

Vector calculus

1. scalar and vector functions.
2. Gradient, Directional Derivatives
3. Divergence.
4. curl.
5. line integrals
6. Double integrals
7. Green's Theorem

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 Differentiation

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 Integration

scalar point function

A function that gives a real number (scalar value) to every point of a part of the region of space.

e.g: $f(x, y, z) = xy + yz + zx$

$f(1, 2, 3) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 = 11$
(scalar).

vector point function

A function that gives a vector to every point of a part of the region of space.

e.g: $\vec{f}(x, y, z) = xy\hat{i} + yz\hat{j} + zx\hat{k}$

$\vec{f}(1, 2, 3) = 2\hat{i} + 6\hat{j} + 3\hat{k}$

operator del (∇)

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

(scalar fun) f

$\vec{f}$ (vector fun)

(gradient)

$\nabla f = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) f$

$= \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}$

$\nabla \cdot \vec{f} = \text{divergent}$
 $\nabla \times \vec{f} = \text{curl}$

Derivative of scalar funⁿ / Gradient

$$\text{grad } f = \left| \nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} \right|$$

eg: If $f = xz^2 - 5yz + xz$, then find grad f
or find ∇f at $(1, -1, 2)$

Ans $\rightarrow$

$$\begin{aligned} \nabla f &= \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} \\ &= \hat{i} (z^2 + z) + \hat{j} (-5z) + \hat{k} (2xy + z - 5y) \end{aligned}$$

$$\begin{aligned} \nabla f \big|_{(1, -1, 2)} &= \hat{i} (4 + 2) + \hat{j} (-10) + \hat{k} (2 \cdot 2 + 2 - 5) \\ &= 6\hat{i} - 10\hat{j} + 10\hat{k} \end{aligned}$$

Q: Find ∇f if $f = xy^3z^2 - 4$

Directional Derivative

D.D of f in the direction of $\vec{a}$

$$= \frac{\nabla f \cdot \vec{a}}{|\vec{a}|}$$

eg: Find the directional derivative of

$f = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ in the

direction of $2\hat{i} + 3\hat{j} + 6\hat{k}$

$$\text{Ans } \nabla f = \hat{i}(4z^3 - 6xy^2z) + \hat{j}(-6x^2yz) + \hat{k}(12xz^2 - 3x^2y^2)$$

$$\nabla f \Big|_{(2,-1,2)} = \hat{i}(32-24) + \hat{j}(48) + \hat{k}(96-12) \\ = 8\hat{i} + 48\hat{j} + 84\hat{k}$$

$$\text{Let } \vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$|\vec{a}| = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$\therefore D \cdot D = \frac{\nabla f \cdot \vec{a}}{|\vec{a}|} = \frac{16+144+504}{7} = \frac{664}{7}$$

e.g: find the directional derivative of

$$f = x^2 + y^2 + z^2 \text{ at } A(1, -2, 1) \text{ in the direction of } \vec{AB}$$

where. $B(2, 6, -1)$

Ans

$$\vec{AB} = (2, 6, -1) - (1, -2, 1)$$

$$= (1, 8, -2) = \hat{i} + 8\hat{j} - 2\hat{k}$$

$$\nabla f = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla f \Big|_{(1,-2,1)} = 2\hat{i} - 4\hat{j} + 2\hat{k}$$

$$\vec{a} = 2\hat{i} + 8\hat{j} - 2\hat{k}$$

$$|\vec{a}| = \sqrt{4+64+4} = \sqrt{72}$$

$$\therefore D.D = \frac{\nabla f \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{2 - 32 - 4}{\sqrt{69}} = \frac{-34}{\sqrt{69}}$$

Normal vector ($\vec{N}$)

Normal vector of a function $f = \text{grad } f$

$$\text{unit normal vector } \hat{N} = \frac{\vec{N}}{|\vec{N}|}$$

Properties of Gradient

1. If f is a ^{constant} scalar point function,

$$\text{then } \nabla f = \vec{0}$$

2. If f_1 and f_2 are two scalar point function

$$\text{then (i) } \nabla(f_1 \pm f_2) = \nabla f_1 \pm \nabla f_2$$

$$(ii) \nabla(c f_1) = c \nabla f_1$$

$$(iii) \nabla(f_1 f_2) = f_1 \nabla f_2 + f_2 \nabla f_1$$

$$(iv) \nabla\left(\frac{f_1}{f_2}\right) = \frac{f_2 \nabla f_1 - f_1 \nabla f_2}{(f_2)^2}, \quad f_2 \neq 0$$

eg: Find grad f when $f = 5x^2y - 5y^2z + \frac{5}{2}z^2x$
at point $P(1, 1, 1)$. Also find its directional

derivative in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

Ans $\rightarrow f = 5x^2y - 5y^2z + \frac{5}{2}z^2x$

$$\nabla f = (10xy + \frac{5}{2}z^2) \hat{i} + (5x^2 - 10yz) \hat{j} + (-5y^2 + 5zx) \hat{k}$$

$$\nabla f|_{(1,1,1)} = \frac{25}{2} \hat{i} - 5 \hat{j}$$

The direction of the given line is

$$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k} \quad (\text{Taking dms})$$

$$\Rightarrow |\vec{a}| = \sqrt{4+4+1} = 3$$

$$\therefore D \cdot D = \frac{\nabla f \cdot \vec{a}}{|\vec{a}|} = \frac{25 + 10}{3} = \frac{35}{3}$$

e.g: Find the directional derivative of

$$f(x, y, z) = e^{2x} \cos(yz) \text{ at } (0, 0, 0).$$

in the direction of the tangent to the curve

$$(x, y, z) = (a \sin t, a \cos t, at) \text{ at } t = \frac{\pi}{4}$$

Ans $\rightarrow f(x, y, z) = e^{2x} \cos yz$

$$\nabla f = \hat{i} (2e^{2x} \cos yz) + \hat{j} (-ze^{2x} \sin yz) + \hat{k} (-ye^{2x} \sin yz)$$

$$\left. \frac{df}{dt} \right|_{(0,0,0)} = 2\hat{i}$$

Given curve is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
 $= (a \sin t)\hat{i} + (a \cos t)\hat{j} + at\hat{k}$

Tangent to the curve is

$$\frac{d\vec{r}}{dt} = a \cos t \hat{i} - a \sin t \hat{j} + a \hat{k}$$

$$\left. \frac{d\vec{r}}{dt} \right|_{t=\frac{\pi}{4}} = a \cos \frac{\pi}{4} \hat{i} - a \sin \frac{\pi}{4} \hat{j} + a \hat{k}$$

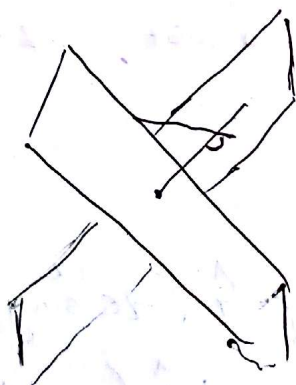
$$= \frac{a}{\sqrt{2}} \hat{i} - \frac{a}{\sqrt{2}} \hat{j} + a \hat{k}$$

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2} = a\sqrt{2}$$

$$\therefore \text{Unit D.D} = \frac{\frac{df}{dt} \cdot \frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|} = \frac{2 \cdot \frac{a}{\sqrt{2}}}{a\sqrt{2}} = 1$$

Q: Find the angle between the surfaces / planes

$$x^2 + y^2 + z^2 = 9 \text{ and } z = x + y - 3 \text{ at the point } L(2, -1, 2)$$



angle between planes
 $=$ angle between
 their normal
 $=$ angle between
 their gradients.

$$\text{let } \vec{F}_1 = x^2 + y^2 + z^2 - 9$$

$$\text{grad } \vec{F}_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad } \vec{F}_1 \Big|_{(2, -1, 2)} = 4\hat{i} - 2\hat{j} + 4\hat{k} = \vec{n}_1$$

$$\text{let } \vec{F}_2 = x^2 + y^2 - z - 3$$

$$\text{grad } \vec{F}_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\text{grad } \vec{F}_2 \Big|_{(2, -1, 2)} = 4\hat{i} - 2\hat{j} - \hat{k} = \vec{n}_2$$

$\therefore$ Angle between $\vec{n}_1$ & $\vec{n}_2$

$$= \cos^{-1} \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right)$$

$$= \cos^{-1} \left(\frac{16 + 4 - 4}{\sqrt{16 + 4 + 16} \sqrt{16 + 1 + 1}} \right)$$

$$= \cos^{-1} \left(\frac{16}{6\sqrt{21}} \right) = \cos^{-1} \left(\frac{8}{3\sqrt{21}} \right)$$

Derivative of vector point functions

$$\begin{cases} \nabla \cdot \vec{F} = \text{divergence} \\ \nabla \times \vec{F} = \text{curl} \end{cases}$$

Divergence

The divergence of a vector point function $\vec{F}$ is denoted by $\text{div } \vec{F}$ and

defined by

$$\begin{aligned} \text{div } \vec{F} &= \nabla \cdot \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \vec{F} \\ &= \hat{i} \cdot \frac{\partial \vec{F}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{F}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{F}}{\partial z} \end{aligned}$$

or

$$\text{If } \vec{F} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\begin{aligned} \text{div } \vec{F} &= \nabla \cdot \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \\ &= \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \end{aligned}$$

e.g: Let $\vec{F} = xyz \hat{i} + 3x^2y \hat{j} + (xz^2 - y^2z) \hat{k}$

Find $\text{div } \vec{F}$ at $(2, -1, 1)$

$$\begin{aligned} \text{Ans} \rightarrow \text{div } \vec{F} &= \frac{\partial (xyz)}{\partial x} + \frac{\partial (3x^2y)}{\partial y} + \frac{\partial (xz^2 - y^2z)}{\partial z} \\ &= yz + 3x^2 + 2xz - y^2 \end{aligned}$$

$$\begin{aligned} \text{div } \vec{F} \Big|_{(2, -1, 1)} &= -1 + 12 + 4 - 1 \\ &= 14 \end{aligned}$$

Curl :
The curl of a vector point function $\vec{F}$ is denoted by $\text{curl } \vec{F}$ and defined by

$$\begin{aligned} \text{curl } \vec{F} &= \nabla \times \vec{F} \\ &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \vec{F} \\ &= \hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z} \end{aligned}$$

$$\text{or} \quad \text{If } \vec{F} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

eg. Find the curl of $\vec{F} = 2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}$ at $(1,1,1)$

Ans
→

$$a_1 = 2xz^2, \quad a_2 = -yz, \quad a_3 = 3xz^3$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \hat{i} \left\{ \frac{\partial}{\partial y}(3xz^3) - \frac{\partial}{\partial z}(-yz) \right\}$$

$$- \hat{j} \left\{ \frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2) \right\} + \hat{k} \left\{ \frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2) \right\}$$

$$= \hat{i} (0 - (-y)) - \hat{j} (3z^3 - 4xz) + \hat{k} (0 - 0)$$

$$= y\hat{i} - (3z^3 - 4xz)\hat{j}$$

$$\text{curl } \vec{F} \big|_{(1,1,1)} = \hat{i} - (3 - 4)\hat{j} = \hat{i} + \hat{j}$$

Q. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that

$$(i) \text{ div } \vec{r} = 3$$

$$(ii) \text{ curl } \vec{r} = \vec{0}$$

Ans
→

$$(i) \text{ div } \vec{r} = \nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\ = 1 + 1 + 1 = 3$$

$$\begin{aligned}
 \text{(ii) } \text{curl } \vec{r} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
 &= \hat{i} \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) - \hat{j} \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) + \hat{k} \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \\
 &= \hat{i}(0) - \hat{j}(0) + \hat{k}(0) \\
 &= \vec{0}
 \end{aligned}$$

eg: Find $\text{curl}(\text{curl } \vec{v})$ where

$$\vec{v} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k} \text{ at } (1,1,1)$$

Ans $\rightarrow$ we have already found,

$$\text{curl } \vec{v} = y\hat{i} - (3z^3 - 4xz)\hat{j}$$

and $\text{curl } \vec{v} \Big|_{(1,1,1)} = \hat{i} - \hat{j}$

$$\text{now } \text{curl}(\text{curl } \vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 4xz - 3z^3 & 0 \end{vmatrix}$$

$$= \hat{i} \left(0 - \frac{\partial (4xz - 3z^3)}{\partial z} \right) - \hat{j} (0 - 0) + \hat{k} \left(\frac{\partial (4xz - 3z^3)}{\partial x} - \frac{\partial y}{\partial y} \right)$$

$$= \hat{i} (0 - (4x - 9z^2)) - \hat{j} (0 - 0) + \hat{k} (4z - 1)$$

$$= (9z^2 - 4x)\hat{i} + (4z - 1)\hat{k} = 5\hat{i} + 3\hat{k} \text{ at } (1,1,1)$$

Properties of divergence and curl

1. For a constant vector $\vec{a}$,

$$\text{div } \vec{a} = 0, \quad \text{curl } \vec{a} = \vec{0}$$

$$2. \text{div } (\vec{A} + \vec{B}) = \text{div } \vec{A} + \text{div } \vec{B}$$

$$\text{or } \nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$$

$$3. \text{curl } (\vec{A} + \vec{B}) = \text{curl } \vec{A} + \text{curl } \vec{B}$$

$$\text{or } \nabla \times (\vec{A} + \vec{B}) = (\nabla \times \vec{A}) + (\nabla \times \vec{B})$$

4. If $\vec{A}$ is a vector function and ϕ is a scalar function

$$\text{div } (\phi \vec{A}) = \phi \text{div } \vec{A} + (\text{grad } \phi) \cdot \vec{A}$$

$$\text{or } \nabla \cdot (\phi \vec{A}) = \phi (\nabla \cdot \vec{A}) + (\nabla \phi) \cdot \vec{A}$$

5. If $\vec{A}$ is a vector function and ϕ is a scalar function

$$\text{curl } (\phi \vec{A}) = (\text{grad } \phi) \times \vec{A} + \phi \text{curl } \vec{A}$$

$$\text{or } \nabla \times (\phi \vec{A}) = (\nabla \phi) \times \vec{A} + \phi (\nabla \times \vec{A})$$

⑧ If $\text{div } \vec{F} = 0$, then $\vec{F}$ is called solenoidal vector function.

⑨ If $\text{curl } \vec{F} = \vec{0}$, then $\vec{F}$ is called irrotational vector.

e.g: Prove that the vector $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$ is irrotational

Sol:

$$\text{curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \hat{i} \left(0 - \frac{\partial (y^2 + x^2y)}{\partial z} \right) - \hat{j} \left(0 - \frac{\partial (x^2 + xy^2)}{\partial z} \right) + \hat{k} \left(\frac{\partial (y^2 + x^2y)}{\partial x} - \frac{\partial (x^2 + xy^2)}{\partial y} \right)$$

$$= \hat{i} (0 - 0) + \hat{j} (0 - 0) + \hat{k} (2xy - 2xy)$$

$$= \vec{0}$$

as $\text{curl } \vec{A} = \vec{0}$, so $\vec{A}$ is irrotational.

e.g: If the vector $\vec{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$ is solenoidal, find the value of a .

Ans Given $\text{div } \vec{F} = 0$ (solenoidal)

$$\Rightarrow \frac{\partial}{\partial x}(ax^2 + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2xz^2) = 0$$

$$\Rightarrow 2axy + 2xy + 2xy = 0$$

$$\Rightarrow 2axy + 4xy = 0$$

$$\Rightarrow 2xy(a+2) = 0$$

$$\Rightarrow a+2 = 0$$

$$\Rightarrow a = -2$$

Physical meanings of gradient, divergence and curl

$\Rightarrow$ gradient of a scalar field ϕ is a vector that normal (perpendicular) to that surface ϕ .

$\Rightarrow$ Divergence gives the rate of outflow per unit volume at a point of any fluid.

$\Rightarrow$ curl: Angular velocity at a point of a body = $\frac{1}{2}$ (curl of linear velocity at that point of body)

Repeated operations by ∇

Let ϕ be a scalar function and $\vec{V}$ be a vector function

$$1. \text{div}(\text{grad } \phi) = \nabla^2 \phi \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$2. \text{curl}(\text{grad } \phi) = \nabla \times (\nabla \phi) = \vec{0}$$

$$3. \text{div}(\text{curl } \vec{V}) = \nabla \cdot (\nabla \times \vec{V}) = 0$$

$$4. \text{curl}(\text{curl } \vec{V}) = \text{grad}(\text{div } \vec{V}) - \nabla^2 \vec{V}$$

$$\text{or} \quad \nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$$