 \end{vmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \begin{vmatrix} 2 & 6 & -1 \\ 1 & 3 & 8 \\ 1 & 3 & 8 \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$= 0$$

The rank is not 3.

Let us take minors of 2×2 matrix (submatrix of 2)

$$B_1 = \begin{bmatrix} 6 & 1 \\ 4 & 2 \end{bmatrix}$$

$$|B_1| = \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= 6 \times 2 - 1 \times 4 = 12 - 4$$

$$= 8 \neq 0.$$

$\therefore$ The rank of the given Matrix is 2.

Row Reduced Echelon Matrix :-

→ A matrix of order $m \times n$ is called a row reduced Echelon Matrix if the following conditions satisfied;

- (i) The first non-zero elements in each non-zero row is unity (1) which is called the leading entry of the row.
- (ii) All the non-zero rows if any precede the zero rows.
- (iii) The number of zeros preceding the first non-zero element in a row is less than the no of such zeros in the succeeding row.

Important Results :-

→ The rank of a matrix in echelon form is equal to the number of non-zero rows of the Matrix.

EX :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Rank of $A = 2$

• Vector spaces:-

• Definition:-

→ A non-empty set V is said to be a vector space over a field F . If in V there are defined a binary operating addition and scalar multiplication, such that the following axioms are satisfied.

(i) closure axiom for addition:-

$$u, v \in V \Rightarrow (u+v) \in V \quad \text{for real, } 2 \in \mathbb{R}, 3 \in \mathbb{R} \Rightarrow 2+3=5 \in \mathbb{R}.$$

(ii) commutative axiom for addition:-

$$u, v \in V \Rightarrow u+v = v+u \quad 2+3 = 3+2 = 5$$

(iii) Associative axiom for addition:-

$$u, v, w \in V \Rightarrow u+(v+w) = (u+v)+w$$

$$2+(3+5) = (2+3)+5$$

(iv) Existence of additive identity element in V :-

→ There exists an element 0 in V such that

$$\boxed{v+0 = 0+v = v} \quad \text{for all } v \in V$$

$$5+0 = 0+5 = 5$$

↓
additive
Identity.

(v) Existence of additive Inverse element in V :-

→ For every v in V , there exists $-v$ in V .

such that $v+(-v) = (-v)+v = 0$

$$5+(-5) = (-5)+5 = 0$$

↓
-5 is the additive inverse of 5.

• Multiplication:-

(vi) closure axiom for scalar Multiplication

$$u \in V, \alpha \in F \Rightarrow \alpha u \in V$$

↓
Scalar

(vii) Scalar Multiplication is distributive over vector addition :-
 $u, v \in V, \alpha \in F \Rightarrow \alpha(u+v) = \alpha u + \alpha v$.

(viii) Scalar Multiplication is distributive over scalar addition :-
 $u \in V, \alpha, \beta \in F \Rightarrow (\alpha + \beta)u = \alpha u + \beta u$

(Scalar Multiplication is distributive over scalar addition)

(ix) $\alpha, \beta \in F$ and $u \in V \Rightarrow \alpha(\beta u) = (\alpha\beta)u$
 $= (\alpha u)\beta$
 $= \beta(\alpha u)$

(x) $u \in V$ and $1 \in F \Rightarrow 1 \cdot u = u$

• REMARK :-

→ Vector space is an abelian group under addition.

↓
 (Commutative property holds.)

→ The field F may be the set of real numbers or the set of complex numbers.

• Real vector space :-

→ If the field F is the set of real numbers, then the vector space is known as real vector space.

• complex vector space :-

→ If the field F is the set of complex numbers, then the vector space is known as complex vector space.

→ EX :- Let R^+ be the set of all positive real numbers, Define the operations addition and scalar Multiplication as follows:-

(i) $u+v = u \cdot v \quad \forall u, v \in R^+$

(ii) $\alpha u = u\alpha \quad \forall u \in R^+$ and real scalar α .

(iii) Prove that R^+ is a real vector space.

→ Given that R^+ is the set of all positive real numbers.
 → Addition and scalar Multiplication on R^+ are defined as follows:-

$$u+v = u \cdot v \text{ for all } u, v \in R^+$$

$$\alpha u = u \alpha \text{ for all } u \in R^+ \text{ and real scalar } \alpha.$$

→ To prove that R^+ is a real vector space i.e. to prove that for all $u, v, w \in R^+$ and α, β scalars

(i) $u+v \in R^+$

(ii) $u+v = v+u$

(iii) $u+(v+w) = (u+v)+w$

(iv) $\exists$ an element 1 in R^+ such that

$$u+1 = 1+u = u \quad \forall u \in R^+.$$

(v) for every u in R^+ , $\exists \frac{1}{u}$ in R^+ such that

$$\boxed{u + \frac{1}{u} = \frac{1}{u} + u = 1}$$

(vi) $\alpha u \in R^+$

(vii) $(\alpha + \beta)u = \alpha u + \beta u$

(viii) $\alpha(u+v) = \alpha u + \alpha v$

(ix) $1u = u$

(x) $u+v = u \cdot v \in R^+$

(2) $u+v = u \cdot v$ ($u \cdot v = v \cdot u$ commut)

$$= v \cdot u$$

(3) $u+(v+w) = u+(v \cdot w)$ (Given that, $u+v = u \cdot v + u, \forall u, v \in R^+$).

$$= u \cdot (v \cdot w)$$

$$= (u \cdot v) \cdot w$$

$$= (u+v) \cdot w$$

$$= (u+w) + w$$

($\because R^+$ is an associative under product)

(4) clearly $1 \in R^+$.

$$\boxed{u+1 = u \cdot 1 = u}$$

$$\boxed{1+u = 1 \cdot u = u}$$

$$\Rightarrow \boxed{u+1 = 1+u = u}$$

$\forall u \in R^+$ (proved)

So, $\exists 1$ in R^+ such that

$$u+1 = 1+u = u$$

(v) let $u \in R^+$

$$\Rightarrow \frac{1}{u} \in R^+$$

$$u + \frac{1}{u} = u \cdot \frac{1}{u} = 1$$

$$\frac{1}{u} + u = \frac{1}{u} \cdot u = 1$$

$$\Rightarrow u + \frac{1}{u} = \frac{1}{u} + u = 1$$

So for every u in R^+ $\frac{1}{u}$ is in R^+ such that

$$u + \frac{1}{u} = \frac{1}{u} + u = 1 \quad (\text{proved})$$

(vi) Given that $\alpha u = u \alpha \in R^+$

$$(vii) (\alpha + \beta)u = u(\alpha + \beta)$$

$$(u\alpha = \alpha u)$$

↓

$$\text{Scalar} = u\alpha \cdot u\beta$$

$$a^m \cdot a^n = a^{m+n}$$

$$= \alpha u \cdot \beta u$$

$$(u+v = u \cdot v)$$

$$= \alpha u + \beta u$$

$$\Rightarrow (u \cdot v = u + v)$$

$$\Rightarrow (\alpha + \beta)u = \alpha u + \beta u \quad (\text{proved}).$$

$$(viii) LHS = \alpha(u+v) = \alpha(u \cdot v) \quad (\because u+v = u \cdot v)$$

$$= (u \cdot v)\alpha$$

$$\alpha u = u \alpha$$

$$= u\alpha \cdot v\alpha$$

$$(ab)^n = a^n \cdot b^n$$

$$= \alpha u \cdot \alpha v$$

$$(u \cdot v = u + v)$$

$$= \alpha u + \alpha v$$

$$= RHS \quad (\text{proved})$$

$$(ix) \alpha \cdot (u\beta) = \alpha(u\beta)$$

$$= (u\beta)\alpha$$

$$= u\alpha\beta$$

$$= (\alpha\beta)u \quad (\text{proved})$$

(X) $1 \cdot u = u1 = u$ (proved) ($\alpha u = u\alpha$)
Hence R^+ is a Real Vector space (proved)

Q Give Examples of one and 2 dimensional subspace of R^3 ?
Ans The elements of R^3 are $[v_1, v_2, v_3]$ thus its substance is also space. since we have to get a one-dimensional and two-dimensional subspace. so let us one and two free variables. so, $[0 \ 0 \ v_3]$ and $[0 \ v_2 \ v_3]$ are the elements in one and 2, dimensional subspace.

→ so, projection of R^3 on a Axis gives one dimensional subspace. and projection of R^3 on a plane gives a two dimensional subspace.

MODULE :- 02

EIGEN VALUES AND EIGEN VECTORS

(38)

→ consider the linear transformation $Y = AX$ which transform the column vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ into column vector $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ where A , is the transformation Matrix, which is given by,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

→ To consider a particular transformation,

$$Y = AX \rightarrow Y = \lambda X$$

→ Where λ is scalar under a vector X is transformed into a vector Y . This implies that,

$$\begin{aligned} Y &= AX \\ \Rightarrow A &= \lambda X \end{aligned}$$

$$\Rightarrow AX = \lambda X$$

$$\text{or } AX - \lambda X = 0$$

$$\text{or } AX - (\lambda I)X = 0 \quad (\because \lambda = \lambda I)$$

$$\Rightarrow (A - \lambda I)X = 0 \quad \text{--- (i)}$$

Where I is an identity Matrix of order n , which represents the following system of Homogeneous equations in $x_1, x_2, \dots, x_n$.

$$\begin{cases} (a_{11} - \lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn} - \lambda)x_n = 0 \end{cases}$$

→ This system of equations has trivial solution $X = 0$ for any λ but we are interested only in the non-trivial solutions of

② which is possible only when .

$$\boxed{|A - \lambda I| = 0} \quad \text{--- (3)}$$

characteristic polynomial

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - \lambda \end{bmatrix} \quad \text{--- (4)}$$

Taking determinant of $A - \lambda I$,

$$\therefore |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - \lambda \end{vmatrix} \quad \text{--- (5)}$$

from (5) and (3), we have,

$$\text{from (3)} \Rightarrow |A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - \lambda \end{vmatrix} = 0 \quad \text{--- (6)}$$

→ This is called the characteristic eqn of the square matrix A

• NOTE:- The roots $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ of the eqn $P_n(\lambda) = 0$ are called the eigen values or characteristic values or roots or latent roots of the Matrix A .

→ The roots can be single or repeated real or complex.

(a) Characteristic Roots or Eigen values:-

(a) For a given square Matrix A , $(A - \lambda I)$ Matrix is called the characteristic Matrix, where λ is scalar and I is the unit Matrix.

EX:- Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}_{3 \times 3}$.

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \quad \text{--- (1)}$$

[Characteristic Matrix]

(b) Characteristic polynomial:-

→ The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of Matrix A .

EX:- For Example from (1)

Let, $A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} = \text{characteristic Matrix.}$

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

Expanding along column ① we have,

$$= (2-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 2 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 3-\lambda & 1 \end{vmatrix}$$

$$\Rightarrow A - \lambda I = (2-\lambda) [(3-\lambda)(2-\lambda) - 2 \times 1] - [2(2-\lambda) - 2 \times 1] + [(2 \times 1) - (3-\lambda)1]$$

$$= (2-\lambda) [6 - 3\lambda - 2\lambda + \lambda^2 - 2] - [4 - 2\lambda - 2] + [2 - 3 + \lambda]$$

$$= (2-\lambda) [\lambda^2 - 5\lambda + 4] - [2 - 2\lambda] + [-1 + \lambda]$$

$$= 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 2 + 2\lambda - 1 + \lambda$$

$\Rightarrow |A - \lambda I| = -\lambda^3 + 7\lambda^2 - 11\lambda + 5$ which is a characteristic polynomial of the Matrix A.

(c) characteristic equation :-

$\rightarrow$ The eqn $|A - \lambda I| = 0$ is called the characteristic eqn of the Matrix A.

Ex :- $|A - \lambda I| = 0$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0 \quad \text{--- (i)}$$

which is the characteristic equation,

(d) characteristics roots or eigen values :-

$\rightarrow$ The roots of characteristics eqn $|A - \lambda I| = 0$ are called characteristic roots of Matrix A

Example :- from (i)

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

using Trial Method,

$$\lambda = 1, 1^3 - 7 \times 1^2 + 11 \times 1 - 5$$

$$= 1 - 7 + 11 - 5$$

$$= -6 + 6$$

$$= 0$$

$$ax^2 + bx + c \geq 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x^2 + (\alpha + \beta)x + \alpha\beta \geq 0$$

$\therefore (\lambda-1)$ is one factor of $\lambda^3 - 7\lambda^2 + 11\lambda - 5$

$$\begin{array}{r}
 \lambda-1 \left| \begin{array}{r} \lambda^3 - 7\lambda^2 + 11\lambda - 5 \\ \lambda^3 - \lambda^2 \\ \hline -6\lambda^2 + 11\lambda - 5 \\ -6\lambda^2 + 6\lambda \\ \hline 5\lambda - 5 \\ 5\lambda - 5 \\ \hline 0 \end{array} \right| \begin{array}{r} \lambda^2 - 6\lambda + 5 \end{array}
 \end{array}$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 - 6\lambda + 5) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 - \lambda - 5\lambda + 5) = 0$$

$$\Rightarrow (\lambda-1)[\lambda(\lambda-1) - 5(\lambda-1)] = 0$$

$$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-5) = 0$$

$$\Rightarrow \lambda-1=0 \text{ or } \lambda-1=0 \text{ or } \lambda-5=0$$

$$\Rightarrow \boxed{\lambda=1} \text{ or } \boxed{\lambda=1} \text{ or } \boxed{\lambda=5}$$

$$\therefore \boxed{\lambda = 1, 1, 5}$$

→ The required eigen values or characteristic roots are 1, 1, 5 (Ans)

• Some Important properties of eigen values :-

(1) Any square Matrix A and its transpose A have the same eigen values.

• REMARK :- Trace of Matrix :- $\boxed{\lambda^2 - \text{Tr} A \lambda + |A| = 0}$

→ The sum of the elements on the principal diagonal of a Matrix is called the trace of the Matrix.

(2) The sum of the eigen values of a Matrix is equal to the trace of the Matrix i.e. $\lambda_1 + \lambda_2 + \dots + \lambda_n = \text{Trace of Matrix}$

(3) The product of the eigen values of a Matrix A is equal to the determinant of A. i.e. $\boxed{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n = |A|}$ det A

(4) If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are the eigen values of A , then the eigen values of

(i) kA are $k\lambda_1, k\lambda_2, \dots, k\lambda_n$

(ii) A^m are $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$

(iii) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \dots, \frac{1}{\lambda_n}$

(5) Find the characteristic roots of the Matrix?

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Ans Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic Matrix of the given Matrix is,

$$A - \lambda I = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$$

The characteristic eqn is :-

$$\therefore |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda) \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} - (-2) \begin{vmatrix} -2 & -1 \\ 2 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & 3-\lambda \\ 2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)[(3-\lambda)(3-\lambda) - (-1) \times (-1)] + 2[-2(3-\lambda) - 2(-1)] + 2[-2 \times -1 - 2(3-\lambda)] = 0$$

$$\Rightarrow (6-\lambda)[9-6\lambda+\lambda^2-1] + 2[-6+2\lambda+2] + 2[2-6+2\lambda] = 0$$

$$\Rightarrow 54 - 36\lambda + 6\lambda^2 - 6 - 9\lambda + 6\lambda^2 - \lambda^3 + \lambda - 8 + 4\lambda + 4\lambda - 8 = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0 \quad \text{--- (1)}$$

By trial,

$$\lambda = 1, -1 + 12 - 36 + 32 \neq 0$$

$$\lambda = -1, -1 + 12 + 36 + 32 \neq 0$$

$$\lambda = 2, -8 + 48 - 72 + 32 = 0$$

$$\lambda - 2 \mid \frac{-\lambda^3 + 12\lambda^2 - 36\lambda + 32}{0} \quad -\lambda^2 + 10\lambda - 16 \quad (\text{Quotient})$$

Putting 4 in (1) we have from (1)

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$(\lambda - 2)[- \lambda^2 + 10\lambda - 16] = 0$$

$$\Rightarrow -[\lambda - 2][\lambda^2 - 10\lambda + 16] = 0$$

$$\Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0 \quad \text{--- (ii)}$$

$$\Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda^2 - 2\lambda - 8\lambda + 16) = 0$$

$$\Rightarrow (\lambda - 2)[\lambda(\lambda - 2) - 8(\lambda - 2)] = 0$$

$$\Rightarrow (\lambda - 2)[(\lambda - 2)(\lambda - 8)] = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0 \quad \text{--- (iii)}$$

$$\Rightarrow \lambda - 2 = 0, \lambda - 2 = 0, \lambda - 8 = 0$$

$$\Rightarrow \lambda = 2, \lambda = 2, \lambda = 8$$

$\Rightarrow \lambda = 2, 2, 8$ are the characteristic roots or Eigen values.

(Q) The matrix A is defined as $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$
Find the eigen values of $3A^2 + 5A^2 - 6A + 2I$?

Ans

Given that,

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The characteristic eqn of the Matrix A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

Expanding along c,

$$\Rightarrow (1-\lambda) \begin{vmatrix} 3-\lambda & 2 \\ 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) [(3-\lambda)(-2-\lambda) - 0 \times 2] = 0$$

$$\Rightarrow (1-\lambda) [-6 - 3\lambda + 2\lambda + \lambda^2 - 0] = 0$$

$$\Rightarrow (1-\lambda) [\lambda^2 - \lambda - 6] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 3\lambda + 2\lambda - 6) = 0$$

$$\Rightarrow (1-\lambda)[\lambda(\lambda-3) + 2(\lambda-3)] = 0$$

$$\Rightarrow (1-\lambda)(\lambda-3)(\lambda+2) = 0$$

$$\Rightarrow 1-\lambda=0, \lambda-3=0, \lambda+2=0$$

$$\Rightarrow \lambda=1, \lambda=3, \lambda=-2$$

$$\Rightarrow \boxed{\lambda = 1, 3, -2}$$

$$\downarrow \downarrow \downarrow$$

$$\lambda_1 \lambda_2 \lambda_3$$

are the characteristic roots or eigen values of matrix A.

Eigen values of $A^3 = \lambda_1^3, \lambda_2^3, \lambda_3^3$ (using E.V A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m$)
 $= 1^3, 3^3, (-2)^3 = 1, 27, -8$

$$\text{Eigen values of } A^2 = 1^2, 3^2, (-2)^2 \\ = 1, 9, 4$$

$$\text{Eigen values of } A = 1, 3, -2$$

$$\text{Eigen values of } I = 1, 1, 1$$

$$\text{Eigen values of } 3A^3 = 3 \times 1, 3 \times 27, 3 \times -8 \\ = 3, 81, -24$$

$$\text{Eigen values of } 5A^2 = 5 \times 1, 5 \times 9, 5 \times 4$$

$$\text{Eigen values of } -6A = -6 \times 1, -6 \times 3, -6 \times 2$$

$$\text{Eigen values of } 2I = 2 \times 1, 2 \times 1, 2 \times 1$$

$$\text{Eigen values of } 3A^3 + 5A^2 - 6A + 2I = 0$$

$$\text{First Eigen values of this is, } 3(1) + 5(1) - 6(1) + 2 = 4$$

$$\text{Second Eigen values of this, } 3(27) + 5(9) - 6 \times 3 + 2 \times 1 = 110$$

$$\text{Third Eigen value, } 3(-8) + 5 \times 4 + (-6) \times (-2) + 2 \times 1 = 10$$

$\therefore$ The required eigen values are 4, 110, 10 (Ans)

(Q) find all the eigen values and eigen vectors of the Matrix,

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Ans Characteristic eqn of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda) \begin{vmatrix} 1-\lambda & -6 \\ -2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & -6 \\ -1 & -\lambda \end{vmatrix} - 3 \begin{vmatrix} 2 & 1-\lambda \\ -1 & -2 \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda) [(1-\lambda)(-\lambda) - (-2)(-6)] - 2 [2 \times (-\lambda) - (-1)(-6)] - 3 [2 \times (-2) - (-1)(1-\lambda)] = 0$$

$$\Rightarrow (-2-\lambda) [-\lambda + \lambda^2 - 12] - 2 [-2\lambda - 6] - 3 [-4 + (1-\lambda)] = 0$$

$$\Rightarrow 2\lambda - 2\lambda^2 + 24 - \lambda^2 - \lambda^3 + 12\lambda - 4\lambda + 12 + 9 + 12 + 9 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\lambda = 1, 1 + 1 - 21 - 45 \neq 0$$

$$\lambda = 2, 8 + 4 - 42 - 45 \neq 0$$

$$\lambda = 3, 27 + 9 - 63 - 45 \neq 0$$

$$\lambda = -3, -27 + 9 + 63 - 45 = 0$$

$$\lambda - (-3) = 0$$

$$\Rightarrow \lambda + 3 = 0$$

$\lambda - (-3)$ is a factor of the.

$$\begin{array}{r|l} \lambda + 3 & \lambda^3 + \lambda^2 - 21\lambda - 45 \\ & \underline{\lambda^3 + 3\lambda^2} \\ & -2\lambda^2 - 21\lambda - 45 \\ & \underline{-2\lambda^2 - 6\lambda} \\ & -15\lambda - 45 \\ & \underline{-15\lambda - 45} \\ & 0 \end{array}$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\Rightarrow (\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow \lambda + 3 = 0$$

$$\Rightarrow \boxed{\lambda = -3}$$

OR $\lambda^2 - 2\lambda - 15 = 0$

$$\Rightarrow \lambda^2 - 5\lambda + 3\lambda - 15 = 0$$

$$\Rightarrow \lambda(\lambda - 5) + 3(\lambda - 5) = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 3) = 0$$

$$\Rightarrow \lambda - 5 = 0 \text{ or } \lambda + 3 = 0$$

$$\Rightarrow \lambda = 5 \text{ or } \lambda = -3$$

The roots of λ are $-3, -3, 5$.

let us take, $\lambda_1 = -3, \lambda_2 = -3, \lambda_3 = 5$.

eigen values are $-3, -3, 5$

When $\lambda = -3$ — (i) To find out Eigen Vectors.

$$[A - \lambda I] \lambda = 0.$$

$$\Rightarrow \begin{bmatrix} -2-\lambda & 2-3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ — (ii)}$$

By putting $\lambda = -3$

$$\Rightarrow \begin{bmatrix} 2+3 & 2-3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ — (iii)}$$

$$\sim \boxed{x_1 + 2x_2 - 3x_3 = 0} \text{ — (iv)}$$

Let $x_3 = 0$, $x_2 = -1$ put them in Eqn (iv)

$$x_1 + 2x(-1) - 3(0) = 0$$

$$\Rightarrow x_1 - 2 - 0 = 0$$

$$\Rightarrow x_1 = 2$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Let $x_2 = 0$, $x_3 = 1$ put them in Eqn (iv)

We have,

$$x_1 + 2x_2 - 3x_3 = 0$$

$$\Rightarrow x_1 + 2 \times 0 - 3 \times 1 = 0$$

$$\Rightarrow x_1 - 3 = 0$$

$$\Rightarrow x_1 = 3$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda = 5$

Putting $\lambda = 5$ in eqn(2)

We have,

$$\sim \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (v)}$$

$$\sim \begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 2R_1$$

$$R_3 \rightarrow R_3 - 7R_1$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 2R_2$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & 16 & 32 \\ 0 & -8 & -16 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & 16 & 32 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow -x_1 - 2x_2 - 5x_3 = 0$$

$$\Rightarrow 16x_2 + 32x_3 = 0$$

Let $x_3 = kx$, then

$$\boxed{x_2 = -2kx}$$

$$16x_2 + 32x_3 = 0$$

$$\Rightarrow 16x_2 + 32kx = 0$$

$$\Rightarrow 16x_2 = -32kx$$

$$\Rightarrow x_2 = -2kx$$

Putting values of $x_2 = -2kx$
and $x_3 = kx$

$$-x_1 - 2x_2 - 5x_3 = 0$$

$$\Rightarrow x_1 + 2x_2 + 5x_3 = 0$$

$$\Rightarrow x_1 + 2x(-2kx) + 5kx = 0$$

$$\Rightarrow x_1 - 4kx + 5kx = 0$$

$$\Rightarrow x_1 + kx = 0$$

$$\Rightarrow x_1 = -kx$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -kx \\ -2kx \\ kx \end{bmatrix} = -kx \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

- Symmetric, skew symmetric and orthogonal Matrix :-
- Symmetric Matrix :- $[a_{ij} = a_{ji}]$ For all i & j using the transpose for We have if A is any square matrix, then $a_{ij} = a_{ji}$ gives $[A = A^T]$

$\Rightarrow$ The matrix is symmetric if it is identical with transpose.

Ex :- $A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}_{3 \times 3}$

$\longrightarrow$ The given Matrix is not symmetric.

$$A^T = \begin{bmatrix} a & x & p \\ b & y & q \\ c & z & r \end{bmatrix}$$

$$A \neq A^T$$

Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

$$A^T = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

$\therefore A$ is said to be a symmetric Matrix.

- Skew-symmetric Matrix :-

$\rightarrow$ A square Matrix is said to be a skew symmetric Matrix if $A^T = -A$ or $A = -A^T$ i.e. $[a_{ij} = -a_{ji}]$

$\rightarrow$ The principal diagonal elements all 0's.

Ex :- $A = \begin{bmatrix} 0 & -p & -q \\ p & 0 & -r \\ q & r & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix}$$

$$= - \begin{bmatrix} 0 & -p & -q \\ p & 0 & -r \\ q & r & 0 \end{bmatrix}$$

$$= -A$$

$$\Rightarrow A^T = -A$$

- Every square Matrix is the sum of symmetric and a skew symmetric Matrix.
- If A is a square Matrix A^T is its transpose Matrix form, then $(A + A^T)$ is a symmetric Matrix.

• To prove $(A + A^T)$ is a symmetric Matrix :-

$$(A + A^T)^T = A^T + (A^T)^T$$

$$= A^T + A$$

$$= A + A^T$$

$$(\because (A+B)^T = A^T + B^T)$$

commutative

$$(\because A+B = B+A)$$

$$\boxed{(A + A^T)^T = A + A^T}$$

$\therefore (A + A^T)$ is a symmetric Matrix.

• To prove $(A - A^T)$ is a skew symmetric Matrix :-

$$\rightarrow (A - A^T)^T = A^T - (A^T)^T$$

$$= A^T - A$$

$$= -(A - A^T)$$

$$\Rightarrow \boxed{D^T = -D} \text{ Where } D = A - A^T$$

Where, $D = A - A^T$ is a skew symmetric Matrix.

→ A square Matrix A is the sum of a symmetric and skew symmetric matrix.

Ans Let A is a square Matrix.

$$A = \left(\frac{1}{2} \times 2\right) A$$

$$= \frac{1}{2} (2A)$$

$$= \frac{1}{2} (A + A)$$

$$= \frac{1}{2} [(A + A^T) + (A - A^T)]$$

$$A = \frac{1}{2} [(A + A^T) + (A - A^T)]$$

square Matrix

$$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= \left(\frac{A + A^T}{2} \right) + \left(\frac{A - A^T}{2} \right)$$



Symmetric
Matrix



Skew symmetric
Matrix

∴ A square Matrix is the sum of a symmetric Matrix and Skew symmetric Matrix (proved).

EX:- Expression $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 2 & -1 \end{bmatrix}$ as sum of a symmetric and skew symmetric Matrix.

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & -1 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 2 & 5 & 4 \\ 5 & 4 & 6 \\ 4 & 6 & -2 \end{bmatrix}$$

$$\frac{1}{2}(A + A^T) = \frac{1}{2} \begin{bmatrix} 2 & 5 & 4 \\ 5 & 4 & 6 \\ 4 & 6 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{5}{2} & 2 \\ \frac{5}{2} & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$A - A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix}$$

$$\frac{1}{2} (A - A^T) = \begin{bmatrix} 0 & -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

So, the split is $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \frac{1}{2} (A + A^T) + \frac{1}{2} (A - A^T)$$

• Orthogonal Matrix :-

→ The matrix is said to be an orthogonal Matrix if,

$$A A^T = I$$

$$\Rightarrow A^T = A^{-1}$$

• Remark :-

→ The unit matrix is both orthogonal and symmetric Matrix.

• Orthogonal Matrix :-

→ If x and y are a system of unknowns and $y = Ax$ where A is square Matrix, then it is called transformation.

→ i.e. y is the transformation of x , when this A is an orthogonal Matrix, the transformation is called as orthogonal transformation.

• Complex Matrices :-

• Hermitian and skew-Hermitian Matrices and Unitary Matrices and similarity of Matrices :-

$$z = x + iy, \quad x, y \in \mathbb{R}.$$

Complex number

Real part of z

$$\operatorname{Re}(z) = x$$

imaginary part of z

$$\operatorname{Im}(z) = y$$

- Conjugate of a complex number :-
If $x = x + iy$, then its conjugate is represented as.
 $\bar{x} = \overline{x + iy} = x - iy$

- Conjugate of a Matrix :-

→ Let $A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$

Elements	conjugate elements
$1+i$	$\longrightarrow 1-i$
$2-3i$	$\longrightarrow 2+3i$
4	$\longrightarrow 4$
$7+2i$	$\longrightarrow 7-2i$
$-i$	$\longrightarrow +i$
$3-2i$	$\longrightarrow 3+2i$

Conjugate of Matrix $A = \bar{A}$

$$\Rightarrow \bar{A} = \begin{bmatrix} \overline{1+i} & \overline{2-3i} & \overline{4} \\ \overline{7+2i} & \overline{-i} & \overline{3-2i} \end{bmatrix}$$

$$\Rightarrow \bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

- Matrix A^* :- Transpose of the conjugate of a Matrix A is denoted by A^*

$$\text{Let } A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}_{3 \times 3}$$

$$A^* = (\bar{A})^T = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}_{3 \times 2}$$

• Unitary Matrix :-

→ A square Matrix A is said to be unitary matrix if

$$\boxed{A^* A = I}$$

Ex :- $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ — (i)

Hence, $\bar{A} = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

$$\Rightarrow \bar{A} = \begin{bmatrix} \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$\Rightarrow A^* = (\bar{A})^T = \begin{bmatrix} \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix} \text{ — (ii)}$$

$$\therefore A^* A = \begin{bmatrix} \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1^2 - i^2}{4} + \frac{1^2 - i^2}{4} & \frac{(-1-i)(-1+i)}{4} + \frac{(-1-i)(1-i)}{4} \\ \frac{(-1-i)(1+i)}{4} + \frac{(1+i)^2}{4} & \frac{(-1)^2 - (i)^2}{4} + \frac{1^2 - i^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1(-1) + 1(-1)}{4} & \frac{(1-i) - i - 1}{4} + \frac{-1+i-i+1}{4} \\ \frac{-1-i-i(-1)}{4} + \frac{1-1+2i}{4} & \frac{1-(-1)}{4} + \frac{1-(-1)}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+1+1+1}{4} & \frac{1-i+i-1-1+i-i+1}{4} \\ \frac{-1-2i+1+2i}{4} & \frac{1+1+1+1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{4}{4} \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence A is an unitary Matrix.

• Hermitian Matrix :-

→ A square Matrix $A = (a_{ij})$ is called Hermitian Matrix if every i - j th element of A is equal to conjugate complex i - j th element of A .

→ In other words, $\boxed{a_{ij} = \overline{a_{ji}}}$

$$\text{Ex :- } A = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 5-i & 4+2i & 5 \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

• REMARK :-

→ Necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^*$ i.e. conjugate transpose of A .

$$\therefore \boxed{A = (A)^T = A^*}$$

• Skew Hermitian Matrix :-

→ A square Matrix $A = (a_{ij})$ will be called a square Hermitian Matrix if every i - j th element of A is = to -ve conjugate complex of i - j th element of A .

→ In other words, $\boxed{a_{ij} = -\overline{a_{ji}}}$

→ All the elements in the principal diagonal will be of the form $\boxed{a_{ij} + \overline{a_{ji}} = 0}$ — (i)

if $a_{ij} = a + ib$ then $\overline{a_{ji}} = a - ib$

$$\rightarrow \therefore a + ib + a - ib = 0$$

$$\Rightarrow 2a = 0$$

$$\Rightarrow a = 0$$

So a_{ij} is pure imaginary or $a_{ij} = 0$

• REMARK :-

- (1) Hence all the diagonal elements of a skew Hermitian matrix are either zero or purely imaginary.

EX :-

$$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(-4+5i) & -(-2i) & 3i \end{bmatrix} \rightarrow \text{principal diagonal elements}$$

- (2) The necessary and sufficient condition for a Matrix A to be skew Hermitian is that,

$$\boxed{A^* = -A} \quad \text{or} \quad \boxed{(\bar{A})^T = -A}$$

(Q) Express the Matrix A

$$A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix} \text{ as the sum of a Hermitian and a skew Hermitian Matrix}$$

Ans

$$\bar{A} = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix}$$

$$A^* = (\bar{A})^T = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}$$

$$\text{Now, } \frac{A+A^*}{2} = \frac{1}{2} \begin{bmatrix} 1+i+1-i & 2-2i & 5-5i-1-i \\ 2i+2 & 2+i+2-i & 4+2i-4 \\ -1+i+5+5i & -4+4-2i & 7+7 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 2(1-i) & 4-6i \\ 2(1+i) & 4 & 2i \\ 2+3i & -2i & 14 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 2(1-i) & 4-6i \\ 2(1+i) & 4 & 2i \\ 4+6i & -2i & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix} = P \text{ (say)}$$

which is Hermitian.

$$\rightarrow \text{Similarly, } \frac{A-A^*}{2} = \begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix} = Q \text{ (say)}$$

which is a skew Hermitian Matrix

$$\text{Therefore, } A = \frac{A+A^*}{2} + \frac{A-A^*}{2}$$

$$= P + Q$$

Diagonalising a square matrix :-

- A square matrix of order $n \times n$ is called diagonalising matrix if a matrix 'P' of order $n \times n$ such that $P^{-1}AP$ is a diagonal matrix.
- When such a matrix P exists, we say that P diagonalises A.

Theorem :- For any square matrix A of order (n) having linear independent eigen values of a matrix P such that $P^{-1}AP$ is a diagonal Matrix.

Proof :-

→ We choose a ^{square} 3×3 matrix.

Let us take its eigen values be $\lambda_1, \lambda_2, \lambda_3$ and the corresponding eigen vectors be x_1, x_2 and x_3 respectively

i.e. $x_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, x_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, x_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$

so that,

$$Ax_1 = \lambda_1 x_1$$

$$Ax_2 = \lambda_2 x_2$$

$$Ax_3 = \lambda_3 x_3$$

Now consider the square Matrix,

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = P \text{ (say)}$$

then we have,

$$\begin{aligned} A \cdot P &= A \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \\ &= \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$= PD$ Where D is the diagonal matrix.

$$\boxed{AP = PD}$$

$$\Rightarrow P^{-1}(AP) = P^{-1}(PD)$$

$$= (P^{-1}P)D \quad (\because AA^{-1} = I)$$

$$= ID$$

$$= D \text{ (proved)}$$

• Modal Matrix :-

→ The matrix P which diagonalizes the given matrix A , is called Modal Matrix.

• Spectral Matrix :-

→ The resulting diagonal matrix D is called spectral Matrix.

• Remark :-

→ The modal matrix P is found by grouping the Eigen vectors of A into square matrix and diagonal matrix has the eigen values of A as its elements.

• Similar matrices :-

→ Let A and B be square matrices of same order, then matrix A is said to be similar to B if $\exists$ an invertible matrix P , such that

$$\boxed{A = P^{-1}BP}$$

$$\Rightarrow PA = (PP^{-1})BP$$

$$\Rightarrow PA = IBP$$

$$\Rightarrow \boxed{PA = BP}$$

Post multiplying by P^{-1} on both sides of the relation,

$$\boxed{PA = BP}$$

$$\Rightarrow (PA)P^{-1} = (BP)P^{-1}$$

$$\Rightarrow PAP^{-1} = B(PP^{-1})$$

$$\Rightarrow PAP^{-1} = BI$$

$$\Rightarrow PAP^{-1} = B$$

Hence, A is similar to B if and only if B is similar to A, such a matrix P is called similarity matrix.

• Remark :-

→ If A is similar to B and B is similar to C, then A is similar to C.

1) let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Find the similarity transformation but diagonalises matrix A.

(Given that,

$$A = \begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Characteristic eqn is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -(2+\lambda) & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(2+\lambda)[- \lambda + \lambda^2 - 12] + 4\lambda + 12 + 12 - 3 + 3\lambda = 0$$

$$\Rightarrow -2(-\lambda + \lambda^2 - 12) - \lambda(-\lambda + \lambda^2 - 12) + 4\lambda + 21 + 3\lambda = 0$$

$$\Rightarrow 2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 21 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\Rightarrow (\lambda - 5)(\lambda^2 + 6\lambda + 9) = 0$$

$$\Rightarrow \lambda(\lambda - 5)(\lambda^2 + 2 \cdot 3\lambda + 3^2) = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 3)^2 = 0$$

$$\Rightarrow \lambda - 5 = 0 \text{ or } (\lambda + 3)(\lambda + 3) = 0$$

$$\Rightarrow \lambda = 5, \lambda + 3 = 0, \lambda + 3 = 0$$

$$\Rightarrow \lambda = 5, \lambda = -3, \lambda = -3$$

$$\therefore \boxed{\lambda = 5, -3, -3}$$

Eigenvalues are $\lambda_1 = 5, \lambda_2 = -3, \lambda_3 = -3$

Eigenvectors for $\lambda = 5$

$$\begin{bmatrix} -2 & -5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 2R_1$$

$$R_3 \rightarrow R_3 - 7R_1$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -5 \\ 2+2(-1) & -4+2(-2) & -6+2(-5) \\ -7-7(-1) & 2-7(-2) & -3-7(-5) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 2R_2$$

P.T.O

$$\Rightarrow \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16+2(-8) & 32+2(-16) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -x - 2y - 5z &= 0 \quad \text{--- (1)} \\ -8y - 16z &= 0 \quad \text{--- (2)} \end{aligned}$$

let us take,

$$\boxed{z = 1}$$

$$-8y - 16z = 0$$

$$\Rightarrow -8y - 16 \times 1 = 0$$

$$\Rightarrow -8y = 16$$

$$\Rightarrow y = -2$$

putting the values of $y = -2$ and $z = 1$ in

$$-x - 2y - 5z = 0$$

$$\Rightarrow -x - 2 \times (-2) - 5 \times 1 = 0$$

$$\Rightarrow -x + 4 - 5 = 0$$

$$\Rightarrow -x - 1 = 0$$

$$\Rightarrow \boxed{x = -1}$$

$\therefore$ Eigen values are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

For $\lambda = -3$

$$[A - \lambda I] [x] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 - (-3) & 2 & -3 \\ 2 & 1 - (-3) & -6 \\ -1 & -2 & -(-3) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 2-2 \times 1 & 4-2 \times 2 & -6-2(-3) \\ -1+1 & -2+2 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{x + 2y - 3z = 0}$$

Case-I :-

let us take $z=0, y=-1, x+2(-1)-3 \times 0=0$

$$\Rightarrow x-2=0$$

$$\Rightarrow x=2$$

$\therefore$ Eigen vectors are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Case-II :-

let us take $z=1, y=0$

$$x+2 \times 0-3 \times 1=0$$

$$\Rightarrow x+0-3=0$$

$$\Rightarrow x=3$$

$\therefore$ Eigen vectors are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

We have got three Eigen vectors i.e.

$(-1, -2, 1)^T, (2, -1, 0)^T, (3, 0, 1)^T$

$$\text{i.e. } \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Three independent Eigen vectors, writing as three column vectors. we get the required transformation matrix.

$$\text{let } P = \begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\text{and find out } P^{-1} = \frac{1}{8} \begin{bmatrix} -1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & 2 & 5 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{8} \left\{ \begin{bmatrix} -1 & -2 & 3 \\ 2 & -4 & -6 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 6 \end{bmatrix} \right\} \begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -5 & -10 & 15 \\ -6 & 12 & 18 \\ -3 & -6 & -15 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 40 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Ans

$$0 = 1 \times 5 - 0 \times 0 + 0 \times 0$$

$$0 = 0 - 3 + 0 \Rightarrow$$

$$\Rightarrow X = 3$$

$$\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

We have got three given vectors:-
 $(-1, -2, 1), (2, -1, 0), (3, 0, 1)$

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

∴ Three independent given vectors, which are three column vectors. We get the reduced form of matrix

$$\text{Let } A = \begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\text{and find out } |A| = \frac{1}{8} \begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & 5 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & -10 & 15 \\ -6 & 12 & 18 \\ -3 & -6 & -15 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right\} \frac{1}{8} = 9A^{-1}b$$

Line Integral :-

→ The line integral of a vector point function along the curve 'C' is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

C = curve

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}, d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

(Q) Let the vector field be $\vec{F} = x^2 \hat{i} - xy \hat{j}$. Find the line integral from the origin $O(0,0)$ to $P(1,1)$ along the straight line OP.

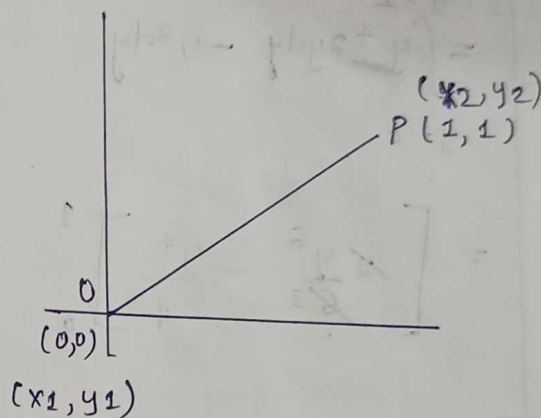
$$\vec{F} = x^2 \hat{i} - xy \hat{j}$$

from $(0,0)$ to $(1,1)$

$$\frac{y - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y = \frac{1}{1} x$$

$$\Rightarrow y = x \quad (C: y = x)$$



$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{y=x} (x^2 \hat{i} - xy \hat{j}) \cdot (dx \hat{i} + dy \hat{j})$$

$$= \int_{y=x} x^2 dx - xy dy$$

$$= \int_{x=0}^{x=1} x^2 dx - x^2 dx = 0$$

(Q) Let the vector field be $\vec{F} = x^2 \hat{i} - xy \hat{j}$. Find the line integral.

from the origin $O(0,0)$ to $P(1,1)$ along the parabola.

$$y^2 = x$$

Ans

$$\vec{F} = x^2 \hat{i} - xy \hat{j}$$

From $(0,0)$ to $(1,1)$

$$y^2 = x$$

$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{y^2=x} (x^2 \hat{i} - xy \hat{j}) \cdot (d\alpha \hat{i} + dy \hat{j})$$

$$= \int x^2 d\alpha - xy dy$$

$$[y=x]$$

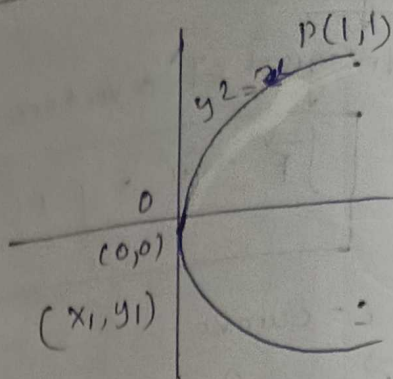
$$y=1$$

$$= \int_{y=0}^1 y^4 2y dy - y^3 dy$$

$$y=0$$

$$= \left[\frac{2y^6}{6} - \frac{y^4}{4} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$



$$x = y^2$$

$$dx = 2y dy$$

(Q)

Let the vector field be $\vec{F} = x^2 \hat{i} - xy \hat{j}$. Find the line integral from the origin $O(0,0)$ to $P(1,1)$ along the x-axis for $(0,0)$ to $(1,0)$ and then $(1,0)$ to $(1,1)$

Ans

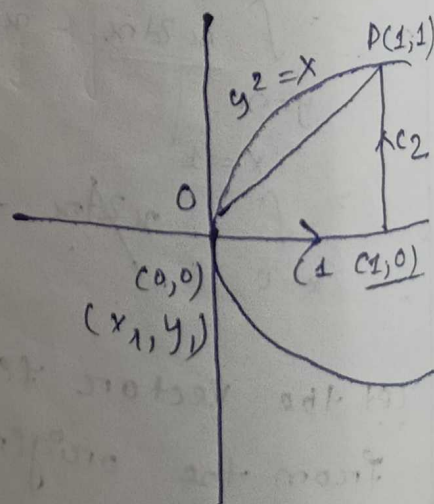
$$\vec{F} = x^2 \hat{i} - xy \hat{j}$$

from $(0,0)$ to $(1,1)$

$$C : C_1 + C_2 : (y=0) + (x=1)$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{y=0} x^2 dx - xy dy \Big|_{y=0}^0 + \int_{x=1} x^2 dx - xy dy$$



$$= \int_0^1 x^2 dx - \int_0^1 y dy.$$

$$x=1 \\ dx=0$$

$$= \left[\frac{x^3}{3} \right]_0^1 - \left[\frac{y^2}{2} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}.$$

(Q) Let the vector field be $\vec{F} = y^2 \hat{i} + 2xy \hat{j}$. Find the line integral from the origin $O(0,0)$ to $P(1,1)$ along the x -axis from $(0,0)$ to $(1,0)$ and then $(1,0)$ to $(1,1)$.

Ans

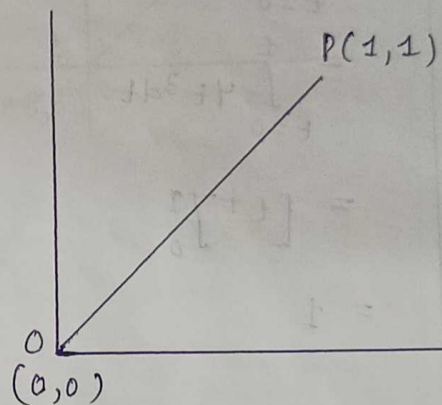
$$\vec{F} = y^2 \hat{i} + 2xy \hat{j}.$$

from $O(0,0)$ to $(1,1)$.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \therefore$$

$$y = \frac{1}{1} x$$

$$\Rightarrow y = x$$



OR in parametric form,

$$x = t, y = t$$

$$dx = dt, dy = dt$$

$$t: 0 \rightarrow 1$$

$$= \int_{t=0}^1 t^2 dt + 2t^2 dt.$$

$$= \int_{t=0}^1 3t^2 dt.$$

$$= \left[3 \times \frac{t^3}{3} \right]_0^1$$

$$= 1$$

(i) C: along the straight line $y = x$.

$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{y=x}^C (y^2 \hat{i} + 2xy \hat{j}) \cdot (dx \hat{i} + dy \hat{j})$$

$$= \int_{y=x} (y^2 dx + 2xy dy)$$

$$= \int_{x=0}^{x=1} x^2 dx + 2x^2 dx.$$

$$= \left[\frac{x^3}{3} + 2 \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{3} + \frac{2}{3}$$

$$= \frac{3}{3} = 1$$

(ii) C : along the parabola $y^2 = x$

$$y = t, x = t^2$$

$$t: 0 \rightarrow 1$$

$$\Rightarrow dy = dt, dx = 2t dt$$

$$\int_{y^2=x} y^2 dx + 2xy dy$$

$$\begin{matrix} y^2 = x \\ x = t^2 \end{matrix}$$

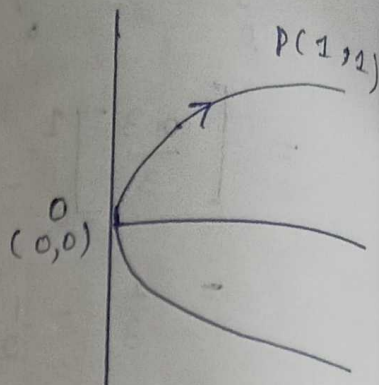
$$= \int_{t=0}^1 2t^2 dt + 2t^2 dt$$

$$= \int_{t=0}^1 2t^3 dt + 2t^3 dt$$

$$= \int_{t=0}^1 4t^3 dt$$

$$= [t^4]_0^1$$

$$= 1$$



(iii)

C : First from O to A then from A to P

$$C = C_1 + C_2$$

C_1 = straight line OA

C_2 = straight line AP .

axis

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

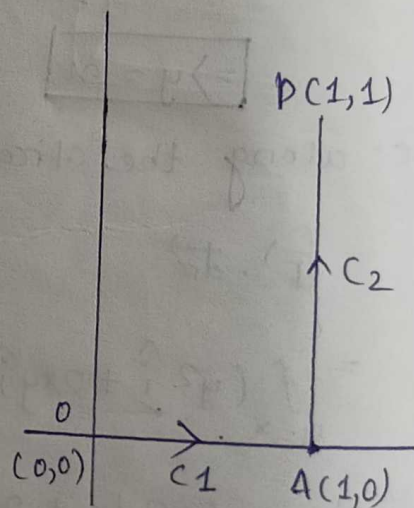
$$= \int_{y=0} y^2 dx + 2xy dy + \int_{x=1} y^2 dx + 2xy dy$$

$$= \int_0^1 0 + 0 + \int_0^1 2y dy$$

$$= \int_0^1 2y dy$$

$$= [y^2]_0^1$$

$$= 1$$



This function is conservative or path independent.

(Q) The value of $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ where $\mathbf{F} = [y^2, -x^2]$ and C be the line segment from $(0,0)$ to $(1,4)$.

Ans C : straight line from $(0,0)$ to $(1,4)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow y = \frac{4}{1}x$$

$$\Rightarrow y = 4x. \quad (x=t)$$

$$x = t, \quad y = 4t, \quad t: 0 \rightarrow 1$$

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

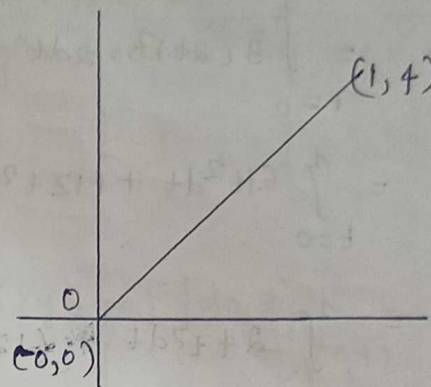
$$= \int_C y^2 dx - x^2 dy$$

$$y = 4x$$

$$= \int_{t=0}^1 16t^2 dt - t^2 4dt$$

$$= \int_{t=0}^1 12t^2 dt$$

$$= \left[4t^3 \right]_0^1 = 4 - 0 = 4.$$



(Q) Calculate the work done in moving a particle in the force field.

$\mathbf{F} = 3x^2 \hat{i} + (2xz - y) \hat{j} + z \hat{k}$ along a straight line from $A(0,0,0)$ to $B(2,1,3)$.

Ans $W = \int_C \mathbf{F} \cdot d\mathbf{r}$

C : straight line from $(0,0,0)$ to $(2,1,3)$

$$\left\{ \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \right\}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$$x = 2t, y = t, z = 3t$$

$$dx = 2dt, dy = dt, dz = 3dt$$

$$W = \int \mathbf{F} \cdot d\mathbf{r}$$

$$= \int [3x^2 dx + (2xz - y)dy + z dz]$$

$$= \int_{t=0}^1 3(2t)^2 \cdot 2dt + (2 \cdot 2t \cdot 3t - t)dt + 3t \cdot 3dt$$

$$= \int_{t=0}^1 24t^2 dt + (12t^2 - t)dt + 9t dt$$

$$= \int_{t=0}^1 24t^2 dt + (12t^2 - t)dt + 9t dt$$

$$= \left[24 \frac{t^3}{3} + 12 \frac{t^3}{3} - \frac{t^2}{2} + \frac{9t^2}{2} \right]_0^1$$

$$= 8 + 4 - \frac{1}{2} + \frac{9}{2}$$

$$= \frac{16 + 8 - 1 + 9}{2}$$

$$= \frac{32}{2}$$

$$= 16$$

Green's Theorem :-

→ If two functions $u(x,y)$ & $v(x,y)$ are single valued and continuous over a region R bounded by the curve C , then.

$$\int_C u \cdot dx + v \cdot dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx \cdot dy$$

(Q) Verify green's theorem in the xy -plane for $\oint_C (xy + y^2) dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Ans We have to show,

$$\oint_C (xy + y^2) dx + x^2 dy = \iint_R \left[\frac{\partial}{\partial x} (x^2) - \frac{\partial}{\partial y} (xy + y^2) \right] dx \cdot dy$$

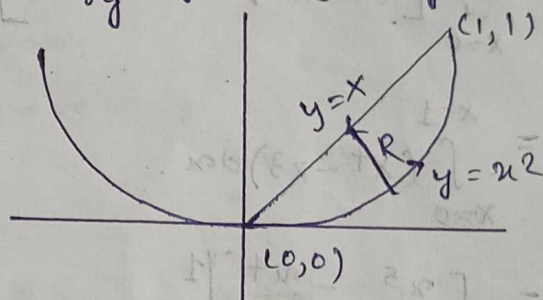
$$y = x^2$$

$$\Rightarrow x^2 = x \Rightarrow x = 0$$

$$\Rightarrow x(x-1) = 0$$

$$\Rightarrow x = 0, 1$$

$$y = 0, 1$$



L.H.S

$$\int_C (xy + y^2) dx + x^2 dy$$

$$= \int_{y=x^2}^{y=x} (xy + y^2) dx + x^2 dy + \int_{y=x}^{y=x^2} (xy + y^2) dx + x^2 dy$$

$$= \int_0^1 (x^3 + x^4) dx + x^2 \cdot 2x dx + \int_1^0 [(x^2 + x^2) dx + x^2] dx$$

$$= \int_0^1 (3x^3 + x^4) dx + \int_1^0 3x^2 dx$$

$$= \left[\frac{3}{4} x^4 + \frac{x^5}{5} \right]_0^1 + [x^3]_1^0$$

$$= \frac{3}{4} + \frac{1}{5} - 1$$

$$= \frac{15 + 4 - 20}{20} = -\frac{1}{20}$$

$$\begin{cases} y = x^2 \\ dy = 2x dx \end{cases}$$

RHS

$$\int_R (2x - x - 2y) dx dy$$

$$= \int_R (x - 2y) dx dy$$

$$= \int_{x=0}^{x=1} \left[\int_{y=x^2}^{y=x} (x - 2y) dy \right] dx$$

$$= \int_{x=0}^{x=1} \left[xy - y^2 \right]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{x=1} [(x^2 - x^2) - (x^3 - x^4)] dx$$

$$= \int_{x=0}^{x=1} (x^4 - x^3) dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$= -\frac{1}{20}$$

(Q)

use green's theorem to find the line integral $\int [\cos y \hat{i} + x(1 - \sin y) \hat{j}] \cdot d\vec{r}$ for the closed curve given by $x^2 + y^2 = 1, z = 0$.

Ans

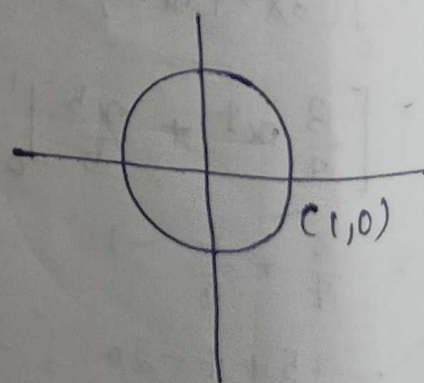
$$\int_C [\cos y \hat{i} + x(1 - \sin y) \hat{j}] \cdot (dx \hat{i} + dy \hat{j})$$

$$= \int_C \frac{\cos y}{u} dx + \frac{x(1 - \sin y)}{v} dy$$

$$= \int_R \left[\frac{\partial}{\partial x} x(1 - \sin y) - \frac{\partial}{\partial y} \cos y \right] dx dy$$

$$= \int_R [1 - \sin y + \sin y] dx dy$$

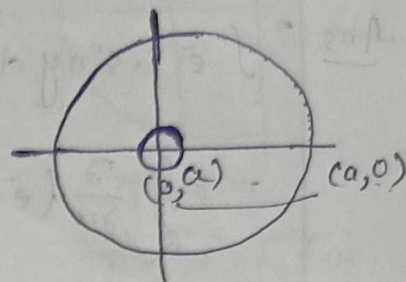
$$= \int_R dx dy = \pi$$



Q. Use Green's theorem to find the value of the integral.

$$\int_C \vec{F} \cdot d\vec{r}, \text{ where } \vec{F} = y^3 \hat{i} - x^3 \hat{j}$$

$$C: x^2 + y^2 = a^2, z=0.$$



Ans

$$\int_C (y^3 \hat{i} - x^3 \hat{j}) \cdot (dx \hat{i} + dy \hat{j})$$

$$= \int_C y^3 dx - x^3 dy$$

$$= \iint_R \left[\frac{d}{dx}(-x^3) - \frac{\partial}{\partial y}(y^3) \right] dx dy \quad 360^\circ = 2\pi$$

$$= \iint_R (-3x^2 - 3y^2) dx dy$$

$$x = r \cos \theta \rightarrow x\text{-axis}$$

$$y = r \sin \theta \rightarrow y\text{-axis}$$

$$dx dy = r dr d\theta$$

$$= -3 \int_{\theta=0}^{2\pi} \int_{r=0}^a [r^2 \cos^2 \theta + r^2 \sin^2 \theta] r dr d\theta$$

$$= -3 \int_{\theta=0}^{2\pi} \int_{r=0}^a r^3 dr d\theta$$

$$= -3 \int_{\theta=0}^{2\pi} \left[\frac{r^4}{4} \right]_0^a d\theta$$

$$= -3 \left[\frac{a^4}{4} \right]_0^a \times [\theta]_0^{2\pi}$$

$$= -\frac{3}{4} a^4 \times 2\pi = -\frac{3}{2} a^4 \pi$$

by green's theorem $\int_C e^{-x} (\sin y \, dx + \cos y \, dy)$ where C is the region with vertices $(0,0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(0, \pi/2)$.

Ans $\int_C e^{-x} \sin y \, dx + e^{-x} \cos y \, dy$

$$= \iint_R \left(\frac{\partial}{\partial x} (e^{-x} \cos y) - \frac{\partial}{\partial y} (e^{-x} \sin y) \right) dx dy$$

$$= \iint_R (-e^{-x} \cos y - e^{-x} \cos y) dx dy$$

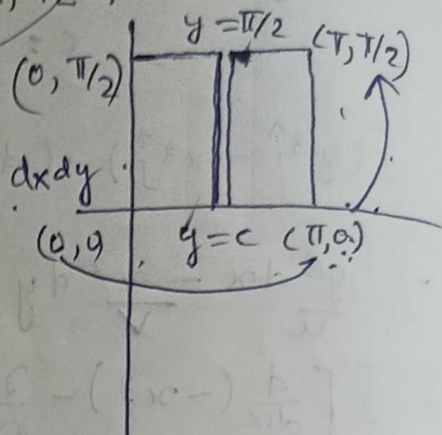
$$= \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} -2e^{-x} \cos y \, dy \, dx$$

$$= -2 \int_{x=0}^{\pi} e^{-x} dx \times \int_{y=0}^{\pi/2} \cos y \, dy$$

$$= -2 \left[-e^{-x} \right]_0^{\pi} \times \left[\sin y \right]_0^{\pi/2}$$

$$= 2 \times (e^{-\pi} - e^0) \times (1 - 0)$$

$$= 2(e^{-\pi} - 1)$$



(Q) Evaluate by green's theorem,

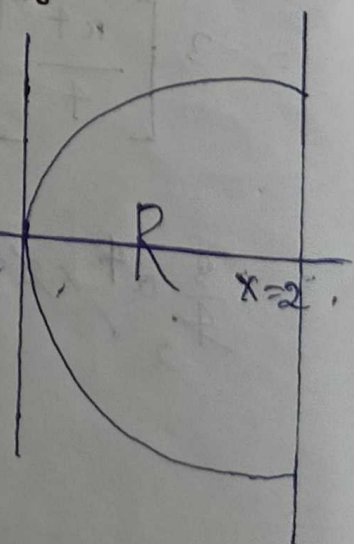
$$\int_C (xy^2 - 2xy) \, dx + (x^2y + 3) \, dy$$

$$C: y^2 = 8x, x=2$$

Ans

$$\iint_R \left[\frac{\partial}{\partial x} (x^2y + 3) - \frac{\partial}{\partial y} (xy^2 - 2xy) \right] dx dy$$

$$= \iint_R (2xy - 2xy + 2x) dx dy$$



$$= \int_0^2 \int_{-2\sqrt{2}\sqrt{x}}^{2\sqrt{2}\sqrt{x}} 2x \, dy \, dx$$

$$= \int_0^2 \left[2xy \right]_{-2\sqrt{2}\sqrt{x}}^{2\sqrt{2}\sqrt{x}} \cdot dx$$

$$= \int_0^2 2x (2\sqrt{2}\sqrt{x} + 2\sqrt{2}\sqrt{x}) \, dx$$

$$= \int_0^2 8\sqrt{2} x\sqrt{x} \, dx$$

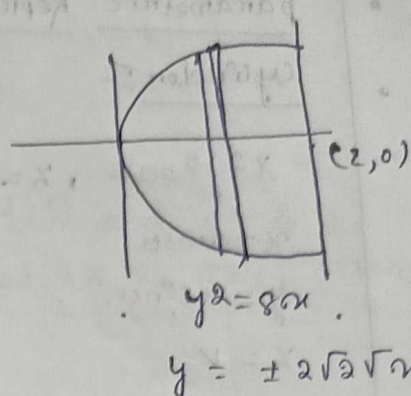
$$= 8\sqrt{2} \int_0^2 x^{3/2} \, dx$$

$$= 8\sqrt{2} \left[\frac{x^{5/2}}{5/2} \right]_0^2$$

$$= \frac{16\sqrt{2}}{5} \cdot 2^{5/2}$$

$$= \frac{16\sqrt{2}}{5} \cdot 4\sqrt{2}$$

$$= \frac{128}{5}$$



Dt :- 20/07/23

parametric Representation of surface :-

Cylinder :-

$$x^2 + y^2 = a^2, \quad z = b, \quad z = c$$

$$x = a \cos u$$

$$y = a \sin u$$

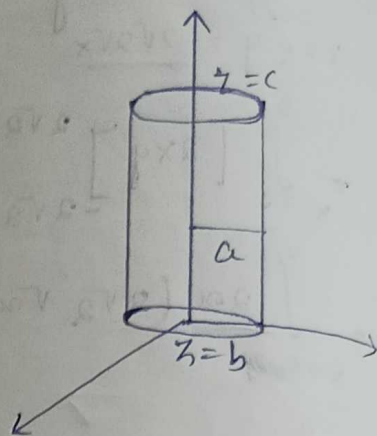
$$z = v$$

$u, v \rightarrow$ parameters

$$r(u, v) = [a \cos u, a \sin u, v]$$

$$0 \leq u \leq 2\pi$$

$$b \leq v \leq c$$



Sphere :-

$$x^2 + y^2 + z^2 = a^2$$

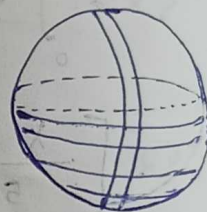
$$x = a \cos v \cdot \cos u$$

$$y = a \cos v \cdot \sin u$$

$$z = a \sin v$$

$$0 \leq u \leq 2\pi$$

$$r(u, v) = [a \cos v \cdot \cos u, a \cos v \cdot \sin u, a \sin v]$$



Cone :-

$$z = \sqrt{x^2 + y^2}$$

$$x = u \cos v$$

$$y = u \sin v$$

$$z = u$$

$$u = h$$

$$0 \leq v \leq 2\pi$$

$$r(u, v) = [u \cos v, u \sin v, u]$$



Surface Normal Vector $r(N)$:-

$r(u, v) \rightarrow$ Eqn of surface in parametric form.

$$N = r_u \times r_v$$

$$r_u = \frac{\partial r}{\partial u}$$

$$r_v = \frac{\partial r}{\partial v}$$

Surface Integration:-

(79)

$$\iint_S \vec{F} \cdot \hat{n} ds = \iint_R \vec{F}(u,v) \cdot \vec{N} du dv.$$

(Q) Evaluate $\iint_S \vec{A} \cdot \hat{n} ds$ where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and 'S' is the surface of the cylinder $x^2 + y^2 = 16$ included on the first octant between $z = 0, z = 5$.

Ans Parametric representation of the surface

$$x = 4 \cos u$$

$$y = 4 \sin u$$

$$z = v$$

$$0 \leq v \leq 5, 0 \leq u \leq \pi/2$$

$$\vec{r}(u,v) = [4 \cos u, 4 \sin u, v]$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

$$\vec{r}_u = [-4 \sin u, 4 \cos u, 0]$$

$$\vec{r}_v = [0, 0, 1]$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 \sin u & 4 \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\vec{N} = \hat{i} (4 \cos u) - \hat{j} (-4 \sin u) + \hat{k} (0 - 0)$$

$$\vec{N} = 4 \cos u \hat{i} + 4 \sin u \hat{j} + 0$$

$$\vec{N} = 4 \cos u, 4 \sin u, 0.$$

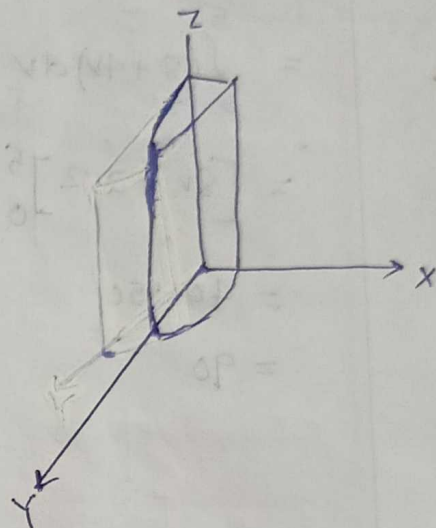
$$\vec{A} = [z, x, -3y^2z]$$

$$= [v, 4 \cos u, -48 \sin u]$$

$$\vec{A} \cdot \vec{N}$$

$$= 4v \cos u + 16 \sin u \cdot \cos u.$$

$$\iint_R \vec{A} \cdot \vec{N} du dv$$



$$= \int_{v=0}^5 \int_{u=0}^{\pi/2} [4v \cos u + 8 \sin 2u] \cdot du dv.$$

$$= \int_{v=0}^5 \left[4v \sin u + \frac{4}{2} (-\cos 2u) \right]_0^{\pi/2} dv.$$

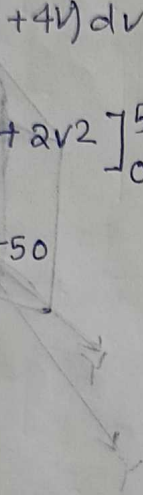
$$= \int_{v=0}^5 (4v + 4) - (-4) dv$$

$$= \int_0^5 (8 + 4v) dv$$

$$= [8v + 2v^2]_0^5$$

$$= 40 + 50$$

$$= 90$$



$$[v, u \cos v, \sin 2v] = (v, u) \pi$$

$$v \cdot u \times u \cdot \pi = u$$

$$[0, \sin 2v, \cos 2v] = u \pi$$

$$[1, 0, 0] = v \pi$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \sin 2v & \cos 2v \\ 1 & 0 & 0 \end{vmatrix} = \sin 2v \hat{i} - \cos 2v \hat{j}$$

$$(0, 0) \hat{i} + (u \cos v, \sin 2v) \hat{j} - (u \cos v) \hat{k} = u$$

$$0 + \sin 2v + \sin 2v = u$$

$$2 \sin 2v = u$$

$$[2 \sin 2v, u, 0] = \hat{i}$$

$$[u \cos v, \sin 2v, v] = \hat{j}$$

$$u \cdot \hat{i}$$

$$u \cdot \hat{j} = u$$

$$u \cdot \hat{k} = u$$

Dt: 21/7/23...

(Q) compute the flux of water through the parabolic cylinder $S: y = x^2$, $0 \leq x \leq 2$, $0 \leq z \leq 3$ if the velocity vector, (51)

$$\vec{V} = [3z^2, 6, 6xz]$$

Ans converting the surface to parametric form
let $x = u$, $y = u^2$, $z = v$

$$r(u, v) = [u, u^2, v]$$

$$r_u = [1, 2u, 0]$$

$$r_v = [0, 0, 1]$$

$$N = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2u\hat{i} - \hat{j} + 0\hat{k}$$

$$= [2u, -1, 0]$$

$$\vec{V}(u, v) = [3v^2, 6, 6uv]$$

$$\vec{V} \cdot N = 6uv^2 - 6$$

$$\iint_R \vec{V} \cdot N \, du \, dv = \int_{u=0}^2 \int_{v=0}^3 (6uv^2 - 6) \, du \, dv$$

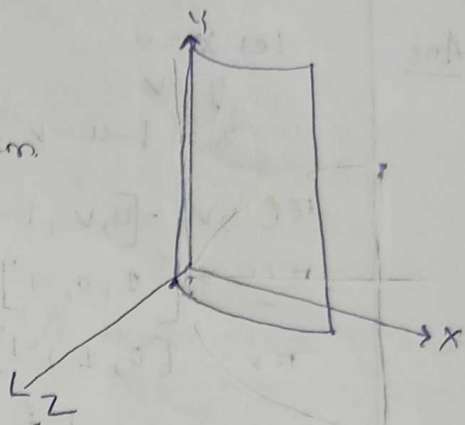
$$= \int_{u=0}^2 \left[2uv^3 - 6v \right]_0^3 \, du$$

$$= \int_{u=0}^2 (54u - 18) \, du$$

$$= [27u^2 - 18u]_0^2$$

$$= 108 - 36$$

$$= 72$$



(Q)

Evaluate the surface integral when $F = [x^2, 0, 3y^2]$ and 's' is the portion of the plane $x+y+z=1$ in the first octant.

Ans

$$\text{Let } x=u$$

$$y=v$$

$$z=1-u-v$$

$$r(u,v) = [u, v, 1-u-v]$$

$$r_u = [1, 0, -1]$$

$$r_v = [0, 1, -1]$$

$$N = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -\hat{i} + \hat{j} + \hat{k}$$

$$F(u,v) = [u^2, 0, 3v^2]$$

$$\vec{F} \cdot N = [-u^2 + 3x^2]$$

$$\iint_R \vec{F} \cdot N \, du \, dv = \int_{u=0}^1 \int_{v=0}^{1-u} u^2 + 3v^2 \, du \, dv$$

$$= \int_{u=0}^1 [u^2 v + v^3]_0^{1-u} \, du$$

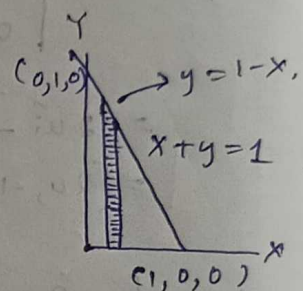
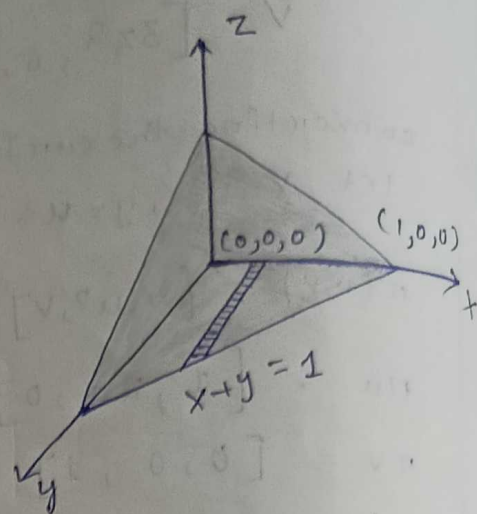
$$= \int_{u=0}^1 [u^2(1-u) + (1-u)^3] \, du$$

$$= \int_{u=0}^1 (u^2 - u^3 + 1 - 3u + 3u^2 - u^3) \, du$$

$$= \int_{u=0}^1 (-2u^3 + 4u^2 - 3u + 1) \, du$$

$$= \left[-\frac{2u^4}{4} + \frac{4u^3}{3} - \frac{3}{2}u^2 + u \right]_0^1$$

$$= -\frac{1}{2} + \frac{4}{3} - \frac{3}{2} + 1$$



$$= \frac{-3+8-9+6}{6}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

Dt 8-22/07/23.

Q1) Evaluate $\iint_S \vec{A} \cdot \hat{n} ds$ where

$\vec{A} = (x+y^2)\hat{i} - 2xz\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x+y+2z=6$, in the first octant.

Ans

$$S: 2x+y+2z=6$$

$$\Rightarrow \frac{2x}{6} + \frac{y}{6} + \frac{2z}{6} = 1$$

$$\Rightarrow \frac{x}{3} + \frac{y}{6} + \frac{z}{3} = 1$$

$$\text{Let } x=u, y=v$$

$$z = \frac{6-2u-v}{2}$$

$$r(u,v) = \left[u, v, \frac{6-2u-v}{2} \right]$$

$$r_u = \left[1, 0, -1 \right]$$

$$r_v = \left[0, 1, -\frac{1}{2} \right]$$

$$N = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \end{vmatrix}$$

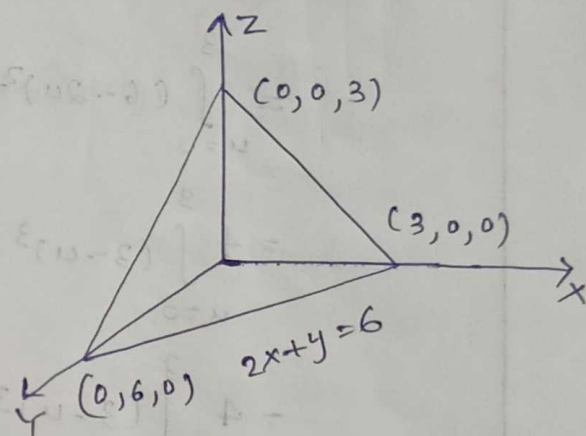
$$= \hat{i} + \frac{1}{2}\hat{j} + \hat{k}$$

$$= \left[1, \frac{1}{2}, 1 \right]$$

$$\vec{A} = (x+y^2)\hat{i} - 2xz\hat{j} + 2yz\hat{k}$$

$$= (u+v^2)\hat{i} - 2u\left(\frac{6-2u-v}{2}\right)\hat{j} + 2v\left(\frac{6-2u-v}{2}\right)\hat{k}$$

$$= \left[(u+v^2), (-2u), 6v-2uv-v^2 \right]$$



$$\begin{aligned}\vec{A} \cdot \vec{N} &= x + \sqrt{z} - x + 6v - 2uv - \sqrt{z} \\ &= 6v - 2uv\end{aligned}$$

$$\begin{aligned}\iint_S \vec{A} \cdot \vec{n} \, ds &= \iint_R \vec{A}(u, v) \cdot \vec{N} \cdot du \, dv \\ &= \int_{u=0}^3 \int_{v=0}^{6-2u} [6v - 2uv] \cdot du \, dv \\ &= \int_{u=0}^3 \left[3v^2 - uv^2 \right]_0^{6-2u} du = \int_{u=0}^3 [3(6-2u)^2 - u(6-2u)^2] du \\ &= \int_{u=0}^3 (6-2u)^2 [3-u] du \\ &= 4 \int_{u=0}^3 (3-u)^3 (3-u) du \\ &= 4 \int_{u=0}^3 (3-u)^3 du \\ &= 4 \left[\frac{(3-u)^4}{4 \times (-1)} \right]_0^3 \\ &= - \left[(3-u)^4 \right]_0^3 \\ &= -[0 - 81] \\ &= 81\end{aligned}$$

Stoke's Circulation Theorem :-

→ If 'S' is an open surface bounded by a closed non intersecting curve 'C' and if a vector field $\vec{F}$ has continuous partial derivatives.

$$\text{Then, } \oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds.$$

DE: - 24/07/23

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Verify Stoke's theorem for

$$\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$$

where 's' is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and 'c' is its Boundary.

Ans

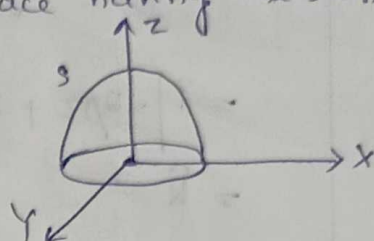
$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n} ds = \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{n} ds.$$

S_1 and S_2 are 2 different surface having the same boundary.

$$\oint_C \vec{F} \cdot d\vec{r}$$

circle - xy plane.

upper half - z -axis.



$$= \int_C (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k} \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C (2x-y)dx - yz^2dy - y^2zdz.$$

$$= C: x^2 + y^2 = 1, z=0.$$

$$x = \cos t$$

$$dx = -\sin t dt$$

$$y = \sin t$$

$$dy = \cos t dt$$

$$z=0$$

$$dz=0$$

$$= \int_C (2x-y)dx - yz^2dy - y^2zdz.$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t dt)$$

$$= \int_0^{2\pi} [-2\sin t \cos t + \sin^2 t] dt$$

$$= \int_0^{2\pi} \left[-\sin 2t + \frac{1 - \cos 2t}{2} \right] dt$$

$$= \left[\frac{\cos 2t}{2} + \frac{1}{2}t - \frac{\sin 2t}{4} \right]_{t=0}^{2\pi}$$

$$= \left(\frac{1}{2} + \pi - 0\right) - \left(\frac{1}{2}\right) = \pi$$

$$\iint_S \text{curl } \vec{F} \cdot \vec{n} \, dS.$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz & -y^2z \end{vmatrix}$$

$$= \hat{i} | -yz + yz | - \hat{j} | 0 - 0 | + \hat{k} | 0 + 1 |$$

$$= \hat{k}$$

$$= \iint_S \hat{k} \cdot \vec{n} \, dS.$$

$$x = \cos v \cos u,$$

$$y = \cos v \sin u,$$

$$z = \sin v.$$

$$\vec{r}(u, v) = [\cos v \cos u, \cos v \sin u, \sin v]$$

$$\vec{r}_u = [-\cos v \sin u, \cos v \cos u, 0]$$

$$\vec{r}_v = [-\sin v \cos u, -\sin v \sin u, \cos v]$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v.$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos v \sin u & \cos v \cos u & 0 \\ -\sin v \cos u & -\sin v \sin u & \cos v \end{vmatrix}$$

$$= \hat{i} |\cos^2 v \cos u| - \hat{j} |-\cos^2 v \sin u| + \hat{k} |\cos v \sin v \sin^2 u + \cos v \sin v \cos^2 u|$$

$$\vec{N} = \cos u \cos^2 v \hat{i} + \sin u \cos^2 v \hat{j} + \sin v \cos v \hat{k}$$

$$\iint_R \hat{k} \cdot \vec{n} \, dS.$$

$$= \iint_R \hat{k} \cdot \vec{N} \, du \, dv.$$

$$= 2\pi \int_0^{\pi/2} \int_0^{2\pi} \sin v \cdot \cos v \, du \, dv$$

$$= \int_0^{2\pi} du \times \int_0^{\pi/2} \frac{\sin 2v}{2} \, dv$$

$$= [u]_0^{2\pi} \times \left[-\frac{\cos 2v}{4} \right]_0^{\pi/2}$$

$$= 2\pi \times \frac{1+1}{4}$$

$$= 2\pi \times \frac{2}{4}$$

$$= \pi$$

$$\underline{\text{LHS} = \text{RHS}}$$

Stokes' Circulation Theorem proved.

Verify Stokes Theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated around the square in the plane, $z=0$, and bounded by the line $x=0, y=0, x=a, y=a$.

Ans

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds$$

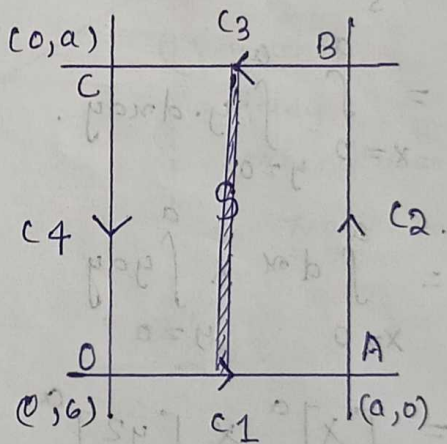
$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} +$$

$$\int_{C_4} \vec{F} \cdot d\vec{r}$$

$$= \int_{y=0}^a x^2 dx + xy dy + \int_{x=a}^0 x^2 dx + xy dy + \int_{y=a}^0 x^2 dx + xy dy + \int_{x=0}^a x^2 dx + xy dy$$

$$= \int_0^a x^2 dx + \int_0^a ay dy + \int_a^0 x^2 dx + 0 + \int_a^0 x^2 dx + 0 + \int_0^a x^2 dx + 0$$



$$= a \left[\frac{y^2}{2} \right]_0^a$$

$$= \frac{a}{2} (a^2 - 0)$$

$$= \frac{a^3}{2}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \hat{i} |0| - \hat{j} |0-0| + \hat{k} |y-0|$$

$$= y \hat{k}$$

$$= \iint_S y \hat{k} \cdot \hat{k} \, dx \, dy$$

$$= \int_{x=0}^a \int_{y=0}^a y \, dx \, dy$$

$$= \int_{x=0}^a dx \times \int_{y=0}^a y \, dy$$

$$= \left[x \right]_0^a \times \left[\frac{y^2}{2} \right]_0^a$$

$$= a \times \frac{a^2}{2}$$

$$= \frac{a^3}{2}$$

Hence, STOKES Theorem is Verified.

Gauss Divergence Theorem:-

→ The surface integral of the normal component of a vector point function $\vec{F}$ taken over a closed surface 'S' enclosing a volume 'V' is equal to the volume integral of the divergence of $\vec{F}$ taken throughout the volume 'V' bounded by S.

$$\iiint_S \vec{F} \cdot \hat{n} ds = \iiint_V (\nabla \cdot \vec{F}) dV$$

(Q) Verify the divergence theorem of the vector field $\vec{F} = 4xz\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z \geq 0$, $z \leq 3$.

Ans. $\iiint_S \vec{F} \cdot \hat{n} ds$

$$= \iint_{S_1} \vec{F} \cdot \hat{n} ds + \iint_{S_2} \vec{F} \cdot \hat{n} ds + \iint_{S_3} \vec{F} \cdot \hat{n} ds$$

$$\cdot \iint_{S_1} \vec{F} \cdot \hat{n} ds$$

$$= \int_{z=3} \int (4xz\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot \hat{k} dxdy$$

$$= \int_{z=3} \int z^2 dxdy$$

$$= \int_{z=0} \int 9 dxdy$$

$$= 9 \times (\text{Area of } S_1)$$

$$= 9 \times \pi \times 4$$

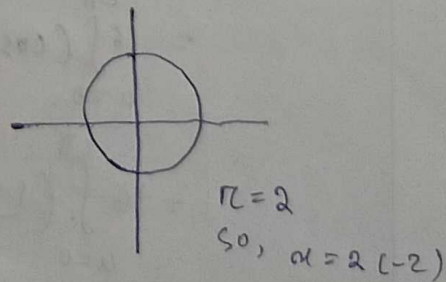
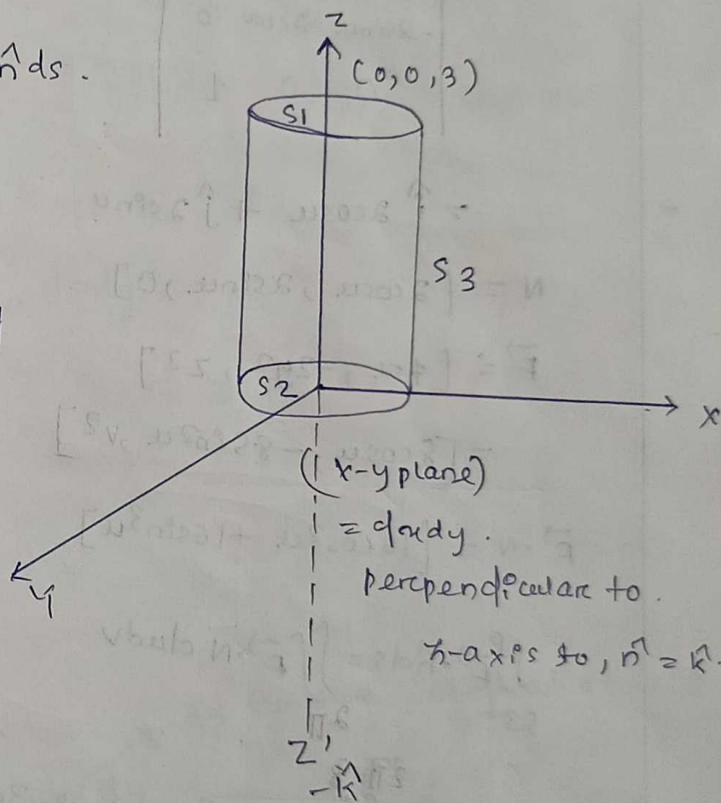
$$= 36\pi$$

$$\cdot \iint_{S_2} \vec{F} \cdot \hat{n} ds$$

$$= \int_{z=0} \int (4xz\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot (-\hat{k}) dxdy$$

$$= \int_{z=0} \int -z^2 dxdy$$

$$= \int_{z=0} \int 0^2 dxdy = 0$$



$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds$$

using parametric form,

$$x = 2\cos u, y = \sin u, z = v.$$

$$\vec{r}(u, v) = [2\cos u, \sin u, v]$$

$$\vec{r}_u = [-2\sin u, \cos u, 0]$$

$$\vec{r}_v = [0, 0, 1]$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \hat{i} 2\cos u + \hat{j} 2\sin u$$

$$\vec{N} = [2\cos u, 2\sin u, 0]$$

$$\vec{F} = [4x, -2y^2, z^2]$$

$$= [8\cos u, -8\sin^2 u, v^2]$$

$$\vec{F} \cdot \vec{N} = [16\cos^2 u - 16\sin^3 u]$$

$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds = \iint \vec{F} \cdot \vec{N} \, du \, dv$$

$$= \int_{u=0}^{2\pi} \int_{v=0}^3 (16\cos^2 u - 16\sin^3 u) \, du \, dv$$

$$= 16 \int_{u=0}^{2\pi} (\cos^2 u - \sin^3 u) \, du \times \int_{v=0}^3 dv$$

$$= 16 \int_{u=0}^{2\pi} \left(\frac{1 + \cos 2u}{2} - \frac{3\sin u - \sin^3 u}{4} \right) du \times [v]_0^3$$

$$= 16 \times 3 \int_{u=0}^{2\pi} \left(\frac{1}{2} + \frac{\cos 2u}{2} - \frac{3}{4} \sin u + \frac{1}{4} \sin^3 u \right) du$$

$$= 48 \left[\frac{1}{2} u + \frac{\sin 2u}{4} + \frac{3}{4} \cos u - \frac{1}{12} \cos 3u \right]_0^{2\pi}$$

$$= \frac{48}{2} \times 2\pi$$

$$= 48\pi$$

$$\iint_S \vec{F} \cdot \hat{n} ds = 36\pi + 0 + 48\pi = 84\pi.$$

$$\iiint_S (\nabla \cdot \vec{F}) dv.$$

$$\nabla \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (4x\hat{i} - 4y^2\hat{j} + 2z\hat{k})$$

$$= 4 - 4y + 2z.$$

$$\iiint_{x=-2, y=-\sqrt{4-x^2}, z=0}^{x=2, y=\sqrt{4-x^2}, z=2} (4 - 4y + 2z) dx dy dz.$$

$$= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4z - 4yz + z^2]_0^2 dx dy.$$

$$= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9) dx dy.$$

$$= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) dx dy.$$

$$= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21 dx - \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 12y dx dy$$

$$= 21 \times 4\pi$$

$$= 84\pi$$

odd function integration is 0.

DL 8-27/7/23

(Q) use divergence theorem to evaluate $\iint_S \vec{F} \cdot d\vec{s}$ where $\vec{F} = x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$

Ans

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_V (\nabla \cdot \vec{F}) dV$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} x^3 + \frac{\partial}{\partial y} y^3 + \frac{\partial}{\partial z} z^3$$

$$= 3x^2 + 3y^2 + 3z^2$$



$$\iiint_V (3x^2 + 3y^2 + 3z^2) dx dy dz$$

Representation of the sphere in parametric form

$$x = r \cos v \cos u$$

$$y = r \cos v \sin u$$

$$z = r \sin v$$

$$= \int_{r=0}^a \int_{u=0}^{2\pi} \int_{v=0}^{\pi} [3r^2 \cos^2 v \cdot \cos^2 u + 3r^2 \cos^2 v \sin^2 u + 3r^2 \sin^2 v] r^2 \sin v dr du dv$$

radius sphere

$$= 3 \int_{r=0}^a \int_{u=0}^{2\pi} \int_{v=0}^{\pi} [r^2 \cos^2 v (\cos^2 u + \sin^2 u) + r^2 \sin^2 v] r^2 \sin v dr du dv$$

Longitudinal

$$= 3 \int_{r=0}^a \int_{u=0}^{2\pi} \int_{v=0}^{\pi} r^4 \cdot \sin v dr du dv$$

$$= 3 \int_{r=0}^a r^4 dr \times \int_{u=0}^{2\pi} du \times \int_{v=0}^{\pi} \sin v dv$$

$$= 3 \times \left[\frac{r^5}{5} \right]_0^a \times [u]_0^{2\pi} \times [-\cos v]_0^{\pi}$$

$$= \frac{3}{5} a^5 \times 2\pi \times (1+1)$$

$$= \frac{12}{5} \pi a^5$$

• Dot product :-

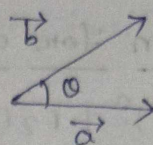
$$\vec{a} \cdot \vec{b} = ab \cos \theta$$

$\theta \rightarrow$ angle between $\vec{a}$ & $\vec{b}$

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$



(upward to the ground)

$\vec{b} \times \vec{a} \rightarrow$ Toward to the ground.

• cross product :-

$$\vec{a} \times \vec{b} = ab \sin \theta \hat{n}$$

$\hat{n} =$ unit vector perpendicular to $\vec{a}$ & $\vec{b}$ (parallelogram)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

• Modulus of a Vector :-

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

• unit vector :-

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\hat{a} = \frac{a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

• scalar triple product :- $\vec{a}(\vec{b} \times \vec{c}), [\vec{a} \vec{b} \vec{c}]$

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\vec{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$$

$$[\vec{a} \vec{b} \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\rightarrow$ It represents the volume of a parallelepiped.

- Vector Triple product :-

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

- Vector function for a single scalar variable :-

$$\vec{r}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$$

- Derivative of a vector value function :-

$$\frac{d}{dt} \vec{r}(t) = \frac{d}{dt} F_1(t)\hat{i} + \frac{d}{dt} F_2(t)\hat{j} + \frac{d}{dt} F_3(t)\hat{k}$$

(Q) $\vec{r}(t) = t^2\hat{i} + 2t\hat{j} + 3t^3\hat{k}$

find $\frac{d}{dt} \vec{r}(t)$ at $t = 1 \text{ sec}$.

$$\frac{d}{dt} \vec{r}(t) = 2t\hat{i} + 2\hat{j} + 9t^2\hat{k}$$

$$\frac{d}{dt} \vec{r}(1) = 2\hat{i} + 2\hat{j} + 9\hat{k}$$

- Scalar point function :-

→ If a scalar quantity ϕ is a function of a point, that depends on its position in space then $\phi(x, y, z)$ is called a scalar point function.

- Vector point function :-

→ If a vector quantity $\vec{V}$ depends on its value on its position (x, y, z) in space. Then $\vec{V}(x, y, z)$ is called a vector point function.

- General Rules of Vector differentiation :-

$$\frac{d}{dt} (\vec{A} + \vec{B}) = \frac{d}{dt} \vec{A} + \frac{d}{dt} \vec{B}$$

$$\frac{d}{dt} (u\vec{A}) = \frac{du}{dt} \vec{A} + u \frac{d}{dt} \vec{A}$$

$$\frac{d}{dt} (\vec{A} \cdot \vec{B}) = \frac{d\vec{A}}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt} (\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt} [\vec{A} \cdot (\vec{B} \times \vec{C})] = \frac{d}{dt} \vec{A} \cdot \vec{B} \times \vec{C} + \vec{A} \cdot \frac{d}{dt} \vec{B} \times \vec{C} + \vec{A} \cdot \vec{B} \times \frac{d}{dt} \vec{C}$$

$$\frac{d}{dt} [\vec{A} \times \vec{B} \times \vec{C}] = \frac{d}{dt} \vec{A} \times \vec{B} \times \vec{C} + \vec{A} \times \frac{d}{dt} \vec{B} \times \vec{C} + \vec{A} \times \vec{B} \times \frac{d}{dt} \vec{C}$$

Important Result :-

- (1) The necessary and sufficient condition for $\vec{V}(t)$ to have a constant Magnitude is

$$\vec{V} \cdot \frac{d\vec{V}}{dt} = 0$$

- (2) The necessary and sufficient condition for any vector $\vec{V}$ to have constant direction is :-

$$\vec{V} \times \frac{d\vec{V}}{dt} = 0$$

∇ operator (nabla) :-

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

The gradient of a scalar point function ϕ :-

$$\nabla \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \phi$$

$$\Rightarrow \nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \rightarrow \text{grad } \phi$$

✓ If $\vec{r}$ represent the position vector of any point $P(x, y, z)$ in the space then find $\nabla |\vec{r}|^2$ or ∇r^2 .

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

$$\nabla r^2 = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)$$

$$= \hat{i} \frac{\partial}{\partial x} (x^2 + y^2 + z^2) + \hat{j} \frac{\partial}{\partial y} (x^2 + y^2 + z^2) + \hat{k} \frac{\partial}{\partial z} (x^2 + y^2 + z^2)$$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$

$$= 2(x\hat{i} + y\hat{j} + z\hat{k}) = 2\vec{r}$$

4. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$. find ∇u , ∇v and ∇w , show these vectors are coplanar.

Ans

$$\begin{aligned}\nabla u &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x + y + z) \\ &= \hat{i} \frac{\partial}{\partial x} (x + y + z) + \hat{j} \frac{\partial}{\partial y} (x + y + z) + \hat{k} \frac{\partial}{\partial z} (x + y + z) \\ &= \hat{i} + \hat{j} + \hat{k}\end{aligned}$$

$$\begin{aligned}\nabla v &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2) \\ &= \hat{i} \frac{\partial}{\partial x} (x^2 + y^2 + z^2) + \hat{j} \frac{\partial}{\partial y} (x^2 + y^2 + z^2) + \hat{k} \frac{\partial}{\partial z} (x^2 + y^2 + z^2) \\ &= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}\end{aligned}$$

$$\begin{aligned}\nabla w &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (xy + yz + zx) \\ &= \hat{i} \frac{\partial}{\partial x} (xy + yz + zx) + \hat{j} \frac{\partial}{\partial y} (xy + yz + zx) + \hat{k} \frac{\partial}{\partial z} (xy + yz + zx) \\ &= \hat{i}(y + z) + \hat{j}(x + z) + \hat{k}(y + x)\end{aligned}$$

$$\begin{aligned}\nabla u \cdot (\nabla u \times \nabla w) &= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & y+x \end{vmatrix} \\ &= 1 \begin{vmatrix} 2y & 2z \\ x+y & y+x \end{vmatrix} - 1 \begin{vmatrix} 2x & 2z \\ y+z & y+x \end{vmatrix} + 1 \begin{vmatrix} 2x & 2y \\ y+z & x+z \end{vmatrix} \\ &= 2y^2 + 2xy - 2zx - z^2 - 2xy - 2x^2 + 2zy + 2z^2 + 2x^2 + 2zx - 2y^2 - 2zy = 0 \\ &\text{OR}\end{aligned}$$

$$\begin{aligned}&= 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & y+x \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} \\ &= 2(x + y + z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0\end{aligned}$$

Note:-

→ The maximum rate of change of ϕ is along the direction of $\nabla\phi$ and its Maximum value will be denoted by $|\nabla\phi|$

(Q) Find gradient of $\pi(\nabla\pi)$ where $\pi = |\vec{r}|$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Ans $\pi = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow \hat{i} \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} + \hat{j} \frac{\partial}{\partial y} \sqrt{x^2 + y^2 + z^2} + \hat{k} \frac{\partial}{\partial z} (\sqrt{x^2 + y^2 + z^2})$$

$$\Rightarrow \hat{i} \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x + \hat{j} \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2y + \hat{k} \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2z$$

$$\Rightarrow \frac{1}{\sqrt{x^2 + y^2 + z^2}} (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\Rightarrow \frac{1}{\pi} x\vec{r} = \frac{\vec{r}}{\pi}$$

(Q) Find a unit vector normal to the surface $\phi = xy + yz + zx$ at the point $(1, 2, 0)$.

Ans We know that unit vector normal to the surface ϕ is denoted by

$$\frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) (xy + yz + zx)$$

$$= \hat{i} \frac{\partial}{\partial x} (xy + yz + zx) + \hat{j} \frac{\partial}{\partial y} (xy + yz + zx) + \hat{k} \frac{\partial}{\partial z} (xy + yz + zx)$$

$$= \hat{i} (y + z) + \hat{j} (x + z) + \hat{k} (y + x)$$

$$[\nabla\phi]_{(1,2,0)} = 2\hat{i} + \hat{j} + 3\hat{k}$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{4+1+9}} = \frac{2\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{14}} \quad (\text{Ans})$$

(Q) Find the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ at $(1, 1, -1)$ in the direction $2\hat{i} + \hat{j} - \hat{k}$. In what direction the directional derivative from the point $(1, 1, -1)$ is maximum and what is its value?

Ans The directional derivative of ϕ along any vector $\vec{a}$ is denoted by $\frac{\nabla \phi \cdot \vec{a}}{|\vec{a}|}$

$$\phi = x^2 - 2y^2 + 4z^2 \text{ at point } (1, 1, -1)$$

$$\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$$

$$\begin{aligned} \nabla \phi &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 - 2y^2 + 4z^2) \\ &= 2x\hat{i} - 4y\hat{j} + 8z\hat{k} \end{aligned}$$

$$[\nabla \phi]_{(1, 1, -1)} = 2\hat{i} - 4\hat{j} - 8\hat{k}$$

$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-1)^2}$$

$$= \sqrt{4 + 1 + 1}$$

$$= \sqrt{6}$$

$$\nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= 2\hat{i} - 4\hat{j} - 8\hat{k} \cdot \frac{2\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}} (4 - 4 + 8)$$

$$= \frac{8}{\sqrt{6}} [\phi \text{ along } \vec{a}]$$

$$|\nabla \phi| = \sqrt{2^2 + (-4)^2 + (-8)^2}$$

$$= \sqrt{4 + 16 + 64}$$

$$= \sqrt{84}$$

The maximum rate of change of ϕ along $\nabla\phi$ so its value at $(1,1,1)$ is $[2x\hat{i} - 4y\hat{j} + 6z\hat{k}]_{(1,1,1)}$.

Q1) If $\phi(x,y,z) = x^2y + y^2x + z^2$. find grad ϕ ($\nabla\phi$) at $(1,1,1)$.

Ans $\phi(x,y,z) = x^2y + y^2x + z^2$

$$\nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) (x^2y + y^2x + z^2)$$

$$= \hat{i} \frac{\partial}{\partial x} (x^2y + y^2x + z^2) + \hat{j} \frac{\partial}{\partial y} (x^2y + y^2x + z^2) + \hat{k} \frac{\partial}{\partial z} (x^2y + y^2x + z^2)$$

$$= \hat{i} (2xy + y^2 + 0) + \hat{j} (x^2 + 2y + 0) + \hat{k} (0 + 0 + 2z)$$

$$= \hat{i} (2xy + y^2) + \hat{j} (x^2 + 2y) + \hat{k} (2z)$$

put $x=1, y=1, z=1$

$$= \hat{i} (2+1) + \hat{j} (1+2) + \hat{k} (2)$$

$$= 3\hat{i} + 3\hat{j} + 2\hat{k}$$

Q2) If $\vec{r}$ is the position vector $r = |\vec{r}|$, find ∇r^2 and ∇r .

Ans

$$r(x,y,z)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla r = \hat{i} \frac{\partial r}{\partial x} + \hat{j} \frac{\partial r}{\partial y} + \hat{k} \frac{\partial r}{\partial z}$$

$$= \hat{i} \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} + \hat{j} \frac{\partial}{\partial y} \sqrt{x^2 + y^2 + z^2} + \hat{k} \frac{\partial}{\partial z} \sqrt{x^2 + y^2 + z^2}$$

$$= \hat{i} \frac{x}{\sqrt{x^2 + y^2 + z^2}} + \hat{j} \frac{y}{\sqrt{x^2 + y^2 + z^2}} + \hat{k} \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{1}{r} (\hat{i}x + \hat{j}y + \hat{k}z) = \frac{\vec{r}}{r}$$

• Divergence of a vector point function :-

→ When the vector differential operator ∇ operate scalarly on a vector point function, the result is a scalar quantity. Thus if $\vec{F}(x, y, z)$ is a vector point function then,

$\boxed{\nabla \cdot \vec{F}}$ denotes the divergence of a vector ($\text{div } \vec{F}$).

$$\left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$\boxed{\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \vec{F}}$$

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

• Note :-

If $\boxed{\nabla \cdot \vec{F} = 0}$, the vector field $\vec{F}$ is called solenoidal.

• physical interpretation :-

→ The divergence of the fluid vector $\vec{V}$ represents the rate of fluid flow through a unit volume.

If $\vec{V}$ represent the electric flow, $\text{div } \vec{V}$ will

represent the amount of electric flux, which crosses the unit volume in unit time.

• Note :- If $\text{div } \vec{V} = 0$,

Then the fluid is said to be incompressible

(Q) find the $\text{div } \vec{r}$.

Ans

$$\begin{aligned} \vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \nabla \cdot \vec{r} &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \\ &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \\ &= 1 + 1 + 1 = 3 \end{aligned}$$

• Divergence of a vector point function :-

→ When the vector differential operator ∇ operate scalarly on a vector point function, the result is a scalar quantity. Thus if $\vec{F}(x, y, z)$ is a vector point function then,

$\boxed{\nabla \cdot \vec{F}}$ denotes the divergence of a vector ($\text{div } \vec{F}$).

$$\left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$\boxed{\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \vec{F}}$$

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

• Note :-

If $\boxed{\nabla \cdot \vec{F} = 0}$, the vector field $\vec{F}$ is called solenoidal

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Then the fluid is said to be incompressible

(Q) find the $\text{div } \vec{r}$.

Ans

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \cdot \vec{r} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = 3$$

(Q) Find the value of λ for which the vector $\vec{u} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+\lambda z)\hat{k}$.

Ans

$$\nabla \cdot \vec{u} = 0$$

$$\Rightarrow \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot ((x+3y)\hat{i} + (y-2z)\hat{j} + (x+\lambda z)\hat{k}) = 0$$

$$\Rightarrow 1+1+\lambda=0$$

$$\Rightarrow 2+\lambda=0$$

$$\Rightarrow \lambda = -2$$

(Q) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that $\nabla \cdot (r^n \vec{r}) = (n+3)r^n$.

Ans

$$\left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k})$$

$$= \frac{\partial}{\partial x} r^n x + \frac{\partial}{\partial y} r^n y + \frac{\partial}{\partial z} r^n z$$

$$= n r^{n-1} \frac{\partial r}{\partial x} x + r^n 1 + \frac{\partial}{\partial y} r^n y + r^n \frac{\partial y}{\partial y} + n r^{n-1} \frac{\partial r}{\partial y} y$$

$$+ r^n 1 + n r^{n-1} \frac{\partial r}{\partial z} z + r^n 1$$

$$= 3r^n + n r^{n-1} \left[x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z} \right]$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$= 3r^n + n r^{n-1} \left[x \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} + y \frac{\partial}{\partial y} \sqrt{x^2 + y^2 + z^2} + z \frac{\partial}{\partial z} \sqrt{x^2 + y^2 + z^2} \right]$$

$$= 3r^n + n r^{n-1} \left[x \times \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} + y \frac{2y}{2\sqrt{x^2 + y^2 + z^2}} + z \frac{2z}{2\sqrt{x^2 + y^2 + z^2}} \right]$$

$$= 3\pi^n + n\pi^{n-1} \left[\frac{x^2}{\pi} + \frac{y^2}{\pi} + \frac{z^2}{\pi} \right]$$

$$= 3\pi^n + n\pi^{n-1} \cdot \frac{1}{\pi} (x^2 + y^2 + z^2)$$

$$= 3\pi^n + n\pi^{n-2} \cdot \pi^2$$

$$= 3\pi^n + n\pi^n$$

$$= (3+n)\pi^n$$

$$= (n+3)\pi^n \text{ (proved)}$$

(Q) In what direction from the point $(2, 1, -1)$ is the directional derivative of $\phi = x^2yz^3$ is a maximum and what is its magnitude.

Ans We know the directional derivative of ϕ is maximum along the direction of its gradient.

$$\nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2yz^3)$$

$$= \hat{i} \frac{\partial}{\partial x} x^2yz^3 + \hat{j} \frac{\partial}{\partial y} x^2yz^3 + \hat{k} \frac{\partial}{\partial z} x^2yz^3$$

$$= \hat{i} 2xyz^3 + \hat{j} x^2z^3 + \hat{k} 3z^2x^2y$$

Put $x=2, y=1, z=-1$

$$\nabla\phi = -4\hat{i} - 4\hat{j} + 12\hat{k}$$

$$|\nabla\phi| = \sqrt{(-4)^2 + (-4)^2 + (12)^2}$$

$$= \sqrt{16 + 16 + 144}$$

$$= \sqrt{176}$$

(Q) What is the direction for the maximum change of the space function $\phi(x, y, z) = x \sin z - y \cos z$ at the origin $(0, 0, 0)$.

Ans
$$\nabla\phi = \hat{i} \frac{\partial}{\partial x} (x \sin z - y \cos z) + \hat{j} \frac{\partial}{\partial y} (x \sin z - y \cos z) + \hat{k} \frac{\partial}{\partial z} (x \sin z - y \cos z)$$

$$= \hat{i} (\sin z - 0) + \hat{j} (0 - \cos z) + \hat{k} (x \cos z + y \sin z)$$

(Q) find a unit vector normal to the surface $\phi = x^2 - y^2 + z$ at the point $(1, -1, 2)$

Ans $\nabla\phi$ is always normal to the surface ϕ

$$\begin{aligned}\nabla\phi &= \hat{i} \frac{\partial}{\partial x} (x^2 - y^2 + z) + \hat{j} \frac{\partial}{\partial y} (x^2 - y^2 + z) + \hat{k} \frac{\partial}{\partial z} (x^2 - y^2 + z) \\ &= \hat{i}(2x) + \hat{j}(-2y) + \hat{k}\end{aligned}$$

$$(\nabla\phi)(1, -1, 2) = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$a^n = \frac{\vec{a}}{|\vec{a}|}$$

unit vector normal to ϕ is

$$= \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{4+4+1}} = \pm \frac{2\hat{i} + 2\hat{j} + \hat{k}}{3}$$

(Q) find the change of $\phi = xyz$ in the direction normal to the surface, $yx^2 + xy^2 + z^2y - 3$ at the point $(1, 1, 1)$.

Ans first we need to find the direction normal to the surface.

$$yx^2 + xy^2 + z^2y = 3$$

$$\psi = yx^2 + xy^2 + z^2y - 3$$

$$\begin{aligned}\nabla\psi &= \hat{i} \frac{\partial}{\partial x} (yx^2 + xy^2 + z^2y - 3) + \hat{j} \frac{\partial}{\partial y} (yx^2 + xy^2 + z^2y - 3) \\ &\quad + \hat{k} \frac{\partial}{\partial z} (yx^2 + xy^2 + z^2y - 3)\end{aligned}$$

$$= (2xy + y^2)\hat{i} + (x^2 + 2xy + z^2)\hat{j} + 2yz\hat{k}$$

$$\text{put } x=1, y=1, z=1$$

$$= 3\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\nabla\phi \propto \frac{3\hat{i} + 4\hat{j} + 2\hat{k}}{|3\hat{i} + 4\hat{j} + 2\hat{k}|}$$

$$\nabla\phi = \hat{i} \frac{\partial}{\partial x} (xyz) + \hat{j} \frac{\partial}{\partial y} (xyz) + \hat{k} \frac{\partial}{\partial z} (xyz)$$

$$= yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\nabla\phi(1,1,1) = \hat{i} + \hat{j} + \hat{k}$$

$$(\hat{i} + \hat{j} + \hat{k}) \cdot (3\hat{i} + 4\hat{j} + 2\hat{k})$$

$$\frac{\sqrt{9+16+4}}{\sqrt{9+16+4}}$$

$$= \frac{1}{\sqrt{29}} (3+4+2)$$

$$= \frac{9}{\sqrt{29}}$$

Q) find the directional derivative at $\phi = x^2yz + 4xz^2$ at $(1,2,-1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$. Ψ

Ans
$$\frac{\nabla\phi \cdot (2\hat{i} - \hat{j} - 2\hat{k})}{(2\hat{i} - \hat{j} - 2\hat{k})}$$

$$\nabla\phi = \hat{i} \frac{\partial}{\partial x} (x^2yz + 4xz^2) + \hat{j} \frac{\partial}{\partial y} (x^2yz + 4xz^2) + \hat{k} \frac{\partial}{\partial z} (x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z + 0)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$\nabla\phi(1,2,-1) = (-4+4)\hat{i} - \hat{j} + (2-8)\hat{k}$$

$$= 0\hat{i} - \hat{j} - 6\hat{k}$$

$$(0\hat{i} - \hat{j} - 6\hat{k}) \cdot \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}}$$

$$= \frac{1}{3} [0 + 1 + 12]$$

$$= \frac{13}{3}$$

• Scalar point function :-

→ If $\vec{F}$ is a vector point which is irrotational conservative then there exist a scalar point function ϕ such that $\vec{F} = \nabla\phi$

(Q) ^{doubt} Show that $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find scalar ϕ such that $\vec{F} = \nabla\phi$

Ans $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$

$$= \hat{i} |-1+1| - \hat{j} |3z^2 - 3z^2| + \hat{k} |6x - 6x|$$

$$= \vec{0}$$

$\vec{F}$ is irrotational.

$$\vec{F} = \nabla\phi$$

$$\Rightarrow (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k} = \hat{i} \frac{\partial\phi}{\partial x} + \hat{j} \frac{\partial\phi}{\partial y} + \hat{k} \frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 6xy + z^3 \quad \text{--- (1)}$$

$$\Rightarrow \partial\phi = (6xy + z^3) \partial x$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \quad \text{--- (2)}$$

$$\Rightarrow \int \partial\phi = \int (6xy + z^3) \partial x$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \quad \text{--- (3)}$$

$$\Rightarrow \phi = 3x^2y + z^3x + A(y, z)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 + \frac{\partial A}{\partial y} \quad \text{--- (4)}$$

comparing eqn (3) and (4)

$$\frac{\partial A}{\partial y} = -z$$

$$\Rightarrow \int \partial A = - \int z \partial y$$

$$\Rightarrow A = -zy + B(z)$$

$$\phi = 3x^2y + z^3x - zy + B(z)$$

$$\frac{\partial \phi}{\partial z} = 0 + 3z^2x - y + \frac{\partial B}{\partial z} \quad \text{--- (5)}$$

$$\frac{\partial B}{\partial z} = 0$$

$$\Rightarrow B = C \text{ (any constant)}$$

$$\phi = 3x^2y + z^3x - zy + C \quad \underline{\text{Ans}}$$

$$\vec{F} = \nabla \phi$$

$$\begin{aligned} \nabla \phi &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (3x^2y + z^3x - zy + C) \\ &= \hat{i} (6xy + z^3) + \hat{j} (3x^2 - z) + \hat{k} (3z^2x - y) \end{aligned}$$

(Q) A vector field $\vec{F}$ is given by $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$. Show that the field is irrotational and find its scalar potential?

$$\underline{\text{Ans}} \quad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 + x & -(2xy + y) & 0 \end{vmatrix}$$

$$\begin{aligned} &= \hat{i} |0 - 0| - \hat{j} |0 - 0| + \hat{k} |-2y + 2y| \\ &= \vec{0} \end{aligned}$$

$\vec{F}$ is irrotational

$$\vec{F} = \nabla \phi$$

$$\Rightarrow (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j} = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = (x^2 - y^2 + x) \quad \text{--- (1)}$$

$$\frac{\partial \phi}{\partial y} = -(2xy + y) \quad \text{--- (2)}$$

$$\Rightarrow \partial \phi = (x^2 - y^2 + x) \partial x$$

$$\Rightarrow \int \partial \phi = \int (x^2 - y^2 + x) \partial x$$

$$\Rightarrow \phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} + A(y) \text{ --- (3)}$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = 0 - 2xy + \frac{\partial A}{\partial y} \text{ --- (4)}$$

comparing eqn (3) and (4)

$$\frac{\partial A}{\partial y} = -y$$

$$\Rightarrow \int \partial A = - \int y dy$$

$$\Rightarrow A = -\frac{y^2}{2} + C$$

$$\Rightarrow \phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2} + C \text{ --- Ans}$$

Q) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k}$ is conservative field. find its scalar potential and also the work done in moving the particle from $(1, -2, 1)$ to $(3, 1, 4)$.

Ans

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3z^2x \end{vmatrix}$$

$$= \hat{i} \left| \frac{\partial}{\partial y} 3z^2x - \frac{\partial}{\partial z} x^2 \right| - \hat{j} \left| \frac{\partial}{\partial x} 3z^2x - \frac{\partial}{\partial z} (2xy + z^3) \right| + \hat{k} \left| \frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} (2xy + z^3) \right|$$

$$= \hat{i} (0 - 0) - \hat{j} (3z^2 - 3z^2) + \hat{k} (2x - 2x)$$

$$= \vec{0}$$

$$\vec{F} = \nabla \phi$$

$$(2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k} = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 2xy + z^3 \quad \text{--- (1)}$$

$$\Rightarrow \partial \phi = (2xy + z^3) \partial x$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = x^2 \quad \text{--- (2)}$$

$$\frac{\partial \phi}{\partial z} = 3z^2x \quad \text{--- (3)}$$

Integrating both sides,

$$\int \partial \phi = (2xy + z^3) \partial x$$

$$\Rightarrow \phi = x^2y + xz^3 + A(y, z)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = x^2 + 0 + \frac{\partial A}{\partial y} \quad \text{--- (4)}$$

Comparing eqn ② and ④

$$\frac{\partial A}{\partial y} = 0$$

$\Rightarrow A$ is a function of z only

$$\Rightarrow A = B(z)$$

Now,

$$\phi = x^2y + xz^3 + B(z)$$

$$\frac{\partial \phi}{\partial z} = 0 + 3xz^2 + \frac{\partial B}{\partial z} \quad \text{--- ⑤}$$

Comparing eqn ③ and ⑤

$$\frac{\partial B}{\partial z} = 0$$

$\Rightarrow B$ is a constant.

$$\Rightarrow B = c.$$

Putting the value of B in ϕ

$$\phi = x^2y + xz^3 + c \quad \text{Ans}$$

Vector identities :-

- (1) $\text{div}(\phi \vec{F}) = \text{grad } \phi \cdot \vec{F} + \phi \text{div } \vec{F}$
- (2) $\text{curl}(\phi \vec{F}) = \text{grad } \phi \times \vec{F} + \phi \text{curl } \vec{F}$
- (3) $\text{div}(\text{grad } \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

proof :-

$$\begin{aligned} \nabla \cdot \nabla \phi &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial z} \right) \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \end{aligned}$$

$$(4) \text{curl}(\text{grad } \phi) = 0$$

proof :-

$$\begin{aligned} \nabla \times \nabla \phi &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} \\ &= \hat{i} \left| \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial y} \right) \right| - \hat{j} \left| \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial x} \right) \right| \\ &\quad + \hat{k} \left| \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) \right| \\ &= \vec{0} \end{aligned}$$

$$(5) \text{div}(\text{curl}(\vec{F})) = 0$$

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

proof :-

$$\nabla \cdot (\nabla \times \vec{F})$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i} \left| \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right| - \hat{j} \left| \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right| + \hat{k} \left| \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right|$$

$$\left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left(\hat{i} \left| \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right| - \hat{j} \left| \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right| + \hat{k} \left| \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right| \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \frac{\partial}{\partial x} \cdot \frac{\partial F_3}{\partial y} - \frac{\partial}{\partial x} \cdot \frac{\partial F_2}{\partial z} - \frac{\partial}{\partial y} \cdot \frac{\partial F_3}{\partial x} + \frac{\partial}{\partial y} \cdot \frac{\partial F_1}{\partial z} + \frac{\partial}{\partial z} \cdot \frac{\partial F_2}{\partial x} - \frac{\partial}{\partial z} \cdot \frac{\partial F_1}{\partial y}$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= \vec{0}$$

(6) $\text{curl}(\text{curl } \vec{F}) = \text{grad}(\text{div } \vec{F}) - \nabla^2 \vec{F}$

proof :-

$$\nabla \times (\nabla \times \vec{F}) = (\nabla \cdot \vec{F}) \nabla - (\nabla \cdot \nabla) \vec{F}$$

$$= \nabla \text{div } \vec{F} - \nabla^2 \vec{F}$$

$$= \text{grad}(\text{div } \vec{F}) - \nabla^2 \vec{F}$$

As we know,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

- curve, tangent, arc length :-
- parametric representation of curve, circle :-

$$(x-a)^2 + (y-b)^2 = r^2$$

$$x-a = r \cos t$$

$$\Rightarrow x = a + r \cos t$$

$$y-b = r \sin t$$

$$\Rightarrow y = b + r \sin t$$

t is parameter.

$$r(t) = [a + r \cos t, b + r \sin t, 0]$$

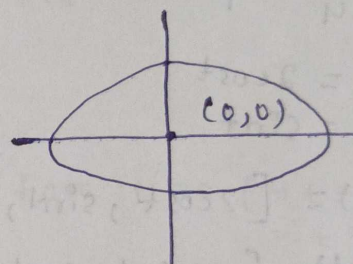
• ellipse :-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x = a \cos t$$

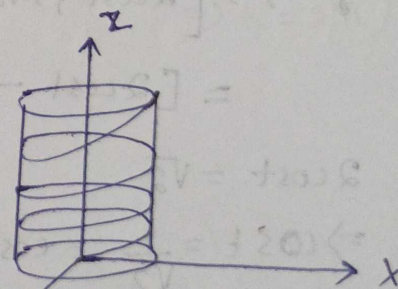
$$y = b \sin t$$

$$r(t) = [a \cos t, b \sin t, 0]$$



- circular helix :-

$$r(t) = [a \cos t, a \sin t, ct]$$



- straight line :-

→ The parametric representation of a straight line passing through the point 'A' having position vector $\vec{a}$ and along the direction of $\vec{b}$ is given by,

$$r(t) = \vec{a} + t\vec{b}$$

Derive the parametric representation of straight line through the point $A(4, 2, 0)$ in the direction of the vector $\vec{b} = \hat{i} + \hat{j}$.

$$\vec{a} = 4\hat{i} + 2\hat{j}$$

$$r(t) = \vec{a} + t\vec{b} \\ = (4\hat{i} + 2\hat{j}) + t(\hat{i} + \hat{j})$$

$$= 4\hat{i} + 2\hat{j} + (t\hat{i} + t\hat{j})$$

$$= (4+t)\hat{i} + (2+t)\hat{j}$$

$$r(t) = [(4+t), (2+t)]$$

• Equation of tangent :-

→ The Equation of tangent of the curve $r(t)$ on a point (p)

$$g(w) = r(t) + wr'(t)$$

w = parameter.

(Q) Find the tangent to the curve ellipse $\frac{1}{4}x^2 + y^2 = 1$ at point $P(\sqrt{2}, \frac{1}{\sqrt{2}})$

Ans $\frac{x^2}{4} + \frac{y^2}{1} = 1$

$$x = 2\cos t$$

$$y = \sin t$$

$$r(t) = [2\cos t, \sin t, 0]$$

$$r'(t) = [-2\sin t, \cos t, 0]$$

$$g(w) = [2\cos t, \sin t, 0] + w[-2\sin t, \cos t, 0]$$

$$= [2\cos t - 2w\sin t, \sin t + w\cos t, 0]$$

$$2\cos t = \sqrt{2}$$

$$\Rightarrow \cos t = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4}$$

$$\Rightarrow t = \frac{\pi}{4}$$

$$= \left[\sqrt{2} \times \frac{1}{\sqrt{2}} - \sqrt{2} w \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} + w \frac{1}{\sqrt{2}}, 0 \right]$$

$$= \left[\sqrt{2} - \sqrt{2}w, \frac{1}{\sqrt{2}} + w \frac{1}{\sqrt{2}}, 0 \right]$$

↓
Tangent
Equation

Length of the curve :-

The length of the curve $r(t)$ from the point $t=a$ to $t=b$ is given by ,

$$l = \int_{t=a}^{t=b} \sqrt{r'(t) \cdot r'(t)} \cdot dt$$

$$r(t) = -a \sin t \hat{i} + a \cos t \hat{j} + 4 \hat{k}$$

$$r'(t) \cdot r'(t) = \left[\sqrt{(-a \sin t)^2 + (a \cos t)^2 + 4^2} \right]^2$$

$$= a^2 \sin^2 t + a^2 \cos^2 t + 16$$

$$= a^2 + 16$$

$$l = \int_{t=0}^{t=2\pi} \sqrt{a^2 + 16} \, dt$$

$$= \sqrt{a^2 + 16} [t]_0^{2\pi}$$

$$= 2\pi \sqrt{a^2 + 16}$$

$$\text{For } t=0 \quad (4t=0) \\ \Rightarrow t=0$$

$$t=2\pi \quad (4t=8\pi) \\ \Rightarrow t=2\pi$$

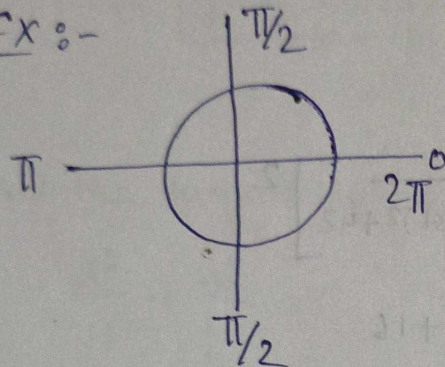
✓ periodic Functions :-

→ A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be periodic if there exists a positive number T such that $f(x+T) = f(x)$ $\forall$ real numbers x and T is called a period of f .

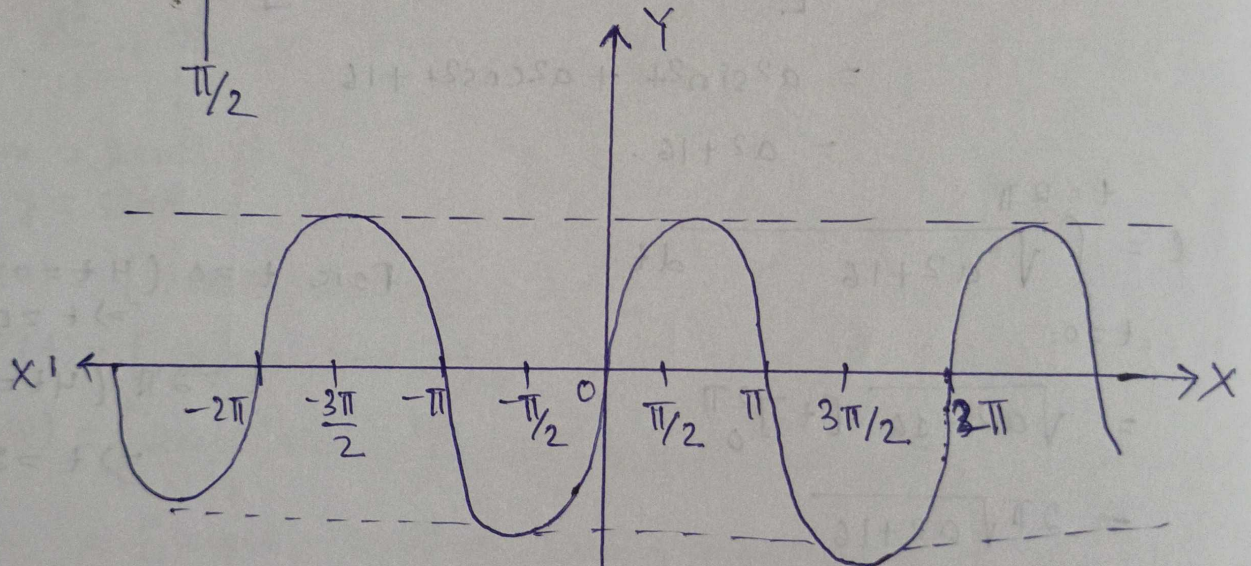
✓ primitive period of f :-

→ If a periodic function has a smallest positive period T , then T is called the primitive period of f .

Ex :-



Range of $\sin x = [-1, 1]$



Let $f(x) = \sin x$.

$$f(x+2\pi) = \sin(2\pi+x)$$

$$= \sin(360^\circ + x)$$

$$= \sin x$$

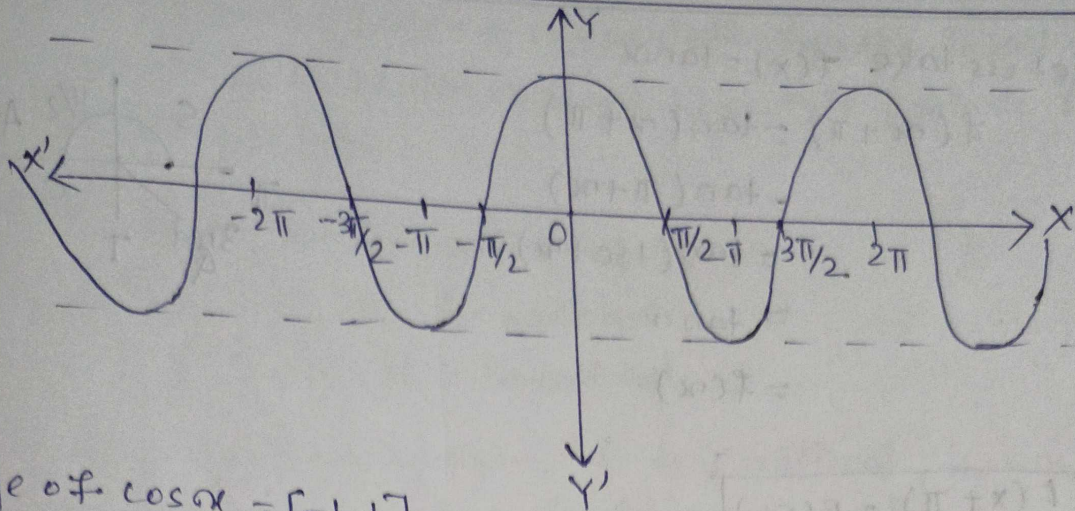
$$= f(x)$$

$\therefore 2\pi$ is the period of $\sin x$

Here, $T = 2\pi$

for $\cos x$:-

P.T.O. $\rightarrow$



Range of $\cos x = [-1, 1]$

Let $f(x) = \cos x$

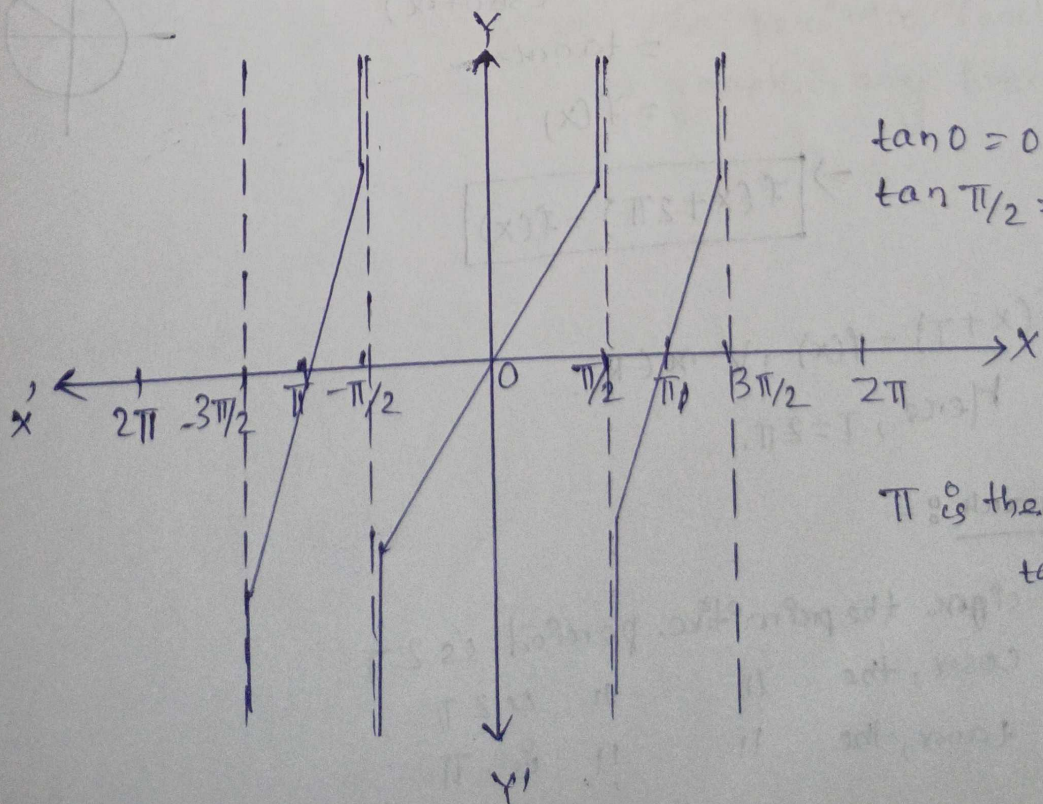
$$\begin{aligned} f(x+2\pi) &= \cos(2\pi+x) \\ &= \cos(360^\circ+x) \\ &= \cos x \\ &= f(x) \end{aligned}$$

$$\Rightarrow \boxed{f(x+2\pi) = f(x)} \quad \forall x \in \mathbb{R}$$

$$\therefore \boxed{f(x+T) = f(x)} \quad \forall x \in \mathbb{R}$$

$$\boxed{T=2\pi} \rightarrow \text{period of } \cos x.$$

$\rightarrow$ period of f

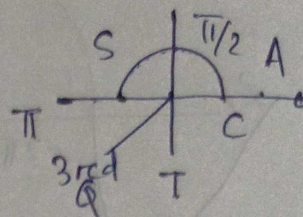


$$\begin{aligned} \tan 0 &= 0 \\ \tan \pi/2 &= \infty \end{aligned}$$

π is the period of $\tan x$.

Let us take $f(x) = \tan x$

$$\begin{aligned}
 f(x+\pi) &= \tan(x+\pi) \\
 &= \tan(\pi+x) \\
 &= \tan(180+x) \\
 &= \tan x \\
 &= f(x)
 \end{aligned}$$



$$\boxed{f(x+\pi) = f(x)}$$

$$f(x+T) = f(x), \forall x \in \mathbb{R}$$

Here, $T = \pi$ is the period of $\tan x$ which is known as primitive period of $\tan x$



This is the smallest +ve period : $(\pi + x)$

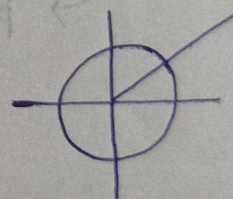
Again,

since

$$\text{Let } f(x) = \tan x$$

$$\begin{aligned}
 f(x+2\pi) &= \tan(2\pi+x) \\
 &= \tan(360+x) \\
 &= \tan x \\
 &= f(x)
 \end{aligned}$$

$$\Rightarrow \boxed{f(x+2\pi) = f(x)}$$



$$f(x+T) = f(x), \forall x \in \mathbb{R}$$

Here, $T = 2\pi$.

Remark :-

for $\sin x$ the primitive period is 2π

$\cos x$, the " " is 2π

$\tan x$, the " " is π

- (2) The constant function $f(x) = c$ is a periodic function. In fact every real number is a period of f and hence this periodic function has no primitive period.

- (3) let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

let w be any rational number, if x is rational, then $x+w$ is also rational and if x is irrational, then $x+w$ is also irrational. Hence,

$$\begin{aligned} f(x+w) &= \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases} \\ &= f(x) \end{aligned}$$

Hence every rational number is a period of f and f has no primitive period.

• Remark :-

- Let f be a periodic function with period w . If the values $f(x)$ are known in an interval of length w then by periodicity, $f(x)$ can be determined for all x . Hence the graph of a periodic function is obtained by periodic repetition of its graph in any interval of length w .

Let us take $f(x) = \tan x$

$$\begin{aligned} f(x+\pi) &= \tan(x+\pi) \\ &= \tan(\pi+x) \\ &= \tan(180^\circ+x) \\ &= \tan x \\ &= f(x) \end{aligned}$$

$$\therefore \boxed{f(x+\pi) = f(x)}$$

→ $f(x+\pi) = f(x) \forall x \in \mathbb{R}$,
→ period of f .

Here $(T = \pi)$ is the period of $\tan x$ which is known as primitive period of $\tan x$.



π is the smallest positive period.

Again :- since

$$\text{let } f(x) = \tan x$$

$$\begin{aligned} f(x+2\pi) &= \tan(2\pi+x) \\ &= \tan(360^\circ+x) \\ &= \tan x \\ &= f(x) \end{aligned}$$

$$\Rightarrow \boxed{f(x+2\pi) = f(x)}$$

$$\hookrightarrow f(x+T) = f(x) \quad \forall x \in \mathbb{R}$$

$$\text{Here } T = 2\pi.$$

- Remark :- For $\sin x$, the primitive period is 2π .
 $\cos x$, the primitive period is 2π .
 $\tan x$, the primitive period is π .

Dt :- 10/6/23

(Q) Let f and g be periodic functions with period w and let a & b be real numbers prove that $(af + bg)$ is also a periodic function with period w .

Ans

Let f and g are periodic with period w .

$$f(x+T) = f(x) \quad \text{--- (1)}$$

$$\& \quad g(x+T) = g(x) \quad \text{--- (2) } \forall x$$

$$\begin{aligned} \text{Now, } (af + bg)(x+T) &= af(x+T) + bg(x+T) \\ &= af(x) + bg(x). \end{aligned}$$

from (1) & (2).

$$\begin{aligned} &= (af + bg) \\ &\Rightarrow \boxed{(af + bg)(x+T) = (af + bg)(x)} \end{aligned}$$

Hence $(af + bg)$ is periodic with period T .

(Q) If T is a period of f , prove that nT where $n \in \mathbb{Z}$ is a period of f .
 $f(x) = f(x+T) = f(x+2T) = \dots$
 $f(x+(n-1)T) = f(x+nT)$

$$\Rightarrow f(x) = f(x+nT)$$

$$\Rightarrow f(x+nT) = f(x)$$

So it follows that (nT) is a period of f .

(Q) Let n be any positive integer. prove that $\sin nx$ is a periodic function with period $2\pi/n$.

Ans Since $\sin x$ is a periodic function with period 2π , we have,

$$\sin(x+2\pi) = \sin x \quad \forall x \quad \text{--- (1)}$$

Now let $g(x) = \sin(nx)$

$$\begin{aligned} \text{Then, } g(x + \frac{2\pi}{n}) &= \sin\left[n\left(x + \frac{2\pi}{n}\right)\right] \\ &= \sin\left(nx + n \cdot \frac{2\pi}{n}\right) \\ &= \sin(nx + 2\pi) \\ &= \sin nx = g(x) \end{aligned}$$

$$g\left(x + \frac{2\pi}{n}\right) = g(x)$$

Compare to,

$$g(x+T) = g(x) \quad \forall x.$$

$$\therefore T = \frac{2\pi}{n}$$

Hence $g(x) = \sin nx$ is a periodic function with period $\left(\frac{2\pi}{n}\right)$

$$\therefore \text{periodic of } \sin 2x = \frac{2\pi}{2} = \pi.$$

(Q) Let $f(x)$ be a periodic function with period ω . prove that for any positive real no (a) , $f(ax)$ is a periodic function with period (ω/a) .

We have $f(x+T) = f(x) \quad \forall x.$

let $g(x) = f(ax)$

$$\text{Now, } g\left(x + \frac{T}{a}\right) = f\left(a\left(x + \frac{T}{a}\right)\right) = f(ax+T) = f(ax) = g(x).$$

$$\Rightarrow g\left(x + \frac{T}{a}\right) = f\left(ax + \frac{T}{a}\right) = f(ax+T) = f(ax) = g(x).$$

$\therefore g(x)$ is a periodic function with period $\frac{\omega}{a}$.

$$g\left(x + \frac{1}{a}\right) = g(x)$$

compare to

$$g(x+T) = g(x)$$

$$\Rightarrow T = \frac{1}{a}$$

Periodic Functions:-

→ We know that a periodic function of period λ is such that:

$$f(x+\lambda) = f(x) \quad \forall x$$

Theorem:- For a periodic function of period 2π , prove that:-

$$\int_a^B f(x) dx = \int_{a+2\pi}^{B+2\pi} f(x) dx$$

Ans For a periodic function of period 2π , we know that $f(t-2\pi) = f(t)$.
Hence, putting $t-2\pi = x$.

$$\Rightarrow t = x + 2\pi$$

For all $x \in B$, we get

$$\int_a^B f(x) dx = \int_{a+2\pi}^{B+2\pi} f(t-2\pi) dt$$

$$= \int_{a+2\pi}^{B+2\pi} f(t) dt$$

$$= \int_{a+2\pi}^{B+2\pi} f(x) dx$$

$$\int_{-\pi}^{\pi} f(x) dx = \int_{a+2\pi}^{a+2\pi} f(x) dx$$

$$\int_a^{a+2\pi} f(x) dx = \int_a^c f(x) dx + \int_{-\pi}^{\pi} f(x) dx + \int_{\pi}^{a+2\pi} f(x) dx$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$= \int_{-\pi}^{\alpha} f(x) dx + \int_{-\pi}^{\pi} f(x) dx + \int_{-\pi}^{\alpha} f(x) dx$$

$$= \int_{-\pi}^{\pi} f(x) dx + \int_{-\pi}^{\pi} f(x) dx - \int_{\alpha}^{\pi} f(x) dx$$

$$= \int_{-\pi}^{\pi} f(x) dx$$

from ①

$$\int_{\alpha}^{\beta} f(x) dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) dx = \int_{\alpha}^{\alpha+2\pi} f(x) dx$$

$$= \int_{\alpha+2\pi}^{\beta+2\pi} f(x) dx$$

$$= \int_{\alpha+2\pi-2\pi}^{\beta+2\pi-2\pi} f(x) dx$$

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} f(v+x) dx$$

Where α, β, γ being any numbers.

let $v+x = t$

When

$$\boxed{\begin{array}{l} x = -\pi, t = v - \pi \\ x = \pi, t = v + \pi \end{array}}$$

$$\therefore \int_{-\pi}^{\pi} f(v+x) dx = \int_{v-\pi}^{v+\pi} f(t) dt$$

$$= \int_{v-\pi}^{-\pi} f(t) dt + \int_{-\pi}^{\pi} f(t) dt + \int_{\pi}^{v+\pi} f(t) dt$$

$$= \int_{v-\pi+2\pi}^{-\pi+2\pi} f(t) dt + \int_{-\pi}^{\pi} f(t) dt - \int_{v+\pi}^{\pi} f(t) dt$$

$$= \int_{v+\pi}^{-\pi} f(t) dt + \int_{-\pi}^{\pi} f(t) dt - \int_{v+\pi}^{\pi} f(t) dt$$

$$= \int_{-\pi}^{\pi} f(t) dt$$

$$\left[\begin{array}{l} \int_{\alpha}^{\beta} f(x) dx \\ = \int_{\alpha+2\pi}^{\beta+2\pi} f(x) dx \end{array} \right]$$

Hence $\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} f(v+x) dx$

(proved)

Even and odd functions :-

Even Functions :- A function $f(x)$ is said to be even if $f(-x) = f(x)$.

EX :- $f(x) = x^2$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$$\Rightarrow \boxed{f(-x) = f(x)}$$

So, x^2 is an even function.

Thus, $\cos x$, $\sec x$, x^2 , x^4 etc are even function.

Remark :- In general all functions having even powers are even functions.

EX :- $\sin^n x$, x^n , $\cos^n x$ where n is an even +ve integer are even functions.

ODD Functions :- A function $f(x)$ is said to be odd function.

$$\boxed{f(-x) = -f(x)}$$

EX :- let $f(x) = x^3$

$$\Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x)$$

$$\Rightarrow \boxed{f(-x) = -f(x)}$$

$\therefore x^3$ is an odd function

Remark :- In general, all functions having odd powers are odd functions.
eg :- $\sin^n x$, x^n , where n is an odd integer.

Thus, $\sin x$, $\csc x$, $\tan x$ are odd functions.

→ $\sin x$ is odd function.

proof :- let $f(x) = \sin x$,

$$f(-x) = \sin(-x)$$

$$\Rightarrow \sin x = -f(x)$$

$$\Rightarrow \boxed{f(-x) = -f(x)}$$

$\therefore \sin x$ is an odd function.

→ $\cos x$ is an even function.

$$\therefore \text{let } f(x) = \cos x$$

$$\Rightarrow f(-x) = \cos(-x)$$

$$= \cos x = f(x)$$

$$\Rightarrow \boxed{f(-x) = f(x)}$$

$\therefore \cos$ is an even function.

Dt :- 23/06/23.

✓ Integrating By parts :-

Formula :-

$$\int u \cdot v \, dx = u \cdot \int v \, dx - \int \frac{d}{dx}(u) \left(\int v \, dx \right) dx.$$

ILATE

(Q) ✓ $\int e^x \cdot x \, dx.$

Exponential algebraic.

Ans $\int x \cdot e^x \, dx.$

$$= x \cdot \int e^x \, dx - \int \frac{d}{dx}(x) \left(\int e^x \, dx \right) dx.$$

$$= x \cdot e^x - \int 1 \cdot e^x \, dx + c.$$

$$= x \cdot e^x - \int e^x \, dx + c$$

$$= x e^x - e^x + c.$$

(Q) ✓ $\int \log x \cdot dx$

Ans $\int \log x \cdot 1 \, dx.$

$$= \log x \cdot \int 1 \, dx - \int \frac{d}{dx}(\log x) \cdot \int 1 \, dx \, dx.$$

$$= x \cdot \log x - \int \frac{1}{x} \cdot x \, dx.$$

$$= x \cdot \log x - x + c.$$

Dt :- 28/06/23.

some useful formula :-

$$(1) \int_{-a}^0 f(x) dx = 0 \text{ if } f(x) \text{ is an odd function.}$$

$$(2) \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(x) \text{ is an even function.}$$

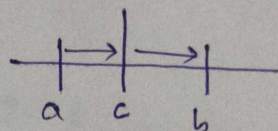
$$(3) \int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(z) dz = f(b) - f(a)$$

$$(4) \int_a^b f(x) dx = - \int_b^a f(x) dx.$$

$$(5) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

where $a < c < b$.

When



$$(6) \int_a^b f(x) dx = \int_a^b f(a+b-x) dx.$$

$$(7) \int_0^a f(x) dx = \int_0^a f(a-x) dx.$$

$$(8) \int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx.$$

$$(9) \int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx & \text{if } f(2a-x) = f(x) \\ & \text{if } f(2a-x) = -f(x). \end{cases}$$

Ex:- $\int_0^{\pi} \sin x dx.$

$$\begin{aligned} \int_0^{2 \cdot \pi/2} \sin x dx &= 2 \int_0^{\pi/2} \sin x dx = 2 \cdot [-\cos x]_0^{\pi/2} = 2 \cdot [-(\cos \frac{\pi}{2} - \cos 0)] \\ &= 2 \cdot [-(0-1)] \\ &= 2 \text{ (proven)} \end{aligned}$$

$$\begin{aligned} \text{let } f(x) &= \sin x \\ f(2 \cdot a - x) &= f(2 \cdot \pi/2 - x) \\ &= \sin(\pi - x) \\ &= \sin x. \end{aligned}$$

$$\boxed{f(2 \cdot (\frac{\pi}{2}) - x) = f(x)}$$

Other important results for solving fourier series problems :-
 where, m & n are integers.

$$(1) \int_{\alpha}^{\alpha+2\pi} \cos x \cdot dx = \int_{\alpha}^{\alpha+2\pi} \sin nx \cdot dx = 0 \text{ where } n \text{ is a positive integer.}$$

$$(2) \int_{\alpha}^{\alpha+2\pi} \cos nx \cdot \cos mx \cdot dx = \int_{\alpha}^{\alpha+2\pi} \sin mx \cdot \sin nx \cdot dx = 0 \text{ where } m \neq n \text{ (} m \text{ \& } n \text{ are positive integers).}$$

$$(3) \int_{\alpha}^{\alpha+2\pi} \sin nx \cdot \cos nx \cdot dx = \int_{\alpha}^{\alpha+2\pi} \cos nx \cdot \sin nx \cdot dx = 0 \text{ where } n \neq 0.$$

$$(4) \int_{\alpha}^{\alpha+2\pi} \cos 2nx \cdot dx = \int_{\alpha}^{\alpha+2\pi} \sin^2 nx \cdot dx = \pi \text{ where } n \neq 0.$$

$$(5) \int_{-\pi}^{\pi} \cos m x \cdot \cos n x \cdot dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n > 0 \\ 2\pi, & m = n = 0 \end{cases}$$

$$(6) \int_{-\pi}^{\pi} \sin m x \cdot \sin n x \cdot dx = \begin{cases} 0, & \text{if } m \neq n \text{ \& } m = n = 0, \\ \pi, & \text{if } m = n > 0. \end{cases}$$

$$(7) \int_{-\pi}^{\pi} \cos m x \cdot \sin n x \cdot dx = 0.$$

Following results will be found useful while attempting problems of fourier series :-

$$(1) \boxed{\sin n\pi = \cos(n + \frac{1}{2})\pi = 0 \quad n \in \mathbb{I}}$$

$$\text{if } n = 1$$

$$\sin nx = \cos(\frac{3\pi}{2}) = 0.$$

$$\boxed{\begin{aligned} \sin \pi &= \sin 2\pi = \sin 3\pi = \dots = 0. \\ \& \cos \frac{\pi}{2} &= \cos \frac{3\pi}{2} &= \cos \frac{5\pi}{2} &= \dots = 0 \end{aligned}}$$

$$(2) \boxed{\cos n\pi = \sin(n + \frac{1}{2})\pi = (-1)^n}, n \in \mathbb{I}$$

$$\text{let } n = 1, \cos \pi = -1$$

$$\sin(\frac{3\pi}{2}) = -1$$

$$\begin{aligned}\cos \pi &= \cos 3\pi = \cos 5\pi = \dots = -1 \\ \cos 0 &= \cos 2\pi = \cos 4\pi = \dots = 1 \\ \sin \pi/2 &= \sin 5\pi/2 = \sin 9\pi/2 = \dots = 1 \\ \sin 3\pi/2 &= \sin 7\pi/2 = \sin 11\pi/2 = \dots = -1\end{aligned}$$

(3)

$$\int e^{ax} \cdot \sin bx \cdot dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx] + C$$

$$\int e^{ax} \cdot \cos bx \cdot dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] + C$$

Dt :- 12/7/23

Fourier series :-

→ A series of the form $a_0 + a_1 \cos x + b_1 \sin x + (a_2 \cos 2x + b_2 \sin 2x) + \dots + (a_n \cos nx + b_n \sin nx) + \dots = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ Where " a_n & b_n " are real constants ~~are~~ is called a trigonometric series. The constants a_n & b_n are known as the real coefficient of the series.

→ Since each term of the trigonometric series is a function of period (2π) .

→ Let $f(x)$ be a periodic function of period 2π which can be repeated as $f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) + \dots + (a_n \cos nx + b_n \sin nx) + \dots$
 $= \frac{a_0}{2} + (a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + a_n \cos nx) \dots$
 $(b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots + b_n \sin nx) \dots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

This is known as Fourier series and its coefficients a_n , b_n & a_0 are called Fourier co-efficients of $f(x)$.
 Where, $n = 1, 2, 3, \dots$

✓ Euler's formula :-

The fourier series of the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

where $a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \cdot dx \quad \& \quad b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \cdot dx$$

These values a_0 , a_n and b_n are known as ~~EULER'S~~ EULER'S FORMULA

① Remark :-

Making $\alpha = 0$, the integral becomes $0 < x < 2\pi$.

Formula (1) reduces to

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \cdot dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \cdot dx$$

② Remark :-

put $\alpha = -\pi$

$$\alpha + 2\pi = -\pi + 2\pi$$

$$= \pi$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \cdot dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \cdot dx$$

$$\int_{\alpha}^{\alpha+2\pi} \rightarrow \int_{-\pi}^{\pi}$$

Working Rule :-

Let $f(x)$ be a periodic function with period 2π .
if the given interval is $[-\pi, \pi]$

STEP-1 :-

check whether $f(x)$ is an even or odd function.

STEP-2 :-

(a) If $f(x)$ is even function then $a_n = 0$ for all n and $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos nx \, dx$ for all $n \geq 0$.

(b) if $f(x)$ is odd function then $a_n = 0$ for all n and $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$ for all $n \geq 0$.

$$\begin{cases} \int_{-\pi}^{\pi} f(x) \, dx \\ = 2 \cdot \int_0^{\pi} f(x) \, dx \\ \rightarrow \text{for Even} \end{cases}$$

STEP-3 :-

→ If $f(x)$ is neither an even nor odd function in $(-\pi, \pi)$. if the given interval is not $(-\pi, \pi)$ then calculate the Fourier co-efficient by using EULERS FORMULA.

Dt :- 19/07/23.

Q.1) Find the Fourier expansion of $f(x) = x$ in $-\pi < x < \pi$
let us take $f(x) = x$ in $-\pi < x < \pi$

$$\therefore f(-x) = -x = -f(x)$$

So, $f(x)$ is an odd function.

According to conditions of Fourier series :-

$$\boxed{a_n = 0} \quad \forall n$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx \cdot dx$$

$$= \frac{2}{\pi} \left[x \int_0^{\pi} \sin nx \, dx - \int_0^{\pi} \left(\frac{d}{dx} \cdot x \cdot \int_0^{\pi} \sin nx \, dx \right) \right]$$

$$= \frac{2}{\pi} \left[x \left(-\frac{1}{n} \cos nx \right) \right]_0^{\pi} - \int_0^{\pi} \left[-\frac{1}{n} \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{n} \cos \pi n \right] - \int_0^{\pi} \left[-\frac{1}{n} \cos \pi n - \frac{1}{n} \cos 0^n \right]$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{n} (-1)^n \right] - \int_0^{\pi} \left[-\frac{1}{n} (-1)^n - \frac{1}{n} (1)^n \right]$$