

PROBLEMS ON MEAN VALUE THEOREM

Ex-1

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals.

- (i) $f(x) = |x|$ on $[-1, 1]$
(ii) $f(x) = 3 + (x-2)^{2/3}$ on $[1, 3]$
(iii) $f(x) = \tan x$ on $[0, \pi]$

Solution :- (i) we have $f(x) = |x| = \begin{cases} -x, & \text{when } -1 \leq x \leq 0 \\ x, & \text{when } 0 \leq x \leq 1 \end{cases}$

Since a polynomial function is every where continuous and differentiable, therefore $f(x)$ is continuous and differentiable for all $x < 0$ and for all $x > 0$ except possible at $x = 0$

So, consider the point $x = 0$

we have $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} -x = 0$ and $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Thus, $f(x)$ is continuous at $x = 0$. Hence $f(x)$ is continuous on $[-1, 1]$

$$\begin{aligned} (\text{LHD at } x=0) &= \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \rightarrow 0^-} \frac{-x - 0}{x} \quad [\because f(x) = -x \text{ for } x < 0 \text{ \& } f(0) = 0] \\ &= \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} -1 = -1 \end{aligned}$$

$$\begin{aligned} (\text{RHD at } x=0) &= \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{x - 0}{x} \quad [\because f(x) = x \text{ for } x > 0] \\ &= \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1 \end{aligned}$$

$$\therefore (\text{LHD at } x=0) \neq (\text{RHD at } x=0)$$

This show that $f(x)$ is not differentiable at $x = 0 \in (-1, 1)$
Thus, the condition of derivability at each point of $(-1, 1)$ not satisfied,
Hence, Rolle's theorem is not applicable to $f(x) = |x|$ on $[-1, 1]$

(ii) we have $f(x) = 3 + (x-2)^{2/3}$, $x \in [2, 3]$
 $f'(x) = \left(\frac{2}{3}\right)(x-2)^{-1/3}$

clearly, $\lim_{x \rightarrow 2} f'(x) = \infty$. So $f(x)$ is not differentiable at $x = 2 \in (1, 3)$

Hence Rolle's theorem is not applicable to $f(x) = 3 + (x-2)^{2/3}$ on the interval $[1, 3]$

(iii) we have $f(x) = \tan x$, $x \in [0, \pi]$. Since $\pi/2 \in [0, \pi]$ and $f(x)$ is continuous at $x = \pi/2$.

So, the condition of continuity at each point of $[0, \pi/2]$ is not satisfied.

Hence, Rolle's theorem is not applicable to $f(x) = \tan x$ on the interval $[0, \pi]$

EX:-2

Discuss the applicability of Rolle's theorem on the function

$$f(x) = \begin{cases} x^2+1, & \text{when } 0 \leq x \leq 1 \\ 3-x, & \text{when } 1 \leq x \leq 2 \end{cases}$$

Solution:-

Since a polynomial function is every where continuous and differentiable, therefore $f(x)$ is continuous and differentiable at all points except possibility at $x=1$.

Now, we consider the differentiability of $f(x)$ at $x=1$

$$\begin{aligned} \text{we have (LHD at } x=1) &= \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \rightarrow 1^-} \frac{(x^2-1) - (1+1)}{x-1} \left[\because f(x) = x^2+1 \text{ for } 0 \leq x \leq 1 \right] \\ &= \lim_{x \rightarrow 1^-} \frac{x^2-1}{x-1} = \lim_{x \rightarrow 1^-} (x+1) = 2 \end{aligned}$$

$$\text{and (RHD at } x=1) = \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \rightarrow 1^+} \frac{(3-x) - (1+1)}{x-1} = \lim_{x \rightarrow 1^+} \frac{-(x-1)}{x-1} = -1$$

$\therefore$ (LHD at $x=1$) $\neq$ (RHD at $x=1$). So, $f(x)$ is not differentiable at $x=1$.

Thus, the condition of differentiability at each point of the given interval is not satisfied.

Hence Rolle's theorem is not applicable to the given function on the interval $[0, 2]$.

EX:-3

Verify Rolle's theorem for the function $f(x) = x^2 - 5x + 6$ on the interval $[2, 3]$

Solution:- Since a polynomial function is every where differentiable and so continuous also. Therefore,

(i) $f(x)$ is continuous on $[2, 3]$

(ii) $f(x)$ is differentiable on $(2, 3)$

$$\text{Also } f(2) = 0 \text{ and } f(3) = 0$$

$$f(2) = f(3)$$

thus all the conditions of Rolle's theorem are satisfied. Now we have to show that there exists some $c \in (2, 3)$ such that $f'(c) = 0$

For this we proceed as follows :-

$$\text{we have } f(x) = x^2 - 5x + 6 \Rightarrow f'(x) = 2x - 5$$

$$f'(x) = 2x - 5 = 0 \Rightarrow x = 2.5$$

Thus $c = 2.5 \in (2, 3)$ such that $f'(c) = 0$

Hence Rolle's theorem is verified.

EX:- 4

Verify Rolle's theorem for the function $f(x) = x(x-3)^2$. $0 \leq x \leq 3$

Solution:-

$$\text{We have } f(x) = x^3 - 6x^2 + 9x$$

We know that a polynomial function is everywhere differentiable and so continuous also. So, $f(x)$ being a polynomial function is continuous on $[0, 3]$ and differentiable on $(0, 3)$

$$\text{Also } f(0) = f(3) = 0$$

Thus, all the conditions of Rolle's theorem are satisfied. Now we have to show that $\exists c \in (0, 3) \ni f'(c) = 0$

$$\text{We have } f(x) = x^3 - 6x^2 + 9x \Rightarrow f'(x) = 3x^2 - 12x + 9$$

$$\therefore f'(x) = 0 \Rightarrow 3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = 1, 3$$

$$\text{Thus } c = 1 \in (0, 3) \ni f'(c) = 0$$

Hence Rolle's theorem is verified.

EX:- 5

Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$ on the interval $[1, 3]$.

Solution:-

Since a polynomial function is everywhere continuous and differentiable therefore $f(x)$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$.

Thus, all the three conditions of Rolle's theorem are satisfied.

Now we have to show that $\exists c \in (1, 3) \ni f'(c) = 0$

$$\text{We have } f(x) = x^3 - 6x^2 + 11x - 6 \Rightarrow f'(x) = 3x^2 - 12x + 11$$

$$\therefore f'(x) = 0 \Rightarrow 3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{12 \pm \sqrt{144 - 132}}{6} = x = 2 \pm \frac{1}{\sqrt{3}}$$

Clearly, both the values of x lie in the interval $(1, 3)$

Thus $c = 2 + \frac{1}{\sqrt{3}} \in (1, 3)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

EX:- 6

Verify Rolle's theorem for the function $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$

Solution:-

Clearly, $f(x)$ is defined for all $x \in [-2, 2]$ and has a unique value for each $x \in [-2, 2]$, so at each point of $[-2, 2]$ the limit of $f(x)$ is equal to the value of the function.

Therefore, $f(x)$ is continuous on $[-2, 2]$

Also, $f'(x) = \frac{-x}{\sqrt{4-x^2}}$ exist for all $x \in (-2, 2)$. So, $f(x)$ is differentiable on $(-2, 2)$

Also, $f(-2) = f(2) = 0$

Thus, all the three conditions of Rolle's theorem is satisfied. Now we have to show that $\exists c \in (-2, 2) \ni f'(c) = 0$

We have, $f(x) = \sqrt{4-x^2} \Rightarrow f'(x) = \frac{-x}{\sqrt{4-x^2}}$

$f'(x) = 0 \Rightarrow \frac{-x}{\sqrt{4-x^2}} = 0 \Rightarrow x = 0$

Since $c = 0 \in (-2, 2)$ such that $f'(c) = 0$. Hence Rolle's theorem is verified.

EX: -7

Verify Rolle's theorem for the function $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ on $[a, b]$, where $0 < a < b$.

Solution:-

We have $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right) = \log(x^2+ab) - \log x(a+b) = \log(x^2+ab) - \log x - \log(a+b)$

Since logarithmic function is differentiable & so continuous on its domain. Therefore, $f(x)$ is continuous on $[a, b]$ & differentiable on (a, b)

Also, $f(a) = \log\left(\frac{a^2+ab}{a(a+b)}\right) = \log 1 = 0$ & $f(b) = \log\left(\frac{b^2+ab}{b(a+b)}\right) = \log 1 = 0$

$\therefore f(a) = f(b)$

Thus, all the three conditions of Rolle's theorem are satisfied. Now we have to show that $\exists c \in (a, b) \ni f'(c) = 0$

We have $f(x) = \log(x^2+ab) - \log x - \log(a+b) \Rightarrow f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$

$\therefore f'(x) = 0 \Rightarrow \frac{x^2-ab}{x(x^2+ab)} = 0 \Rightarrow x^2 = ab \Rightarrow x = \sqrt{ab}$

Since $a < \sqrt{ab} < b$.

$\therefore c = \sqrt{ab} \in (a, b)$ is such that $f'(c) = 0$

Hence Rolle's theorem is verified.

EX: -8

Verify Rolle's theorem for each of the following functions on indicated intervals:

(i) $f(x) = \sin^2 x$ on $0 \leq x \leq \pi$

(ii) $f(x) = \sin x + \cos x - 1$ on $[0, \pi/2]$

(iii) $f(x) = \sin x - \sin 2x$ on $[0, \pi]$

Solution:-

(i) Since $\sin x$ is every where continuous and differentiable and the product of two continuous, (differentiable) function is continuous (differentiable), therefore, $f(x) = \sin^2 x = \sin x \cdot \sin x$ is continuous on $[0, \pi]$ & differentiable on $(0, \pi)$

Also, $f(0) = \sin^2 0 = 0$ & $f(\pi) = \sin^2 \pi = 0$
 $\therefore f(0) = f(\pi)$

Thus, $f(x)$ satisfies all the condition of Rolle's theorem.

Now, we have to show that $\exists c \in (0, \pi) \ni f'(c) = 0$

We have $f(x) = \sin^2 x \Rightarrow f'(x) = 2 \sin x \cdot \cos x = \sin 2x$

$$f'(x) = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = \pi \Rightarrow x = \pi/2$$

Since $c = \pi/2 \in (0, \pi) \ni f'(c) = 0$. Hence Rolle's theorem is verified.

(ii) Since $\sin x$ and $\cos x$ are every where continuous and differentiable.

$f(x) = \sin x + \cos x - 1$ is continuous on $[0, \pi/2]$ & differentiable on $(0, \pi/2)$

$$\text{Also, } f(0) = \sin 0 + \cos 0 - 1 = 0 \text{ \& } f(\pi/2) = \sin \pi/2 + \cos \pi/2 - 1 = 1 - 1 = 0$$

$$\therefore f(0) = f(\pi/2)$$

Thus, $f(x)$ satisfies conditions of Rolle's theorem on $[0, \pi/2]$

$$\therefore \exists c \in (0, \pi/2) \ni f'(c) = 0$$

$$\text{Now } f(x) = \sin x + \cos x - 1 \Rightarrow f'(x) = \cos x - \sin x$$

$$\text{Now } f(x) = \sin x + \cos x - 1 \Rightarrow f'(x) = \cos x - \sin x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4$$

$$\therefore f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4$$

Thus $c = \pi/4 \in (0, \pi/2) \ni f'(c) = 0$. Hence Rolle's theorem is verified.

(iii) Since sine function is every where continuous and differentiable

therefore so are $\sin x$ and $\sin 2x$. consequently

$f(x) = \sin x - \sin 2x$ is continuous on $[0, \pi]$, and differentiable on $(0, \pi)$

$$\text{Also, } f(0) = \sin(0) - \sin(0) = 0 \text{ and } f(\pi) = \sin \pi - \sin 2\pi = 0$$

$$\therefore f(0) = f(\pi)$$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $[0, \pi]$.

consequently $\exists c \in (0, \pi) \ni f'(c) = 0$

$$\text{Now, } f(x) = \sin x - \sin 2x \Rightarrow f'(x) = \cos x - 2 \cos 2x$$

$$\therefore f(x) = 0 \Rightarrow \cos x - 2 \cos 2x = 0 \Rightarrow \cos x - 2(\cos^2 x - 1) = 0$$

$$\Rightarrow 4 \cos^2 x - \cos x - 2 = 0 \Rightarrow \cos x = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$$

$$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } \cos^{-1}(-0.5931)$$

$$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } 180^\circ - \cos^{-1}(0.5931)$$

$$\Rightarrow x = 32^\circ 32' \text{ or } x = 126^\circ 23'$$

$$\text{Thus } c = 32^\circ 32' ; 126^\circ 23' \in (0, \pi) \ni f'(c) = 0$$

Hence Rolle's theorem is verified.

EX:-9

Verify Rolle's theorem for each of the following function on the indicated intervals:

$$(i) f(x) = x(x+3)e^{-x/2} \text{ on } [-3, 0]$$

$$(ii) f(x) = e^x(\sin x - \cos x) \text{ on } [\pi/4, 5\pi/4]$$

Solution:-

Q) Since a polynomial function and an exponential function are everywhere continuous and differentiable, therefore $f(x)$, being product of these two, is continuous on $[-3, 0]$ and differentiable on $(-3, 0)$

$$\text{Also, } f(-3) = -3(-3+3)e^{3/2} = 0 \text{ \& } f(0) = 0$$

$$f(-3) = f(0)$$

Thus $f(x)$ satisfies all three conditions of Rolle's theorem on $[-3, 0]$.

consequently $\exists c \in (-3, 0) \ni f'(c) = 0$

$$\text{Now } f(x) = x(x+3)e^{-x/2} \Rightarrow f'(x) = (2x+3)e^{-x/2} + (x^2+3x)(-1/2)e^{-x/2} = e^{-x/2} \left\{ \frac{-x^2+x+6}{2} \right\}$$

$$\therefore f'(x) = 0 \Rightarrow e^{-x/2} \left\{ \frac{-x^2+x+6}{2} \right\} = 0 \Rightarrow -x^2+x+6 = 0 \Rightarrow x^2-x-6 = 0 \Rightarrow (x-3)(x+2) = 0$$

$$x = -2, 3$$

Thus $c = -2 \in (-3, 0) \ni f'(c) = 0$. Hence Rolle's theorem is verified.

Q) Since an exponential function & sine and cosine function are everywhere continuous & differentiable, therefore $f(x)$ is continuous on $[\pi/4, 5\pi/4]$ and differentiable on $(\pi/4, 5\pi/4)$.

$$\text{Also, } f\left(\frac{\pi}{4}\right) = e^{\pi/4}(\sin \pi/4 - \cos \pi/4) = e^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$

$$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}(\sin 5\pi/4 - \cos 5\pi/4) = e^{5\pi/4}(-\sin \pi/4 + \cos \pi/4) = e^{5\pi/4}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$$

$$f(\pi/4) = f(5\pi/4)$$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $[\pi/4, 5\pi/4]$. consequently $\exists c \in (\pi/4, 5\pi/4) \ni f'(c) = 0$.

$$\text{Now } f(x) = e^x(\sin x - \cos x) \Rightarrow f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x) = 2e^x \sin x$$

$$\therefore f'(x) = 0 \Rightarrow 2e^x \sin x = 0 \Rightarrow \sin x = 0 \text{ (} e^x \neq 0 \text{)}$$

$$\text{Thus } c = \pi \in (\pi/4, 5\pi/4) \ni f'(c) = 0$$

Hence Rolle's theorem is verified.

VERIFICATION OF LAGRANGE'S MEAN VALUE THEOREM:-

Ex:-1

Verify Lagrange's mean value theorem for the function

$$f(x) = (x-3)(x-6)(x-9) \text{ on the interval } [3, 9]$$

Solution:-

$$\text{We have } f(x) = (x-3)(x-6)(x-9) = x^3 - 18x^2 + 99x - 162$$

Since a polynomial function is everywhere continuous and differentiable therefore $f(x)$ is continuous on $[3, 9]$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied.

So $\exists$ at least one real number,

$$c \in (3, 9) \ni f'(c) = \frac{f(9) - f(3)}{9-3}$$

Now, $f(x) = x^3 - 18x^2 + 99x - 162 \Rightarrow f'(x) = 3x^2 - 36x + 99$

$f(5) = (5-3)(5-6)(5-9) = 0$ & $f(3) = 0$

$\therefore f'(x) = \frac{f(5) - f(3)}{5-3} \Rightarrow 3x^2 - 36x + 99 = \frac{0-0}{5-3} \Rightarrow 3x^2 - 36x + 99 = 0$

$\Rightarrow 3x^2 - 36x + 99 = 0 \Rightarrow x = \frac{36 \pm \sqrt{1296 - 1188}}{6} = \frac{36 \pm 12.49}{6} = 8.8 \text{ \& } 4.8$

Thus $c = 4.8 \in (3, 5) \ni f'(c) = \frac{f(5) - f(3)}{5-3}$

Hence Lagrange's mean value theorem is verified.

Ex: -2

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. Also, find a point c in the indicated interval.

(i) $f(x) = x(x-2)$ on $[1, 3]$

(ii) $f(x) = x(x-1)(x-2)$ on $[0, 1/2]$

Solution: -

We have $f(x) = x(x-2) = x^2 - 2x$.

We know that the polynomial function is every where continuous and differentiable. So $f(x)$ being a polynomial, is continuous on $[1, 3]$ and differentiable on $(1, 3)$

Thus $f(x)$ satisfies both the conditions of Lagrange's mean value theorem on $[1, 3]$

So there must exist at least one real number $c \in (1, 3) \ni f'(c) = \frac{f(3) - f(1)}{3-1}$

Now $f(x) = x^2 - 2x \Rightarrow f'(x) = 2x - 2$, $f(3) = 9 - 6 = 3$ & $f(1) = 1 - 2 = -1$

$\therefore f'(x) = \frac{f(3) - f(1)}{3-1} \Rightarrow 2x - 2 = \frac{3 - (-1)}{3-1} \Rightarrow 2x - 2 = 2 \Rightarrow x = 2$

Thus $c = 2 \in (1, 3) \ni f'(c) = \frac{f(3) - f(1)}{3-1}$

Hence Lagrange's mean value theorem is verified for $f(x)$ on $[1, 3]$

(ii) we have $f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$

Since $f(x)$ is a polynomial function and a polynomial function is every where continuous and differentiable. Therefore, $f(x)$ is continuous on $[0, 1/2]$ and differentiable on $(0, 1/2)$

Thus, both the conditions of Lagrange's mean value theorem are satisfied. So there must exist at least one real number.

$c \in (0, 1/2) \ni f'(c) = \frac{f(1/2) - f(0)}{1/2 - 0}$

Now, $f(x) = x^3 - 3x^2 + 2x \Rightarrow f'(x) = 3x^2 - 6x + 2$, $f(0) = 0$ & $f(1/2) = \frac{1}{8} - \frac{3}{4} + 1 = 3/8$

$\therefore f'(x) = \frac{f(1/2) - f(0)}{1/2 - 0} \Rightarrow 3x^2 - 6x + 2 = \frac{3/8 - 0}{1/2 - 0} \Rightarrow 3x^2 - 6x + 2 = 3/4$

$$\Rightarrow 12x^2 - 24x + 5 = 0 \Rightarrow x = \frac{24 \pm \sqrt{336}}{24} = 1 \pm \frac{\sqrt{21}}{6}$$

Since $c = 1 - \frac{\sqrt{21}}{6} \in (0, 1/2) \Rightarrow f'(c) = \frac{f(1/2) - f(0)}{1/2 - 0}$

Hence Lagrange's mean value theorem is verified.

Ex: -4

Verify Lagrange's mean value theorem for the following function on the indicated intervals.

(i) $f(x) = x - 2\sin x$ on $[-\pi, \pi]$ (ii) $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$

(iii) $f(x) = \log_e x$ on $[1, 2]$

(iv) $f(x) = \begin{cases} x+x^2 & \text{if } x \leq 1 \\ 2x & \text{if } x > 1 \end{cases}$ on $[-1, 2]$

Solution:-

(i) Since x and $\sin x$ are everywhere continuous and differentiable, therefore $f(x)$ is continuous on $[-\pi, \pi]$ & differentiable on $(-\pi, \pi)$

Thus, both the conditions of Lagrange's mean value theorem are satisfied. So there exist at least one

$$c \in (-\pi, \pi) \Rightarrow f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$$

Now, $f(x) = x - 2\sin x \Rightarrow f'(x) = 1 - 2\cos x$, $f(\pi) = \pi - 2\sin \pi = \pi$

and $f(-\pi) = -\pi - 2\sin(-\pi) = -\pi$

$$\therefore f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} \Rightarrow 1 - 2\cos c = \frac{\pi - (-\pi)}{\pi - (-\pi)} \Rightarrow 1 - 2\cos c = 1 \Rightarrow \cos c = 0 \Rightarrow c = \pm \frac{\pi}{2}$$

Thus $c = \pm(\frac{\pi}{2}) \in (-\pi, \pi) \Rightarrow f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$

Hence Lagrange's mean value theorem is verified.

(ii) Since $\sin x$ and $\sin 2x$ are everywhere continuous and differentiable, therefore $f(x)$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$

Thus, $f(x)$ satisfies both the conditions of Lagrange's mean value theorem consequently $\exists$ at least one

$$c \in (0, \pi) \Rightarrow f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

Now $f(x) = 2\sin x + \sin 2x \Rightarrow f'(x) = 2\cos x + 2\cos 2x$, $f(0) = 0$ and $f(\pi) = 2\sin \pi + \sin 2\pi = 0$

$$\therefore f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} \Rightarrow 2\cos c + 2\cos 2c = \frac{0 - 0}{\pi - 0} \Rightarrow 2\cos c + 2\cos 2c = 0$$

$$\Rightarrow \cos c + \cos 2c = 0$$

$$\Rightarrow \cos 2c = -\cos c \Rightarrow \cos 2c = \cos(\pi - c)$$

Thus $c = \pi/3 \in (0, \pi) \Rightarrow f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} \Rightarrow 2c = \pi - c \Rightarrow 3c = \pi \Rightarrow c = \pi/3$

Hence Lagrange's mean value theorem is verified.

(iii) Since $f(x) = \log_e x$ is differentiable and so continuous $\forall x > 0$.

So $f(x)$ is continuous on $[1, 2]$ and differentiable on $(1, 2)$

Thus, both the conditions of Lagrange's mean value theorem are satisfied

consequently, $\exists$ some $c \in (1, 2)$ $\ni f'(c) = \frac{f(2) - f(1)}{2 - 1}$

Now, $f(x) = \log_e^2 x \Rightarrow f'(x) = 1/x$, $f(2) = \log_e^2 2$ & $f(1) = \log_e^2 1 = 0$

$$\therefore f'(x) = \frac{f(2) - f(1)}{2 - 1} \Rightarrow \frac{1}{x} = \frac{\log_e^2 2 - 0}{2 - 1} \Rightarrow \frac{1}{x} = \log_e^2 2 \Rightarrow x = \frac{1}{\log_e^2 2} = \log_2^2 e$$

Since $2 < e < 4$. $\therefore \log_2^2 2 < \log_2^2 e < \log_2^2 4$ i.e., $1 < \log_2^2 e < 2$

then $c = \log_2^2 e \in (1, 2)$ $\ni f'(c) = \frac{f(2) - f(1)}{2 - 1}$

Hence Lagrange's mean value theorem is verified.

(iv) Since $2+x^3$ and $3x$ are polynomial functions therefore $f(x)$ is continuous and differentiable for all values of x except possibility at $x=1$.

continuity at $x=1$:- we know $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2+x^3) = 3$

$$\text{and } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 3x = 3 \times 1 = 3$$

$$\text{Also } f(1) = 2+1^3 = 3$$

$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(1)$. So $f(x)$ is continuous at $x=1$.

Differentiability at $x=1$;

$$\begin{aligned} (\text{LHD at } x=1) &= \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \rightarrow 1} \frac{2+x^3-3}{x-1} = \lim_{x \rightarrow 1} \frac{x^3-1}{x-1} \\ &= \lim_{x \rightarrow 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x \rightarrow 1} x^2+x+1 = 1+1+1=3 \end{aligned}$$

$$\text{and } (\text{RHD at } x=1) = \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \rightarrow 1} \frac{3x-3}{x-1} = \lim_{x \rightarrow 1} \frac{3(x-1)}{x-1} = 3$$

$$(\text{LHD at } x=1) = (\text{RHD at } x=1)$$

So $f(x)$ is differentiable at $x=1$.

Thus, $f(x)$ is continuous and differentiable on $[-1, 2]$. So both the conditions of Lagrange's mean value theorem are satisfied. Consequently, there must exist some $c \in (-1, 2)$ $\ni f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$

$$\text{Now } f(x) = \begin{cases} 2+x^3, & x \leq 1 \\ 3x, & x > 1 \end{cases} \Rightarrow f'(x) = \begin{cases} 3x^2, & x \leq 1 \\ 3, & x > 1 \end{cases}$$

$$f(-1) = 2+(-1)^3 = 1 \text{ and } f(2) = 3(2) = 6$$

$$\therefore f'(x) = \frac{f(2) - f(-1)}{2 - (-1)} \Rightarrow f'(x) = \frac{6-1}{2+1} = 5/3$$

Since $f'(x) = 3$ for $x > 1$, the value of x must be less than 1.

$$\begin{aligned} f'(x) &= 5/3 \Rightarrow 3x^2 = 5/3 \quad [\because x < 1 \text{ \& for } x < 1, f'(x) = 3x^2] \\ \Rightarrow x^2 &= 5/9 \Rightarrow x = \pm \sqrt{5}/3 \end{aligned}$$

Since $c = \pm \sqrt{5}/3 \in (-1, 2)$ $\ni f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$. Hence Lagrange's mean value theorem is verified.