

Multiple Integral

$\int$ is used to find length

$\iint$ is used to find Area

$\iiint$ is used to find volume

$$\int_a^b f'(x) dx = f(b) - f(a)$$

Type-1

e.g. $\int_0^1 \int_0^2 (x+y) dx dy$

$$= \int_0^1 \left[\int_0^2 (x+y) dx \right] dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy \right]_0^2 dy$$

$$= \int_0^1 (2 + 2y) dy$$

$$= \left[2y + \frac{2y^2}{2} \right]_0^1$$

$$= \left(2 + \frac{2}{2} \right) - (0 + 0) = 3$$

e.g. $\int_1^2 \int_3^4 (xy + e^y) dy dx = \underline{\hspace{2cm}}$

e.g.: $\int_0^1 \int_0^1 \frac{dx dy}{\sqrt{(1-x^2)(1-y^2)}}$

$$= \int_0^1 \left(\int_0^1 \frac{dx}{\sqrt{1-x^2}} \right) \frac{dy}{\sqrt{1-y^2}}$$

$$= \int_0^1 [\sin^{-1} x]_0^1 \frac{dy}{\sqrt{1-y^2}}$$

$$= \int_0^1 (\sin^{-1} 1 - \sin^{-1} 0) \frac{dy}{\sqrt{1-y^2}}$$

$$= \frac{\pi}{2} \int_0^1 \frac{dy}{\sqrt{1-y^2}}$$

$$= \frac{\pi}{2} [\sin^{-1} y]_0^1 = \frac{\pi}{2} (\sin^{-1} 1 - \sin^{-1} 0)$$

$$= \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

e.g. Prove that

$$\int_1^2 \int_3^4 (xy + e^y) dy dx = \int_3^4 \int_1^2 (xy + e^y) dx dy$$

Ans $\int_1^2 \int_3^4 (xy + e^y) dy dx$

$$= \int_1^2 \left[\frac{xy^2}{2} + e^y \right]_3^4 dx$$

$$= \int_1^2 \left\{ (8x + e^4) - \left(\frac{9x}{2} + e^3 \right) \right\} dx$$

$$= \int_1^2 \left(\frac{7x}{2} + e^4 - e^3 \right) dx$$

$$= \left[\frac{7x^2}{4} + e^4 x - e^3 x \right]_1^2$$

$$= (7 + 2e^4 - 2e^3) - \left(\frac{7}{4} + e^4 - e^3 \right)$$

$$= \frac{21}{4} + e^4 - e^3$$

now, $\int_3^4 \int_1^2 (xy + e^y) dx dy$

$$= \int_3^4 \left[\frac{x^2 y}{2} + x e^y \right]_1^2 dy$$

$$= \int_3^4 \left(2y + 2e^y - \frac{y}{2} - e^y \right) dy$$

$$= \int_3^4 \left(\frac{3y}{2} + e^y \right) dy$$

$$= \left[\frac{3y^2}{4} + e^y \right]_3^4$$

$$= \left(\frac{48}{4} + e^4 \right) - \left(\frac{27}{4} + e^3 \right)$$

$$= \frac{21}{4} + e^4 - e^3$$

so L.H.S = R.H.S

e.g.: Prove that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx$

Q: Evaluate the following

(i) $\int_1^2 \int_0^1 \frac{dx dy}{x^2 + y^2}$

(ii) $\int_0^a \int_0^b (x^2 + y^2) dx dy$

(iii) $\int_0^3 \int_0^1 (x^2 + 3y^2) dy dx$

(iv) $\int_0^3 dx \int_1^2 xy(1+x+y) dy$

(v) $\int_1^a \int_1^b \frac{dy dx}{xy}$

Type-2

$\int_1^2 \int_0^{3y} y dx dy$

First we integrate this w.r.t x and then we put the values in place of x .

$\int_1^2 \left[\int_0^{3y} y dx \right] dy$

$= \int_1^2 [xy]_0^{3y} dy$

$$= \int_1^2 (3y^2 - 0) dy$$

$$= \int_1^2 3y^2 dy = \left[\frac{3y^3}{3} \right]_1^2$$

$$= 8 - 1 = 7$$

Q: $\int_0^1 \int_x^{2x} 5x^2 y \, dy \, dx$

here we first integrate w.r.t. y then

we put the values in place of y

$$= \int_0^1 \left[\int_x^{2x} 5x^2 y \, dy \right] dx$$

$$= \int_0^1 \left[\frac{5x^2 y^2}{2} \right]_x^{2x} dx$$

$$= \int_0^1 \left\{ \frac{5x^2 \cdot 4x^2}{2} - \frac{5x^2 x^2}{2} \right\} dx$$

$$= \int_0^1 \left(\frac{20x^4 - 5x^4}{2} \right) dx$$

$$= \frac{15}{2} \int_0^1 x^4 dx = \frac{15}{2} \left[\frac{x^5}{5} \right]_0^1$$

$$= \frac{15}{2} \left(\frac{1}{5} - 0 \right)$$

$$= \frac{3}{2}$$

Q: evaluate the following

$$(i) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-x}^{1+x} (x^2+y) dy dx$$

$$(ii) \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy dx}{1+x^2+y^2}$$

$$(iii) \int_0^1 dx \int_0^x e^{\frac{y}{x}} dy$$

$$(iv) \int_1^2 \int_0^x \frac{dy dx}{x^2+y^2}$$

$$(v) \int_0^1 \int_y^{y^2+1} x^2 y dx dy$$

$$(vi) \int_0^1 \int_x^{\sqrt{x}} (x^2+y^3) dy dx$$

$$(vii) \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2+y^2) dy dx$$

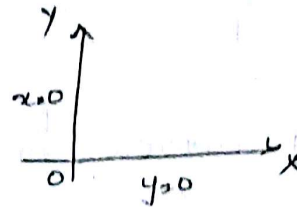
Equation

line / curve

①

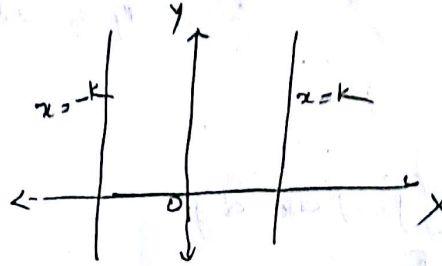
$$x = 0$$

$$y = 0$$



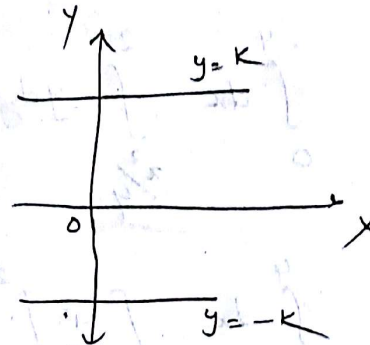
②

$$x = k \text{ (constant)}$$



③

$$y = k \text{ (constant)}$$



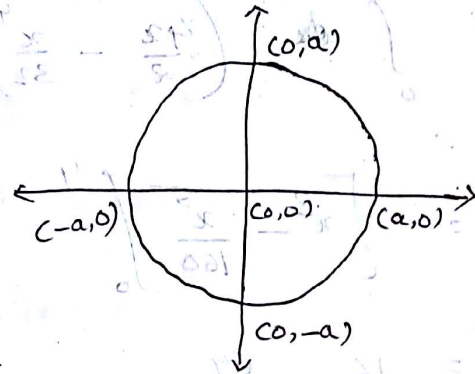
④

$$x^2 + y^2 = a^2$$

(circle)

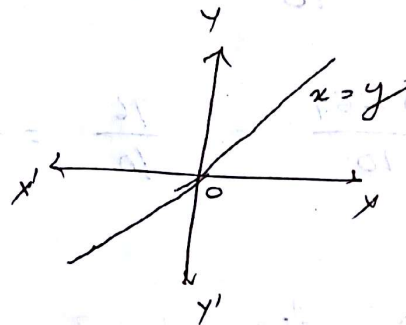
centre = (0, 0)

radius = a



⑤

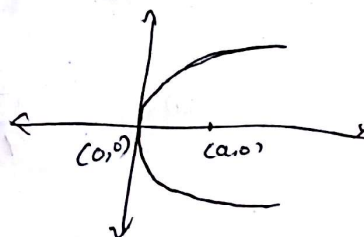
$$x = y \text{ or } y = x$$



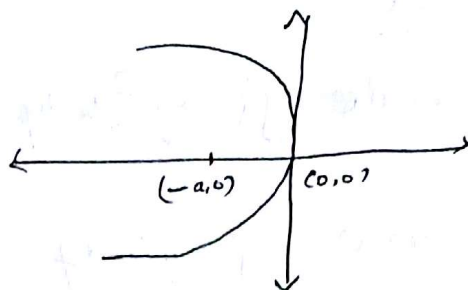
⑥

$$y^2 = ax$$

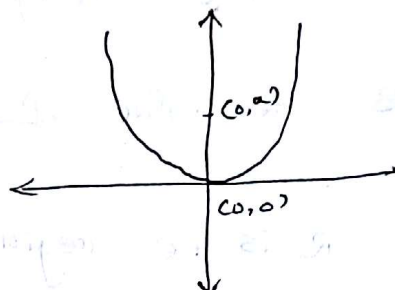
parabola
open towards right



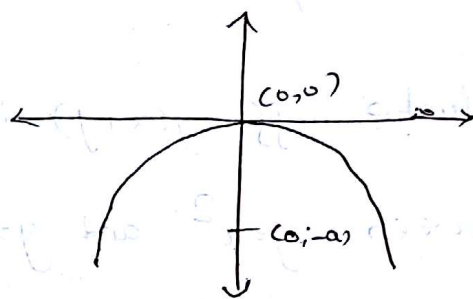
⑦ $y^2 = -ax$
 parabola
 open towards
 left



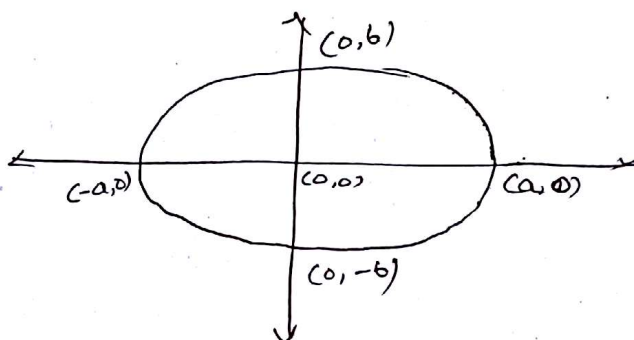
⑧ $x^2 = ay$
 parabola
 open upwards



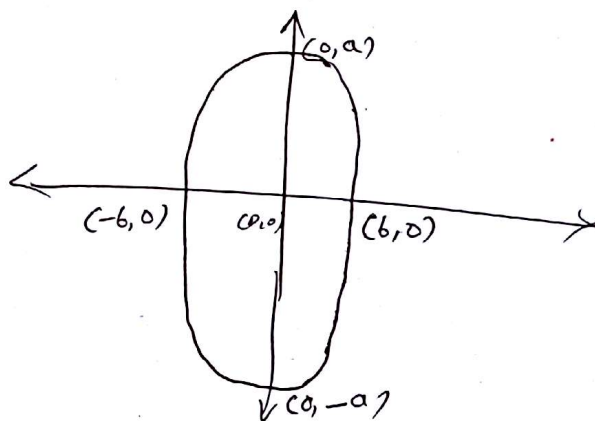
⑨ $x^2 = -ay$
 parabola
 open downwards



⑩ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
 ellipse
 || to major axis
 (x-axis)



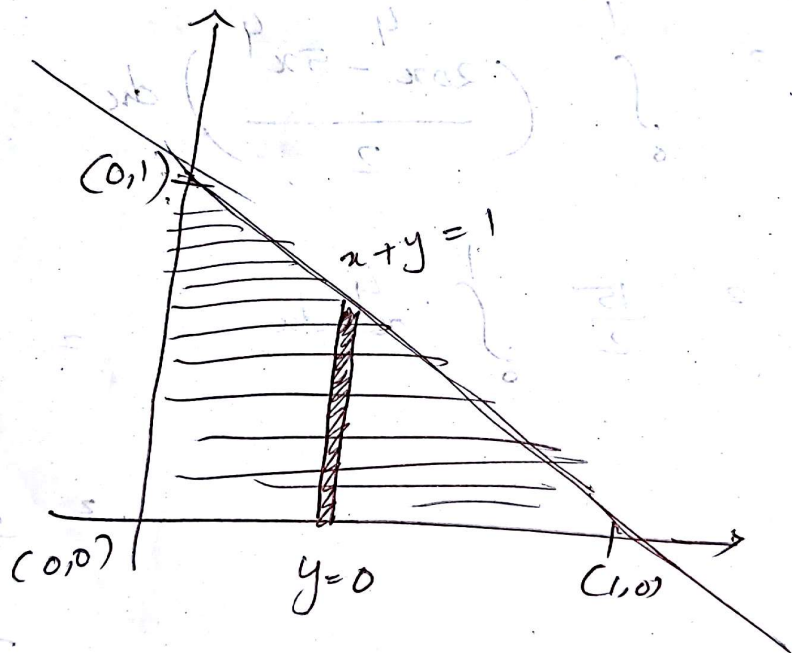
⑪ $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
 ellipse
 || to minor
 axis
 (y-axis)



Type-3 Evaluate $\iint (x^2+y^2) dx dy$ over the region in the (+)ve quadrant for which $x+y \leq 1$

Ans $\rightarrow$

$$x+y=1$$
$$\Rightarrow y=1-x$$



here x varies from 0 to 1

$$\text{so } 0 \leq x \leq 1$$

and y varies from $y=0$ to $y=1-x$

$$\text{so } 0 \leq y \leq 1-x$$

Now
$$\iint (x^2 + y^2) \cdot dx \, dy$$

$$= \int_0^1 dx \int_0^{1-x} (x^2 + y^2) dy$$

$$= \int_0^1 dx \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x}$$

$$= \int_0^1 dx \left\{ x^2 (1-x) + \frac{(1-x)^3}{3} \right\}$$

$$= \int_0^1 \left\{ x^2 - x^3 + \frac{(1-x)^3}{3} \right\} dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{(1-x)^4}{-12} \right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4} + 0 \right) - \left(0 - 0 - \frac{1}{12} \right)$$

$$= \frac{4 - 3 + 1}{12}$$

$$= \frac{2}{12} = \frac{1}{6}$$

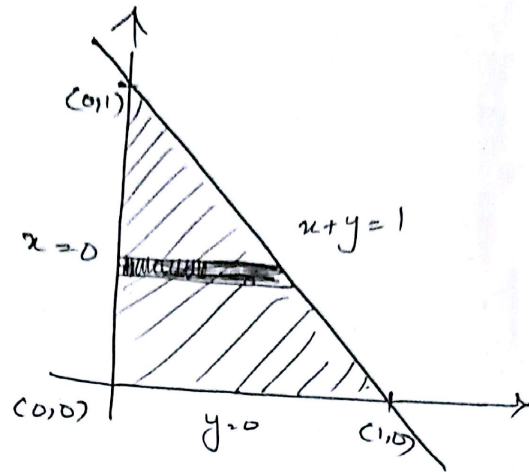
(Ans)

or $\iint (x^2+y^2) dx dy$ over the quadrant
of $x+y \leq 1$

here y varies

from 0 to 1

x varies from
0 to $1-y$



$$\text{so } \int_0^1 dy \int_0^{1-y} (x^2+y^2) dx$$

$$\begin{aligned} x+y &= 1 \\ \Rightarrow x &= 1-y \end{aligned}$$

$$= \int_0^1 dy \left[\frac{x^3}{3} + xy^2 \right]_0^{1-y}$$

$$= \int_0^1 \left\{ \frac{(1-y)^3}{3} + (1-y)y^2 \right\} dy$$

$$= \int_0^1 \left\{ \frac{(1-y)^3}{3} + y^2 - y^3 \right\} dy$$

$$= \left[\frac{(1-y)^4}{-12} + \frac{y^3}{3} - \frac{y^4}{4} \right]_0^1$$

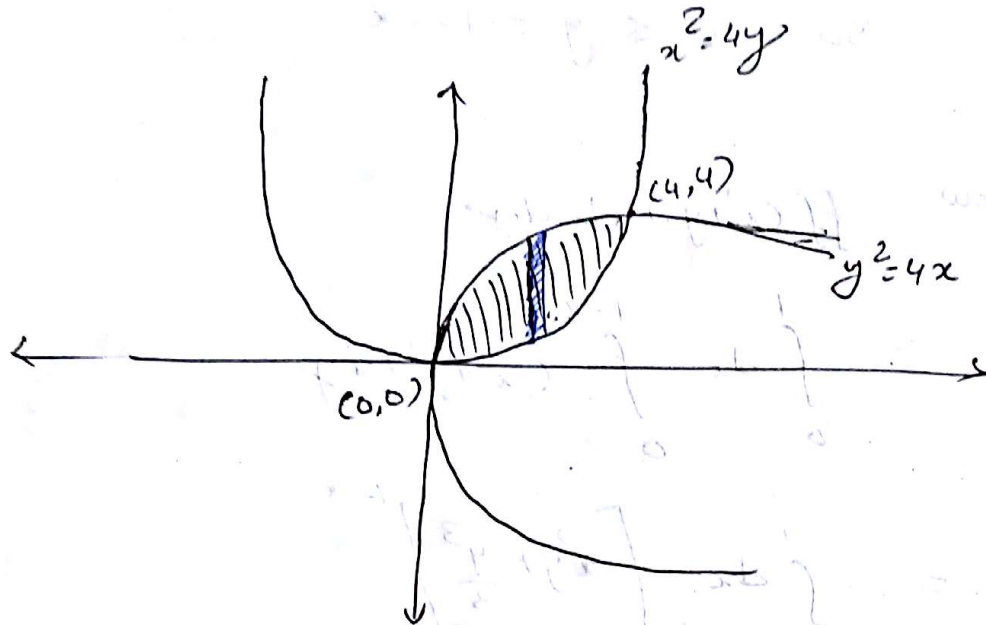
$$= \left(0 + \frac{1}{3} - \frac{1}{4} \right) - \left(-\frac{1}{12} + 0 - 0 \right)$$

$$= \frac{4-3+1}{12} = \frac{2}{12} = \frac{1}{6}$$

Q:2 Evaluate $\iint_R y \, dx \, dy$

where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$

Ans $\rightarrow$



For point of intersection,

$$x^2 = 4y, \quad y^2 = 4x$$

$$\Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x$$

$$\Rightarrow \frac{x^4}{16} = 4x$$

$$\Rightarrow x^3 = 64$$

$$\Rightarrow x = 4, \quad y = 4$$

So the point is $(4, 4)$

Now, x varies from 0 to 4

$$0 \leq x \leq 4$$

$$x^2 = 4y$$

$$\Rightarrow y = \frac{x^2}{4}$$

$$y^2 = 4x$$

$$\Rightarrow y = \sqrt{4x} = 2\sqrt{x}$$

so y varies from $\frac{x^2}{4}$ to $2\sqrt{x}$

$$\Rightarrow \frac{x^2}{4} \leq y \leq 2\sqrt{x}$$

Now, $\iint_R y \, dx \, dy$

$$= \int_0^4 dx \int_{\frac{x^2}{4}}^{2\sqrt{x}} y \, dy$$

$$= \int_0^4 dx \left[\frac{y^2}{2} \right]_{\frac{x^2}{4}}^{2\sqrt{x}}$$

$$= \int_0^4 dx \left(\frac{4x}{2} - \frac{x^4}{32} \right) dx$$

$$= \int_0^4 \left(x^2 - \frac{x^4}{160} \right) dx$$

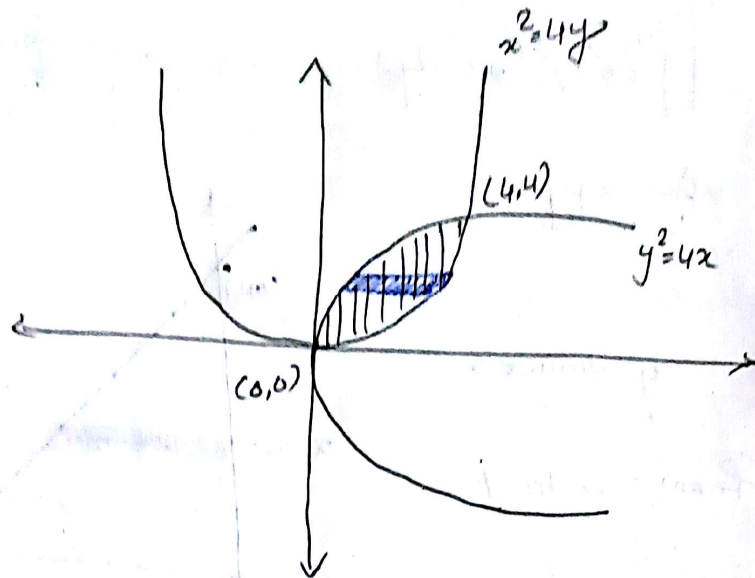
$$= 16 - \frac{64}{10}$$

$$= \frac{160 - 64}{10} = \frac{96}{10} = \frac{48}{5}$$

we can also take $0 \leq y \leq 4$

and $\frac{y^2}{4} \leq x \leq 2\sqrt{y}$

10



now, y varies from 0 to 4
 so $0 \leq y \leq 4$

$$y^2 = 4x$$

$$\Rightarrow x = \frac{y^2}{4}$$

$$x^2 = 4y$$

$$\Rightarrow x = 2\sqrt{y}$$

here x varies from curve $y^2 = 4x$ to curve $x^2 = 4y$
 i.e., from $\frac{y^2}{4}$ to $2\sqrt{y}$

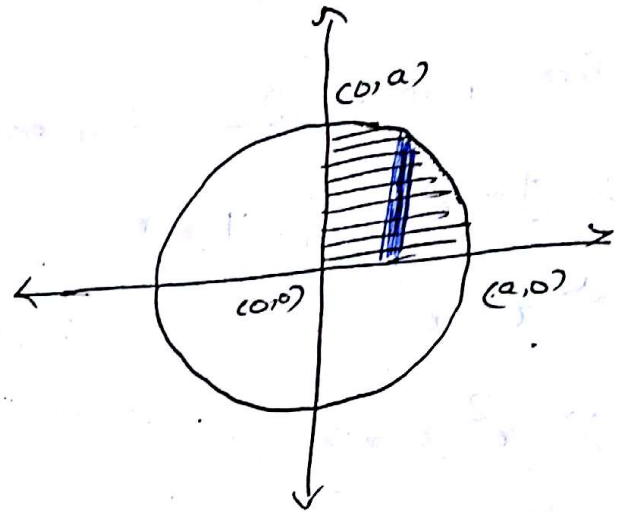
$$\begin{aligned}
 \text{now } & \int_0^4 dy \int_{\frac{y^2}{4}}^{2\sqrt{y}} y \, dx \\
 &= \int_0^4 dy \left[xy \right]_{\frac{y^2}{4}}^{2\sqrt{y}} \\
 &= \int_0^4 \left(2y^{\frac{3}{2}} - \frac{y^3}{4} \right) dy \\
 &= \left[\frac{2y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^4}{16} \right]_0^4 \\
 &= \frac{4 \cdot 4^{\frac{5}{2}}}{5} - \frac{4^4}{16} \\
 &= \frac{128}{5} - 16 \\
 &= \frac{48}{5}
 \end{aligned}$$

Q: $\iint xy \, dx \, dy$ over the (+)ve quadrant of the circle $x^2 + y^2 = a^2$

Ans

x varies from 0 to a

y varies from 0 to $\sqrt{a^2 - x^2}$



$$\text{So } \int_0^a dx \int_0^{\sqrt{a^2 - x^2}} xy \, dy$$

$$x^2 + y^2 = a^2$$

$$\Rightarrow y = \sqrt{a^2 - x^2}$$

$$= \int_0^a dx \left[\frac{xy^2}{2} \right]_0^{\sqrt{a^2 - x^2}}$$

$$= \int_0^a dx \left(x \frac{(a^2 - x^2)}{2} - 0 \right)$$

$$= \frac{1}{2} \int_0^a (a^2 x - x^3) dx$$

$$= \frac{1}{2} \left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a$$

$$= \frac{1}{2} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) = \frac{a^4}{8}$$

Q: Evaluate $\iint xy(x+y) dx dy$ over the area

Ans between $y=x^2$ and $y=x$

For point of intersection,

$$y=x^2, y=x$$

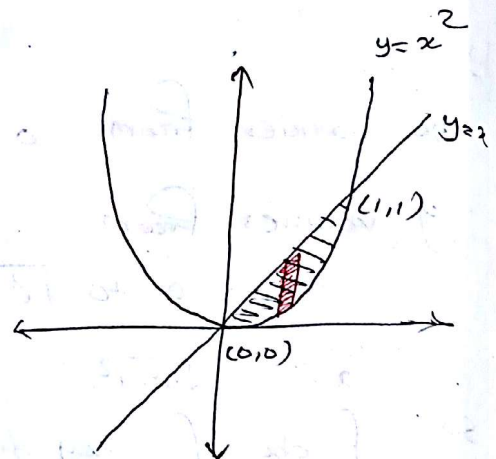
$$\Rightarrow x=x^2$$

$$\Rightarrow x^2-x=0$$

$$\Rightarrow x=0, x=1$$

$$y=0, y=1$$

So the point is (1,1)

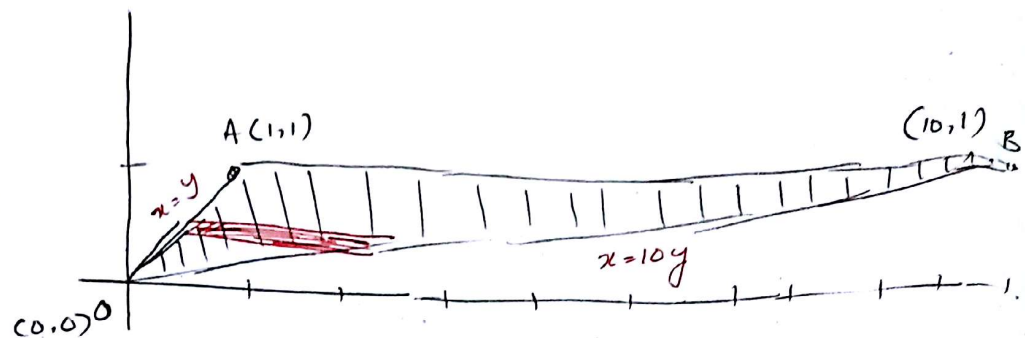


here x varies from 0 to 1, i.e., $0 \leq x \leq 1$

y varies from x^2 to x i.e., $x^2 \leq y \leq x$

$$\begin{aligned} & \int_0^1 \int_{x^2}^x xy(x+y) dy dx \\ &= \int_0^1 \left[\int_{x^2}^x (x^2y + xy^2) dy \right] dx \\ &= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^x dx \\ &= \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} \right) - \left(\frac{x^6}{2} + \frac{x^7}{3} \right) dx \\ &= \int_0^1 \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx \\ &= \left[\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1 = \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{9}{168} = \frac{3}{16} \end{aligned}$$

Q: Evaluate $\iint_S \sqrt{xy-y^2} \, dx \, dy$, where S is a triangle with vertices $(0,0)$, $(10,1)$ and $(1,1)$



here OAB is the triangle with vertices $O(0,0)$
 $A(1,1)$ & $B(10,1)$

Eqⁿ of line joining $(0,0)$ & $(1,1)$ is

$$y-0 = \frac{1-0}{1-0} (x-0) \Rightarrow y=x$$

Eqⁿ of line joining $(0,0)$ & $(10,1)$ is

$$y-0 = \frac{1-0}{10-0} (x-0) \Rightarrow 10y=x$$

$$\text{so } 0 \leq y \leq 1$$

$$y \leq x \leq 10y$$

$$\text{now } \int_0^1 \int_y^{10y} \sqrt{xy-y^2} \, dx \, dy$$

$$= \int_0^1 \left[\frac{2}{3} \frac{(xy-y^2)^{3/2}}{y} \right]_y^{10y} dy$$

$$= \int_0^1 \frac{2}{3y} (9y^2)^{3/2} dy$$

$$= 18 \int_0^1 y^2 dy = 18 \left(\frac{y^3}{3} \right)_0^1 = 18 \left(\frac{1}{3} \right) = 6$$

Polar Co-ordinates :-

> Cartesian : x, y

$$\iint f(x, y) dx dy$$

> Polar : r, θ

$$\iint f(r, \theta) d\theta dr$$

Type-4 Evaluation of double integrals in polar co-ordinates

(i)
$$\int_0^{\pi} \int_0^{a \sin \theta} r d\theta dr$$

First to integrate w.r.t r and then to
put the values in place of r

$$= \int_0^{\pi} d\theta \left[\int_{r=0}^{r=a \sin \theta} r dr \right]$$

$$= \int_0^{\pi} d\theta \left[\frac{r^2}{2} \right]_0^{a \sin \theta}$$

$$= \int_0^{\pi} \left(\frac{a^2 \sin^2 \theta}{2} - 0 \right) d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \sin^2 \theta d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \frac{a^2}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi}$$

$$= \frac{a^2}{4} \left[\left(\pi - \frac{\sin 2\pi}{2} \right) - \left(0 - \frac{\sin 0}{2} \right) \right]$$

$$= \frac{a^2}{4} (\pi - 0) = \frac{1}{4} \pi a^2$$

Q:

$$\int_0^{\pi/2} \int_0^{a \cos \theta} r \sin \theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \sin \theta \, d\theta \left[\int_{r=0}^{r=a \cos \theta} r \, dr \right]$$

$$= \int_0^{\pi/2} \sin \theta \, d\theta \left[\frac{r^2}{2} \right]_0^{a \cos \theta}$$

$$= \int_0^{\pi/2} \sin \theta \left(\frac{a^2 \cos^2 \theta}{2} \right) d\theta$$

$$= -\frac{a^2}{2} \int_0^{\pi/2} \cos^2 \theta \, d(\cos \theta)$$

$$= -\frac{a^2}{2} \left[\frac{\cos^3 \theta}{3} \right]_0^{\pi/2} = -\frac{a^2}{6} (\cos^3 \frac{\pi}{2} - \cos^3 0)$$

$$= -\frac{a^2}{6} (0 - 1) = \frac{a^2}{6}$$

Q: Evaluate the following.

(i) $\int_0^{\pi/2} \int_a^{a(1+\cos \theta)} r \, dr \, d\theta$

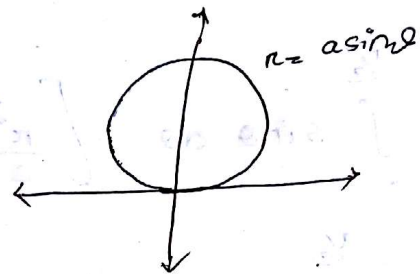
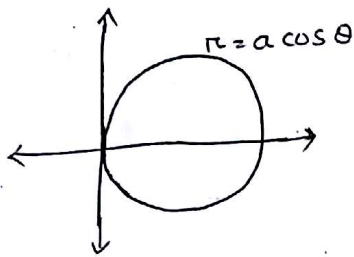
(ii) $\int_{-\pi/2}^{\pi/2} \int_{2 \cos \theta}^{4 \cos \theta} r^3 \, dr \, d\theta$

(iii) $\int_0^{\pi} \int_0^{a(1+\cos \theta)} r \, dr \, d\theta$

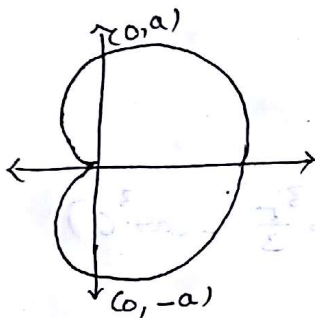
Types of polar graphs

1. Line : $\theta = \text{angle}$.

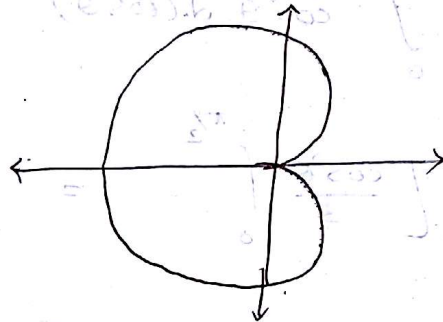
2. circle : $r = a \sin \theta$
 $r = a \cos \theta$.



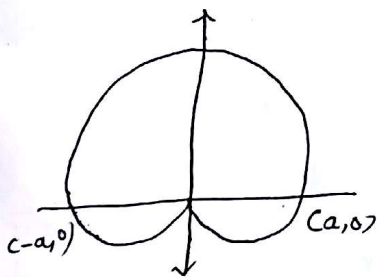
3. Cardioid : $r = a(1 \pm \cos \theta)$, $r = a(1 \pm \sin \theta)$



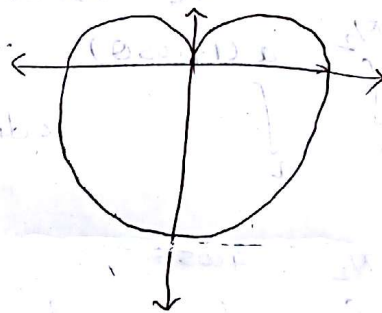
$$r = a(1 + \cos \theta)$$



$$r = a(1 - \cos \theta)$$

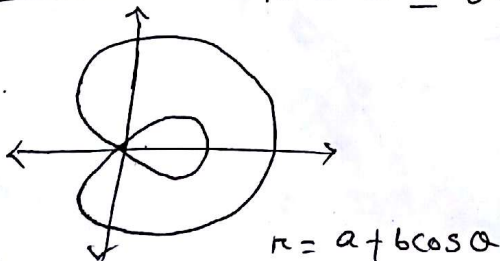


$$r = a(1 + \sin \theta)$$

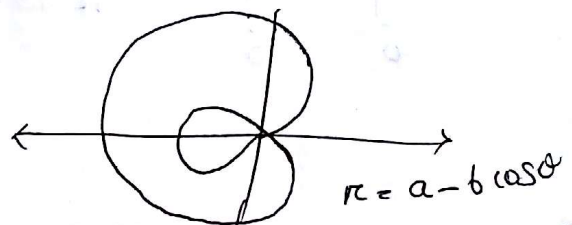


$$r = a(1 - \sin \theta)$$

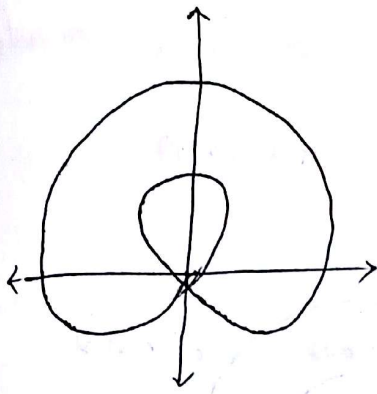
4. Limacons : $r = a \pm b \cos \theta$, $r = a \pm b \sin \theta$



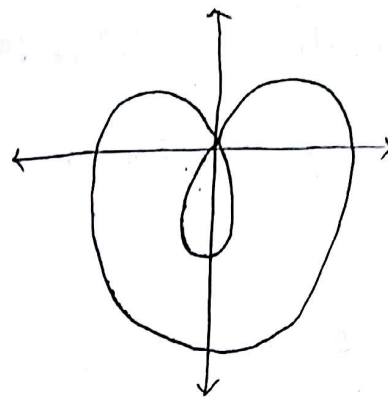
$$r = a + b \cos \theta$$



$$r = a - b \cos \theta$$



$$r = a + b \sin \theta$$



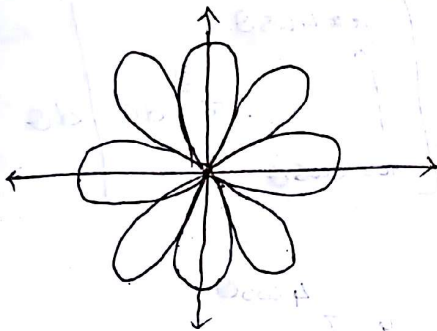
$$r = a - b \sin \theta$$

5. Rose :

$$r = a \sin n\theta, \quad r = a \cos n\theta \quad a, n \neq 0$$

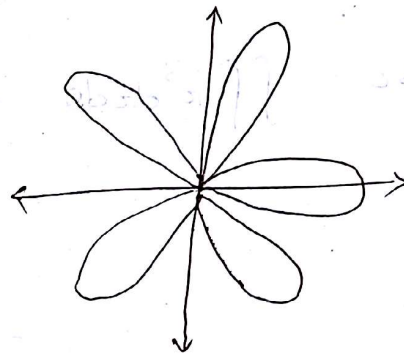
when n is odd, it has n petals

when n is even, it has $2n$ petals.



$$n = 4 \text{ (even)}$$

$$\text{no. of petals} = 8$$

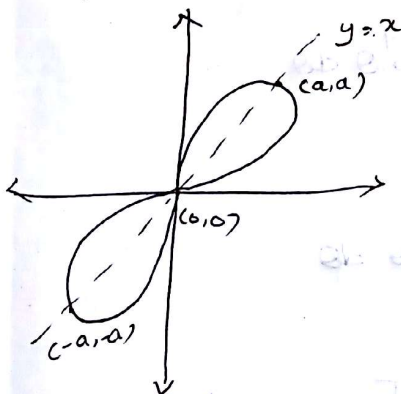


$$n = 5 \text{ (odd)}$$

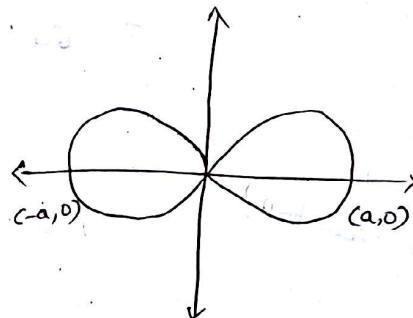
$$\text{no. of petals} = 5$$

6. Lemniscate :

$$r^2 = a^2 \sin 2\theta, \quad r^2 = a^2 \cos 2\theta$$



$$r^2 = a^2 \sin 2\theta$$



$$r^2 = a^2 \cos 2\theta$$

Q: Evaluate $\iint r^3 dr d\theta$, over the area bounded between the circles $r = 2\cos\theta$, $r = 4\cos\theta$.

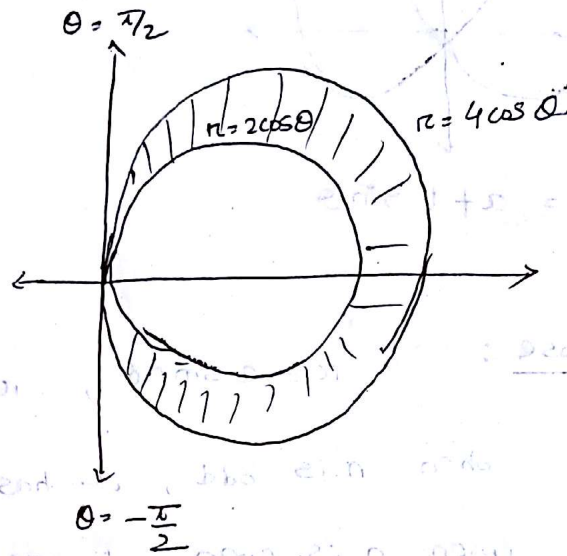
Ans

here θ varies

from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

and r varies

from $2\cos\theta$ to $4\cos\theta$



so

$$\iint r^3 dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_{r=2\cos\theta}^{r=4\cos\theta} r^3 dr \right] d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} (256 \cos^4\theta - 16 \cos^4\theta) d\theta$$

$$= 60 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta$$

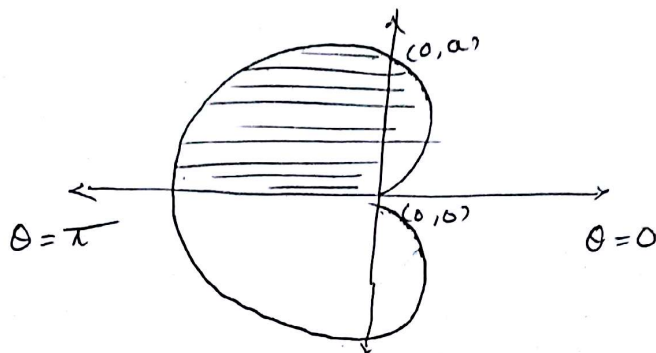
($\because \cos^4\theta$ is even function)

$$= 120 \int_0^{\frac{\pi}{2}} \cos^4\theta d\theta$$

$$= 120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{45}{2} \pi$$

Q: Evaluate $\iint r \sin \theta \, dr \, d\theta$ over the cardioid $r = a(1 - \cos \theta)$ above the initial line.

Ans $\rightarrow$



here θ varies from 0 to π

r varies from 0 to $a(1 - \cos \theta)$

$$\text{so } \iint r \sin \theta \, dr \, d\theta = \int_0^\pi \int_0^{a(1-\cos \theta)} r \sin \theta \, dr \, d\theta$$

$$= \int_0^\pi \sin \theta \, d\theta \left[\frac{r^2}{2} \right]_0^{a(1-\cos \theta)}$$

$$= \frac{a^2}{2} \int_0^\pi \sin \theta (1 - \cos \theta)^2 \, d\theta$$

$$= \frac{a^2}{2} \int_0^\pi 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} (2 \sin^2 \frac{\theta}{2})^2 \, d\theta$$

$$= \frac{a^2}{2} \int_0^\pi 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} 4 \sin^4 \frac{\theta}{2} \, d\theta$$

$$= \frac{8a^2}{2} \int_0^\pi \sin^5 \frac{\theta}{2} \left(\cos \frac{\theta}{2} \, d\theta \right)$$

$$= 4a^2 \int_0^\pi \sin^5 \frac{\theta}{2} \, d \left(\cos \frac{\theta}{2} \right)$$

$$= 8a^2 \left[\frac{\sin^6 \frac{\theta}{2}}{6} \right]_0^\pi = \frac{8a^2}{6} (1 - 0) = \frac{4a^2}{3}$$

change in order of integration

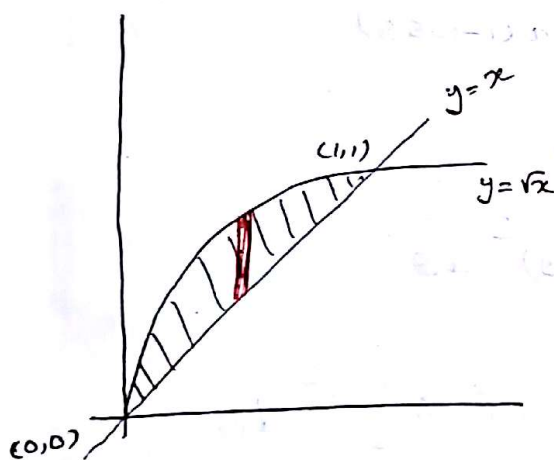
> In the process of double and triple integration, it is sometimes difficult to integrate them directly, and a change in order of integration may be useful.

> Here we convert

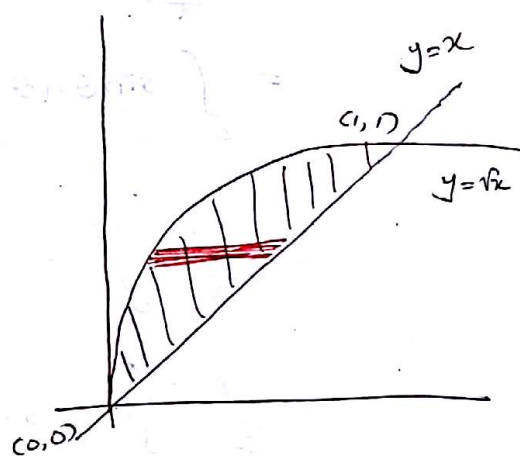
$$\int_{x=a}^b dx \int_{y_1(x)}^{y_2(x)} f(x,y) dy \quad \text{to} \quad \int_{y=c}^d dy \int_{x_1(y)}^{x_2(y)} f(x,y) dx$$

e.g: change the order of integration

$$I = \int_0^1 dx \int_x^{\sqrt{x}} f(x,y) dy$$



$\Rightarrow$



here $0 \leq x \leq 1$

$$\sqrt{x} \leq y \leq x$$

so we write

$$I = \int_0^1 dx \int_x^{\sqrt{x}} f(x,y) dy$$

here $0 \leq y \leq 1$

x varies from curve $x=y^2$ to $x=y$

$$\text{so } y^2 \leq x \leq y$$

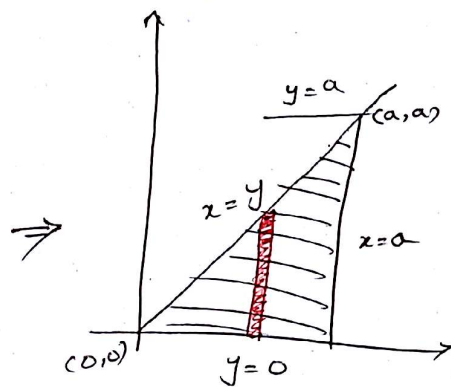
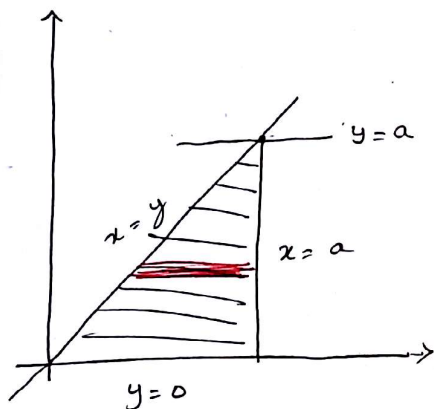
so by changing the order we can write.

$$\int_0^1 dy \int_{y^2}^y f(x,y) dx$$

e.g: change the order of integration in

$$\int_0^a \int_y^a \frac{x}{x^2+y^2} dx dy \quad \text{and hence evaluate that}$$

$$\begin{array}{ll} y=0 & x=y \\ y=a & x=a \end{array}$$



$$\text{here } 0 \leq x \leq a$$

y varies from the line
 $y=0$ to $y=x$

$$\text{so } 0 \leq y \leq x$$

by changing the order we can write

$$\int_0^a \int_y^a \frac{x}{x^2+y^2} dx dy = \int_0^a \int_0^x \frac{x}{x^2+y^2} dy dx$$

$$\begin{aligned}
 &= \int_0^a x dx \int_0^x \frac{1}{x^2+y^2} dy \\
 &= \int_0^a x dx \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right]_0^x \\
 &= \int_0^a x dx \frac{1}{x} (\tan^{-1} 1 - \tan^{-1} 0) \\
 &= \int_0^a \pi/4 dx = \frac{\pi}{4} [x]_0^a = \frac{\pi a}{4}
 \end{aligned}$$

Q: change the order of following integrations.

1. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy dx$

2. $\int_0^1 \int_{4y}^4 e^{x^2} dx dy$

3. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dx dy$

4. $\int_0^{\infty} \int_0^y f dx dy$

5. $\int_0^{\infty} \int_0^x f dy dx$

Ans
→

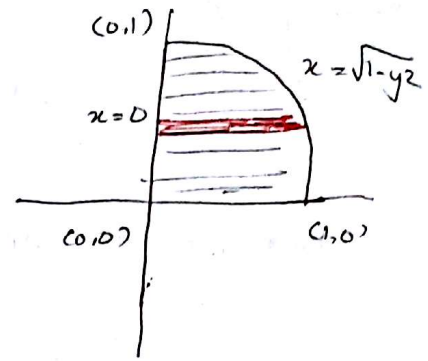
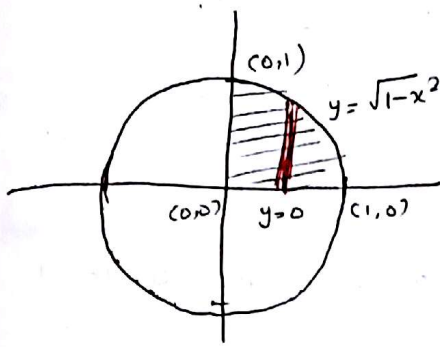
1. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy dx$

$x = 0$ to $x = 1$

$y = 0$ to $y = \sqrt{1-x^2}$

→ $y^2 = 1-x^2$

→ $x^2+y^2=1$ (circle)



here $0 \leq y \leq 1$

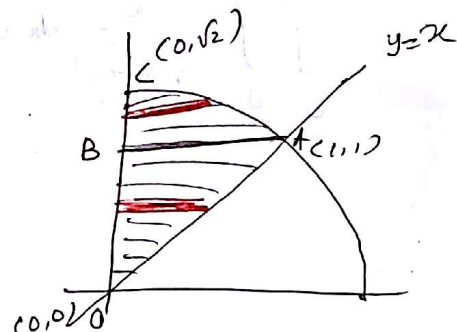
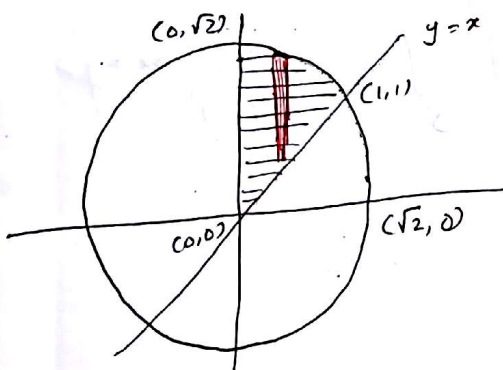
and x varies from line $x=0$ to curve $x = \sqrt{1-y^2}$

so $0 \leq x \leq \sqrt{1-y^2}$

$$\int_0^1 \int_0^{\sqrt{1-y^2}} y^2 dy dx = \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 dx dy$$

3. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dx dy$

here $x=0$ $y=x$
 $x=1$ $y=\sqrt{2-x^2} \Rightarrow x^2+y^2=2$ (circle)



for the horizontal strips, the whole region is divided into two portions OAB and ACB

For portion OAB, y varies from 0 to 1.

x varies from line $x=0$ to line $x=y$.

$$\text{so } 0 \leq x \leq y, \quad 0 \leq y \leq 1$$

For portion ACB,

y varies from 1 to $\sqrt{2}$

x varies from line $x=0$ to curve $x^2+y^2=2$

$$\Rightarrow x = \sqrt{2-y^2}$$

$$\text{so } 0 \leq x \leq \sqrt{2-y^2}, \quad 1 \leq y \leq \sqrt{2}$$

$$\therefore \text{Hence, } \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dx dy$$

$$= \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}} dx dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}} dx dy$$

Evaluate the following integrals by changing the order of integration:

$$1. \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy dx$$

$$2. \int_0^1 \int_{4y}^4 e^{x^2} dx dy$$

$$3. \int_0^4 \int_y^4 \frac{x dx dy}{x^2 + y^2}$$

$$4. \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} x dx dy$$

$$5. \int_0^1 \int_{y^2}^{2-y} xy dx dy$$

$$6. \int_0^a \int_{x^2/a}^{2a-x} xy dy dx$$

$$7. \int_0^b \int_0^{a/b \sqrt{b^2-y^2}} xy dx dy$$

$$8. \int_0^2 \int_{\frac{x^2}{4}}^{3-x} xy dy dx$$

$$9. \int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2 dx dy}{\sqrt{x^4 - 4y^2}}$$

$$10. \int_0^\infty \int_0^y ye^{-\frac{y^2}{x}} dx dy$$

$$11. \int_0^\infty \int_0^x xe^{-\frac{x^2}{y}} dy dx$$

$$12. \int_0^3 \int_1^{\sqrt{4-y}} (x+y) dx dy$$

$$13. \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) dy dx$$

$$14. \int_1^4 \int_1^{\sqrt{x}} (x + y^2) dy dx$$

Answers

$$1. \frac{\pi}{16}$$

$$2. \frac{1}{8}(e^{16} - 1)$$

$$3. \pi$$

$$4. \frac{a^3 \sqrt{2}}{6}$$

$$5. \frac{3}{8}$$

$$6. \frac{3a^4}{8}$$

$$7. \frac{1}{8}a^2b^2$$

$$8. \frac{8}{3}$$

$$9. \frac{2\pi}{3}$$

$$10. \frac{1}{2}$$

$$11. \frac{1}{2}$$

$$12. \frac{241}{60}$$

$$13. \frac{a^3}{28} + \frac{a}{20}$$

$$14. \frac{241}{30}$$

Applications of Multiple Integrals :-

(Area by Double Integration)

$$A = \iint dx dy$$

or

$$A = \iint r dr d\theta$$

e.g : Find the area between the parabolas

$$y^2 = 4ax, \quad x^2 = 4ay$$

For the point
of intersection,

$$x^2 = 4ay$$

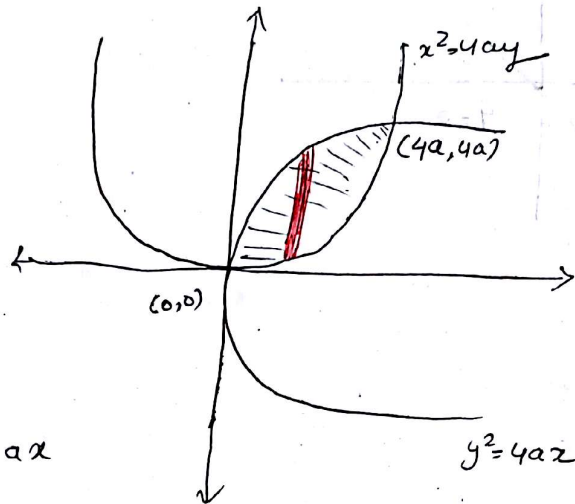
$$\Rightarrow y = \frac{x^2}{4a}$$

$$y^2 = 4ax \Rightarrow \left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow x^4 = (4a)^2 x$$

$$\Rightarrow x^3 = (4a)^3$$

$$\Rightarrow x = 4a$$
$$y = 4a$$



here x varies from 0 to $4a$

y varies from the curve $x^2 = 4ay$ to $y^2 = 4ax$

$$\Rightarrow y = \frac{x^2}{4a} \text{ to } y = \sqrt{4ax}$$

$$\text{so } 0 \leq x \leq 4a \quad \& \quad \frac{x^2}{4a} \leq y \leq \sqrt{4ax}$$

so required area

$$= \int_0^{4a} \int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dy dx$$

$$\begin{aligned}
&= \int_0^{4a} \left[y \right]_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx \\
&= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a} \right) dx \\
&= \left[\sqrt{4a} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{4a(3)} \right]_0^{4a} \\
&= 2 \cdot \frac{\sqrt{4a} (4a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(4a)^3}{12a} \\
&= \frac{4^{\frac{5}{2}} a^2}{3} - \frac{4^3 a^3}{12a} = \frac{32}{3} a^2 - \frac{16}{3} a^2 \\
&= \frac{16}{3} a^2
\end{aligned}$$

Q: Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

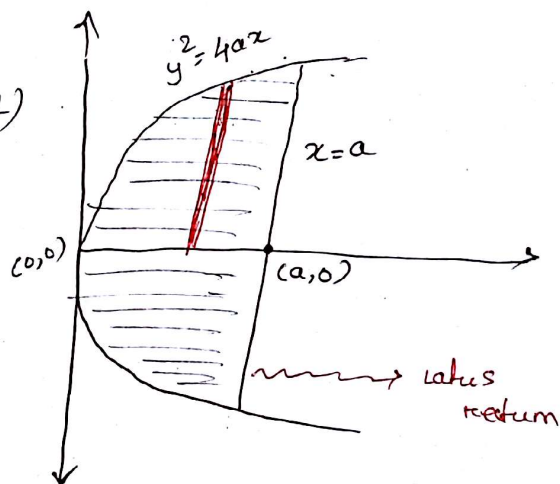
Ans Required area

= 2 (Area in 1st quadrant)

$$= 2 \int_0^a \int_0^{\sqrt{4ax}} dy dx$$

$$= 2 \int_0^a \sqrt{4ax} dx$$

$$= 4\sqrt{a} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^a = \frac{8a^2}{3}$$



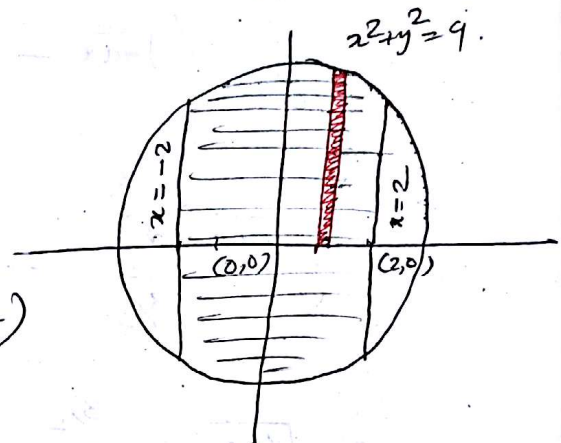
Q: Find the area of the region bounded by the lines $x = -2$, $x = 2$ and the circle $x^2 + y^2 = 9$.

Ans

we can divide the area into 4 parts

Required Area

$$= 4(\text{Area in 1st quadrant})$$



In 1st Q,

x varies from 0 to 2

y varies from line $y = 0$ to

$$y = \sqrt{9 - x^2}$$

$$\text{so Area} = 4 \int_0^2 \int_0^{\sqrt{9-x^2}} dy dx$$

$$= 4 \int_0^2 [y]_0^{\sqrt{9-x^2}} dx$$

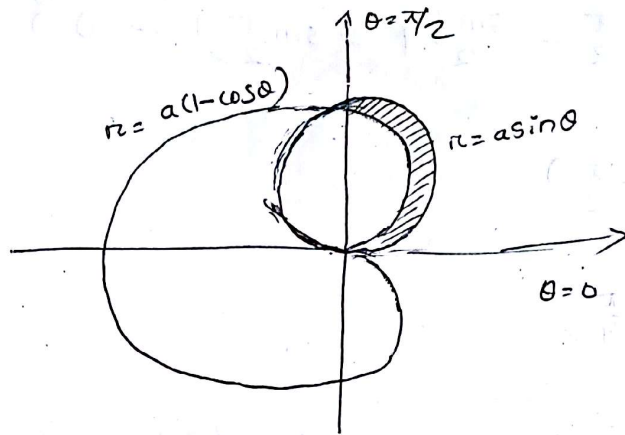
$$= 4 \int_0^2 \sqrt{9-x^2} dx$$

$$= 4 \left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2} \sin^{-1} \frac{x}{3} \right]_0^2$$

$$= 4 \left(\frac{2\sqrt{5}}{2} + \frac{9}{2} \sin^{-1} \frac{2}{3} \right)$$

$$= 4\sqrt{5} + 18 \sin^{-1} \frac{2}{3}$$

Q: Find the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$



here θ varies from 0 to $\pi/2$

r varies from cardioid $a(1 - \cos \theta)$ to circle $a \sin \theta$.

$$\text{So Area} = \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

~~$$= \int_0^{\pi/2} \left(\frac{a^2 \sin^2 \theta}{2} - \frac{a^2 (1 - \cos \theta)^2}{2} \right) d\theta$$~~

$$= \frac{1}{2} \int_0^{\pi/2} [a^2 \sin^2 \theta - a^2 (1 - \cos \theta)^2] d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} (\sin^2 \theta - 1 + 2\cos \theta - \cos^2 \theta) d\theta$$

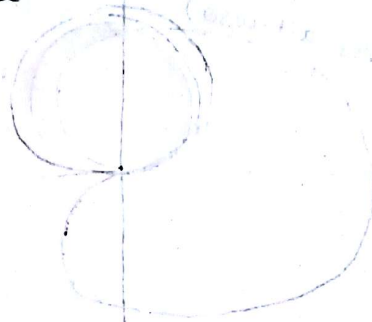
$$= \frac{a^2}{2} \int_0^{\pi/2} (-1 - \cos 2\theta + 2\cos \theta) d\theta$$

$$= \frac{a^2}{2} \left[-\theta - \frac{\sin 2\theta}{2} + 2 \sin \theta \right]_0^{\pi/2}$$

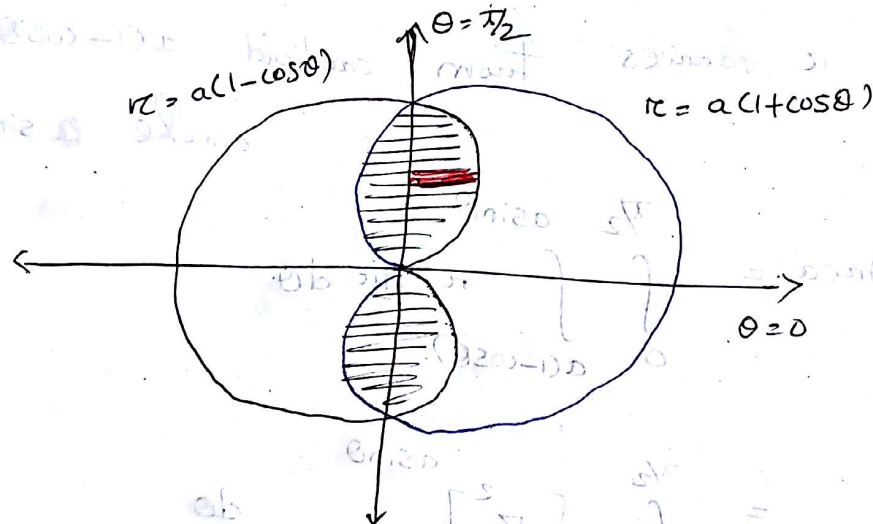
$$= \frac{a^2}{2} \left\{ \left(-\frac{\pi}{2} - \frac{\sin \pi}{2} + 2 \sin \frac{\pi}{2} \right) - 0 \right\}$$

$$= \frac{a^2}{2} \left(2 - \frac{\pi}{2} \right)$$

$$= a^2 \left(1 - \frac{\pi}{4} \right)$$



Q: Find the total area included between two cardioids $r = a(1 + \cos \theta)$ and $r = a(1 - \cos \theta)$



The area can be divided into 4 equal parts

Required Area = 4 (Area in 1st quadrant)

In 1st Q,

θ varies from 0 to $\frac{\pi}{2}$

r varies from 0 to cardioid

$$r = a(1 - \cos \theta)$$

$$\begin{aligned}
 \text{So Area} &= 4 \int_0^{\pi/2} \int_0^{a(1-\cos\theta)} r \, dr \, d\theta \\
 &= 4 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^{a(1-\cos\theta)} d\theta \\
 &= 4 \int_0^{\pi/2} \frac{a^2 (1-\cos\theta)^2}{2} d\theta \\
 &= 2a^2 \int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta) d\theta \\
 &= 2a^2 \int_0^{\pi/2} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2} \right) d\theta \\
 &= 2a^2 \int_0^{\pi/2} \left(\frac{2 - 4\cos\theta + 1 + \cos 2\theta}{2} \right) d\theta \\
 &= a^2 \int_0^{\pi/2} (3 - 4\cos\theta + \cos 2\theta) d\theta \\
 &= a^2 \left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \\
 &= a^2 \left\{ \left(3\frac{\pi}{2} - 4\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi \right) - (0 - 0 + 0) \right\} \\
 &= a^2 \left(\frac{3\pi}{2} - 4 + 0 \right) \\
 &= \frac{a^2}{2} (3\pi - 8)
 \end{aligned}$$

* The multiple integrals are used to calculate mass and area of a plane, surface area & volume of a solid, also centre of gravity, moment of inertia etc.

Line Integrals

Any integral which is to be evaluated along a curve is called a line integral.

or

let $\vec{F}$ be a vector function

C be any curve from

point A to point B



$$\begin{aligned}\text{then line integral} &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_A^B \vec{F} \cdot d\vec{r}\end{aligned}$$

$$\text{where } d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Properties

$$\textcircled{1} \int_A^B \vec{F} \cdot d\vec{r} = - \int_B^A \vec{F} \cdot d\vec{r}$$

$\textcircled{2}$ If the curve has parametric equation

$$\text{i.e., } x = x(t), \quad y = y(t), \quad z = z(t)$$

$$\text{then } \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot \frac{d\vec{r}}{dt} dt$$

Type-1

Q: Find the line integral where $\vec{F} = x^2y^2\hat{i} + y\hat{j}$

and curve C is $y^2 = 4x$ in the xy-plane

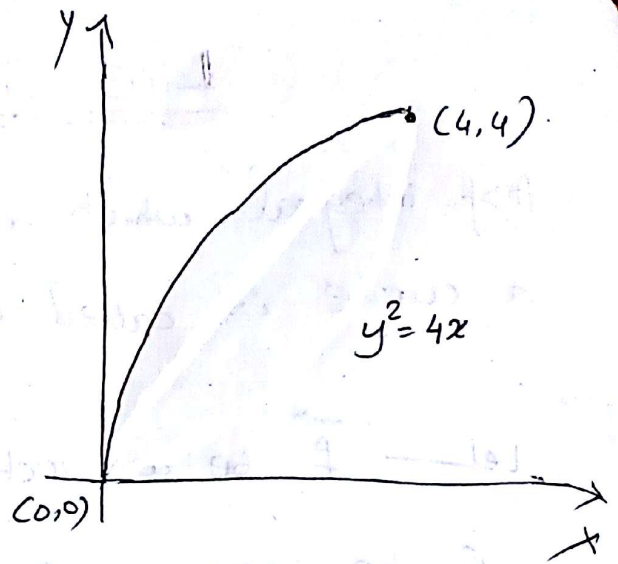
from point (0,0) to (4,4)

$$\vec{F} = x^2 y^2 \hat{i} + y \hat{j}$$

$$d\vec{r} = dx \hat{i} + dy \hat{j}$$

$$\vec{F} \cdot d\vec{r}$$

$$= x^2 y^2 dx + y dy$$



so, line integral

x varies from
0 to 4

y varies from
0 to 4

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C (x^2 y^2 dx + y dy)$$

$$= \int_0^4 x^2 y^2 dx + \int_0^4 y dy$$

$$= \int_0^4 x^2 (4x) dx + \int_0^4 y dy$$

$$(\because y^2 = 4x)$$

$$= 4 \int_0^4 x^3 dx + \int_0^4 y dy$$

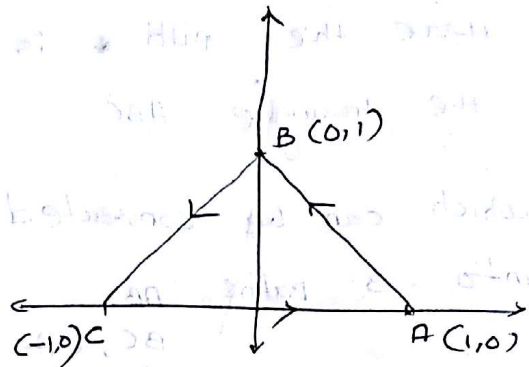
$$= 4 \frac{x^4}{4} \Big|_0^4 + \frac{y^2}{2} \Big|_0^4$$

$$= (256 - 0) + (8 - 0) = 264$$

Q.1 compute the line integral $\int_C (y^2 dx - x^2 dy)$ about the triangle whose vertices are $(1,0)$, $(0,1)$ and $(-1,0)$.

Ans $\rightarrow$ Here the path is triangle ABC

which can be converted into 3 paths AB, BC, CA



In AB Equation of AB is

$$y - 0 = \frac{1-0}{0-1} (x-1)$$

$$\Rightarrow y = 1 - x$$

$$\Rightarrow dy = -dx$$

x varies from 1 to 0

$$\text{So } \int_{AB} y^2 dx - x^2 dy = \int_1^0 (1-x)^2 dx - x^2 (-dx)$$

$$= \int_1^0 (1 + x^2 - 2x + x^2) dx$$

$$= \int_1^0 (2x^2 - 2x + 1) dx$$

$$= \left[\frac{2x^3}{3} - x^2 + x \right]_1^0$$

$$= (0 - 0 + 0) - \left(\frac{2}{3} - 1 + 1 \right) = -\frac{2}{3}$$

In BC eqⁿ of BC is $y-1 = \frac{0-1}{-1-0} (x-0)$

$$\Rightarrow y = 1+x$$

$$\Rightarrow dy = dx$$

x varies from 0 to -1

$$\begin{aligned} \text{so } \int_{BC} y^2 dx - x^2 dy &= \int_0^{-1} (1+x)^2 dx - x^2 dx \\ &= \int_0^{-1} (2x+1) dx \\ &= [x^2 + x]_0^{-1} = (-1) + 1 = 0 \end{aligned}$$

In CA eqⁿ of CA, $y = 0$

$$\Rightarrow dy = 0$$

x varies from -1 to 1

$$\text{so } \int_{CA} y^2 dx - x^2 dy = \int_{-1}^1 (0 - 0) = 0$$

hence $\int_C (y^2 dx - x^2 dy)$

$$= \int_{AB} () + \int_{BC} () + \int_{CA} ()$$

$$= -\frac{2}{3} + 0 + 0 = -\frac{2}{3}$$

Q: Calculate $\int_C \vec{F} \cdot d\vec{r}$ for the data

$$\vec{F} = [y^2, -x^2]$$

C : straight line segment
from $(0,0)$ to $(1,4)$

Ans $\rightarrow \vec{F} = y^2 \hat{i} - x^2 \hat{j}$

$$d\vec{r} = dx \hat{i} + dy \hat{j}$$

$$\vec{F} \cdot d\vec{r} = y^2 dx - x^2 dy$$

$$\text{so } \int_C \vec{F} \cdot d\vec{r} = \int_C y^2 dx - x^2 dy$$

$$= \int_C y^2 dx - \int_C x^2 dy$$

$$= \int_C (4x)^2 dx - \int_C \left(\frac{y}{4}\right)^2 dy$$

$$= \int_0^1 16x^2 dx - \int_0^4 \frac{y^2}{16} dy$$

$$(\because 0 \leq x \leq 1 \\ 0 \leq y \leq 4)$$

$$= 16 \left[\frac{x^3}{3} \right]_0^1 - \frac{1}{16} \left[\frac{y^3}{3} \right]_0^4$$

$$= \frac{16}{3} - \frac{4}{3} = \frac{12}{3} = 4$$

or

$$\vec{F} \cdot d\vec{r} = y^2 dx - x^2 dy$$

$$= (4x)^2 dx - x^2 (4 dx)$$

=

and line is

$$y = 4x$$

$$\Rightarrow dy = 4 dx$$

$$= 16x^2 dx - 4x^2 dx$$

$$= 12x^2 dx$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^1 12x^2 dx \quad (\because 0 \leq x \leq 1)$$

$$= 12 \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{12}{3} - 0 = 4$$

⊛ Work done,

$$W = \int \vec{F} \cdot d\vec{r}$$

where $\vec{F}$ = Force.

$d\vec{r}$ = displacement

Cross product can't be done as work is a scalar quantity.

Q: find the work done in moving a particle in the force field

$$\vec{F} = 3x^2 \hat{i} + (2xz - y) \hat{j} + z \hat{k} \text{ along}$$

the straight line from (0,0,0) to (2,1,3)

Ans $\vec{F} = 3x^2 \hat{i} + (2xz - y) \hat{j} + z \hat{k}$

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

$$\text{work done} = \int \vec{F} \cdot d\vec{r}$$

$$= \int [3x^2 dx + (2xz - y) dy + z dz]$$

Equation of straight line from $(0,0,0)$ to $(2,1,3)$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (let)}$$

$$x=2t, \quad y=t, \quad z=3t$$

$$dx=2dt, \quad dy=dt, \quad dz=3dt$$

$$\text{So work done} = \int_C 3(2t)^2(2dt) + \{2(2t)(3t) - t\} dt + 3t(3dt)$$

$$= \int_0^1 (24t^2 dt + 12t^2 dt - t dt + 9t dt)$$
$$= \int_0^1 (36t^2 + 8t) dt \quad \left(t \text{ varies from } 0 \text{ to } 1 \right)$$

$$= 36 \frac{t^3}{3} + \frac{8t^2}{2} \Big|_0^1$$

$$= \frac{36}{3} + \frac{8}{2} = 12 + 4 = 16$$

Path Independence of Line Integrals :-

$\int_A^B \vec{F} \cdot d\vec{r}$ is said to be path independent if

$$\vec{F} = \nabla \phi$$

$\because \phi$ be any function

$$\Rightarrow \nabla \times \vec{F} = \nabla \times \nabla \phi$$

$$\Rightarrow \boxed{\nabla \times \vec{F} = \vec{0}}$$

Q: Test for path independence.

$$2xy^2 dx + 2x^2y dy + dz$$

$$\vec{F} \cdot d\vec{r} = 2xy^2 dx + 2x^2y dy + dz$$

$$\Rightarrow \vec{F} \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k}) = 2xy^2 dx + 2x^2y dy + dz$$

$$\Rightarrow \vec{F} = 2xy^2 \hat{i} + 2x^2y \hat{j} + \hat{k}$$

Now to test for path independence

(i.e. $\text{curl } \vec{F} = \vec{0}$ or not)

$$\begin{aligned} \text{curl } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 & 2x^2y & 1 \end{vmatrix} \\ &= \hat{i} \left\{ \frac{\partial}{\partial y}(1) - \frac{\partial}{\partial z}(2x^2y) \right\} - \hat{j} \left\{ \frac{\partial}{\partial x}(1) - \frac{\partial}{\partial z}(2xy^2) \right\} \\ &\quad + \hat{k} \left\{ \frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(2xy^2) \right\} \end{aligned}$$

$$= \hat{i} (0-0) - \hat{j} (0-0) + \hat{k} (4xy - 4xy)$$

$$= \vec{0}$$

It is path independent.

(*) closed integration, ($\oint$)

when the path of integration is a closed curve
then the notation of integration is $\oint$ in place of $\int$.

Green's Theorem in the plane

statement:

if $M(x,y)$ and $N(x,y)$ be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of the xy -plane bounded by a closed curve C , then

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

ex:1 verify Green's theorem in the plane for

$$\oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$$

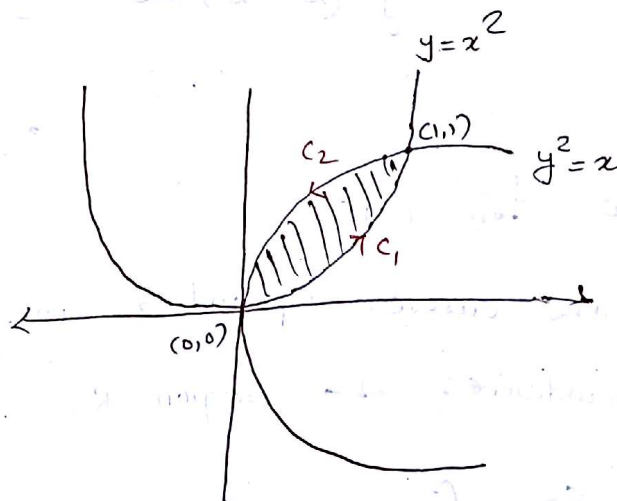
where C is the boundary of the region defined by $y = \sqrt{x}$, $y = x^2$

Ans

$$y = \sqrt{x}$$

$$\Rightarrow y^2 = x \quad \text{and} \quad y = x^2$$

Point of intersection will be $(1,1)$



$$\therefore y = x^2 = (y^2)^2$$

$$\Rightarrow y = y^4$$

$$\Rightarrow y^3 - 1 = 0$$

$$\Rightarrow y = 1, x = 1$$

$$\text{let } M = 3x^2 - 8y^2$$

$$N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y$$

$$\frac{\partial N}{\partial x} = -6y$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y$$

Let R is the region bounded by C ,

$$\text{R.H.S} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$= \iint_R 10y dx dy$$

$$\left(\because 0 \leq x \leq 1 \right. \\ \left. x^2 \leq y \leq \sqrt{x} \right)$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y dy dx$$

$$= \int_0^1 10 \frac{y^2}{2} \Big|_{x^2}^{\sqrt{x}} dx$$

$$= 5 \int_0^1 (x - x^4) dx$$

$$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1$$

$$= 5 \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3}{2}$$

Now for L.H.S,

The curves C_1 and C_2 in the figure are two boundaries of region R .

$$\text{So } \oint_C M dx + N dy = \int_{C_1} M dx + N dy + \int_{C_2} M dx + N dy$$

Along C_1 ,
 $x^2 = y$ and $0 \leq x \leq 1$

$$\Rightarrow 2x dx = dy$$

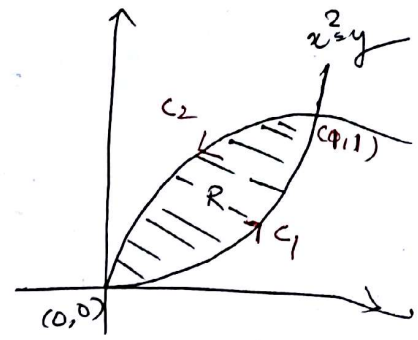
So $\int_{C_1} M dx + N dy$

$$= \int_{C_1} (3x^2 - 8y^2) dx + (4y - 6xy) dy$$

$$= \int_0^1 (3x^2 - 8x^4) dx + (4x^2 - 6x^3) 2x dx$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4) dx$$

$$= \left[x^3 + 2x^4 - 4x^5 \right]_0^1 = 1 + 2 - 4 = -1$$



Along C_2 ,

$$y^2 = x \text{ and } 1 \leq y \leq 0$$

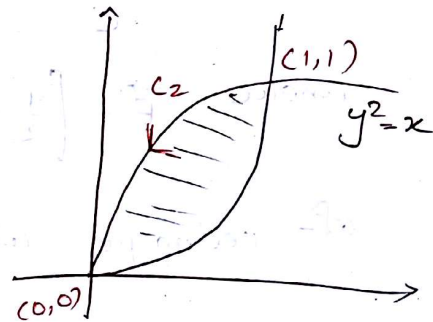
$$\Rightarrow 2y dy = dx$$

So $\int_{C_2} M dx + N dy$

$$= \int_{C_2} (3x^2 - 8y^2) dx + (4y - 6xy) dy$$

$$= \int_1^0 (3y^4 - 8y^2) 2y dy + (4y - 6y^3) dy$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3) dy$$



y varies
 from 1 to 0.

$$= \left[y^6 - 4y^4 + 2y^2 - \frac{6y^4}{4} \right]_1^0$$

$$= 0 - \left(1 - 4 + 2 - \frac{3}{2} \right)$$

$$= \frac{5}{2}$$

L.H.S

$$\therefore \oint_C M dx + N dy = (-1) + \frac{5}{2} = \frac{3}{2} = R.H.S.$$

$\therefore$ Hence, Green's theorem in the plane is verified.

⊗ So Green's theorem is useful for changing a line integral around a closed curve C into a double integral over the region R enclosed by C .

Q: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ using Green's theorem.

where $\vec{F} = \left[\frac{1}{2} xy^4, \frac{1}{2} x^4 y \right]$ and C is the boundary of rectangle with vertices $(0,0), (3,0), (3,2), (0,2)$ in counter clockwise sense.

Ans

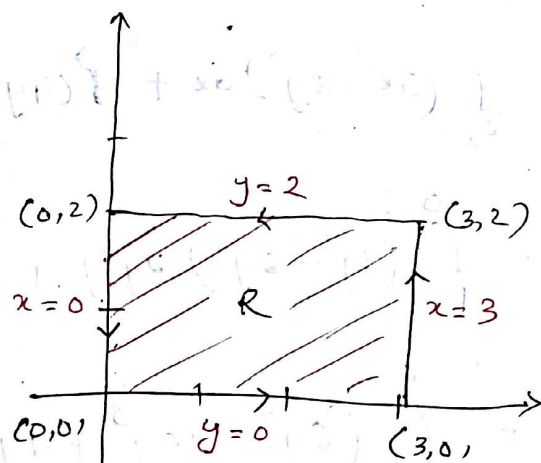
$$\vec{F} = \frac{1}{2} xy^4 \hat{i} + \frac{1}{2} x^4 y \hat{j}$$

$$d\vec{r} = x \hat{i} + y \hat{j}$$

$$d\vec{r} = dx \hat{i} + dy \hat{j}$$

so $\vec{F} \cdot d\vec{r}$

$$= \left(\frac{1}{2} xy^4 \right) dx + \left(\frac{1}{2} x^4 y \right) dy$$



$$\int_C \vec{F} \cdot d\vec{r} = \int_C \frac{1}{2} xy^4 dx + \frac{1}{2} x^4 y dy$$

$$= \int_C M dx + N dy \quad (\text{let})$$

$$M = \frac{1}{2} xy^4$$

$$N = \frac{1}{2} x^4 y$$

$$\frac{\partial M}{\partial y} = \frac{1}{2} x (4y^3)$$

$$= 2xy^3$$

$$\frac{\partial N}{\partial x} = \frac{1}{2} (4x^3) y$$

$$= 2x^3 y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x^3 y - 2xy^3$$

Now according to Green's Theorem,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\Rightarrow \int_C \frac{1}{2} xy^4 dx + \frac{1}{2} x^4 y dy = \iint_R (2x^3 y - 2xy^3) dx dy$$

In R,

$$\left. \begin{array}{l} x = 0 \text{ to } x = 3 \\ y = 0 \text{ to } y = 2 \end{array} \right\}$$

$$= \int_0^2 \int_0^3 (2x^3 y - 2xy^3) dx dy$$

$$= \int_0^2 \left[\frac{2x^4 y}{4} - \frac{2x^2 y^3}{2} \right]_0^3 dy$$

$$= \int_0^2 \left(\frac{81}{2} y - 9y^3 \right) dy$$

$$= \left[\frac{81}{2} \frac{y^2}{2} - \frac{9y^4}{4} \right]_0^2$$

$$= 81 - 36 = 45$$

(Ans)