Subject: B. Tech. PHYSICS (3 – 0 – 0)

Course code 23BS1002 (AS PER BPUT SYLLABUS 2023-24)

Module-1

Waves and Oscillations

Periodic & Oscillatory Motion:-

The motion in which repeats after a regular interval of time is called periodic motion.

1. The periodic motion in which there is existence of a restoring force and the body moves along the same path to and fro about a definite point called equilibrium position/mean position, is called oscillatory motion.
2. In all type of oscillatory motion one thing is common i.e each body (performing oscillatory motion) is subjected to a restoring force that increases with increase in displacement from mean position.
3. **Types of oscillatory motion:-**

   It is of two types such as linear oscillation and circular oscillation.

   Example of linear oscillation:-

   1. Oscillation of mass spring system.
   2. Oscillation of fluid column in a U-tube.
   3. Oscillation of floating cylinder.
   4. Oscillation of body dropped in a tunnel along earth diameter.
   5. Oscillation of strings of musical instruments.

   Example of circular oscillation:-

   1. Oscillation of simple pendulum.
   2. Oscillation of solid sphere in a cylinder (If solid sphere rolls without slipping).
   3. Oscillation of a circular ring suspended on a nail.

4. Oscillation of balance wheel of a clock.
5. Rotation of the earth around the sun.

<u>Oscillatory system</u>:-

1. The system in which the object exhibit to & fro motion about the equilibrium position with a restoring force is called oscillatory system.
2. Oscillatory system is of two types such as mechanical and non- mechanical system.
3. **<u>Mechanical oscillatory system</u>:-**
   - In this type of system body itself changes its position.
   - For mechanical oscillation two things are specially responsible i.e Inertia & Restoring force.
   - E.g oscillation of mass spring system, oscillation of fluid-column in a U-tube, oscillation of simple pendulum, rotation of earth around the sun, oscillation of body dropped in a tunnel along earth diameter, oscillation of floating cylinder, oscillation of a circular ring suspended on a nail, oscillation of atoms and ions of solids, vibration of swings…etc.

4.<u>Non-mechanical oscillatory system</u>:-

In this type of system, body itself doesn‟t change its position but its physical property varies periodically.

e.g:-The electric current in an oscillatory circuit, the lamp of a body which is heated and cooled periodically, the pressure in a gas through a medium in which sound propagates, the electric and magnetic waves propagates undergoes oscillatory change.

Simple Harmonic Motion:-

It is the simplest type of oscillatory motion.

A particle is said to be execute simple harmonic oscillation is the restoring force is directed towards the equilibrium position and its magnitude is directly proportional to the magnitude and displacement from the equilibrium position.

If F is the restoring force on the oscillator when its displacement from the equilibrium position is x, then

$$F \alpha -x$$

Here, the negative sign implies that the direction of restoring force is opposite to that of displacement of body i.e towards equilibrium position.

$$F= -kx ....(1)$$

Where, k= proportionality constant called force constant.

$$Ma=-kx$$

$$M\frac{d2y}{dt2}=-kx$$

$$M\frac{d2y}{dt2}+kx=0$$

$$\frac{d2y}{dt2}+\frac{k}{M}x=0$$

$$\frac{d2y}{dt2}+\omega^2x=0................. (2)$$

Where $\omega^2=\frac{k}{M}$

Here $\omega=\sqrt{\frac{k}{M}}$ is the angular frequency of the oscillation.

Equation (2) is called general differential equation of SHM.

By solving these differential equation

$x=\alpha e^{-i\omega t} + \beta e^{i\omega t}$ ……… (3)

Where α, β are two constants which can be determined from the initial condition of a physical system.

Appling de-Moiver‟s theorem

$x= (\cos\omega t+i\sin\omega t) + \beta(\cos\omega t-i\sin\omega t)$

$x= (\alpha+ \beta) \cos\omega t+ (\alpha - \beta) \sin\omega t$

$x= C\cos\omega t+D\sin\omega t$ ………(4)

Where $C = \alpha + \beta$

& $D= \alpha - \beta$

Let assume,

$C=A\sin \theta$

$D=A\cos \theta$

Putting these value in equation (4)

$x=A\sin \theta\cos\omega t +A\cos \theta\sin\omega t$

$x=A (\sin \theta\cos\omega t + \cos \theta \sin\omega t)$

$x=A \sin(\omega t + \theta)$...............(5)

Where $A=\sqrt{(C^2+D^2)}$ & $\theta = \tan^{-1}(\frac{C}{D})$

Similarly, the solution of differential equation can be given as

x=Acos($\theta + \omega t$) ………(6)

Here A denotes amplitude of oscillatory system, ($+ \omega t$) is called phase and θ is called epoch/initial phase/phase constant/phase angel.

Equation (5) and (6) represents displacement of SHM.

Velocity in SHM:-

x=Asin($\omega t + \theta$)

$\frac{dx}{dt}$=Aωcos($\omega t + \theta$)

v=Aωcos($\omega t + \theta$) ....(7)

The minimum value of v is 0(as minimum value of Asin($\theta + \omega t$)=0 & maximum value is Aω. The maximum value of v is called velocity amplitude.

Acceleration in SHM:-

a= -Aω^2sin($\omega t + \theta$).................. (8)

The minimum value of „a‟ is 0 & maximum value is Aω^2. The maximum valueof „a‟ is called acceleration amplitude.

Also, a= ω^2x (from equation (5))

a a –y

It is also the condition for SHM.

Time period in SHM:-

The time required for one complete oscillation is called the time period (T). It is related to the angular frequency(ω) by.

T=$\frac{2\pi}{\omega}$........................ (9)

Frequency in SHM:-

The number of oscillation per time is called frequency or it is the reciprocal of time period.

$$\upsilon=\frac{1}{T}=\frac{\omega}{2\pi} \text{ ....(10)}$$

Potential energy in SHM:-

The potential energy of oscillator at any instant of time is,

$$U=-\int_0^x F dx$$

$$=-\int_0^x (-kx)\, dx$$

$$=\frac{1}{2} kx^2$$

$$=\frac{1}{2} kA^2\sin^2(\theta+\omega t) \text{ ....(11)}$$

(By using equation (5)).

Kinetic energy in SHM:-

The kinetic energy of oscillator at any instant of time is,

$$K=\frac{1}{2}\left(\frac{dx}{dt}\right)^2$$

$$=\frac{1}{2} Mv^2$$

$$K=\frac{1}{2} MA^2\omega^2 \cos^2(\theta+\omega t) \text{ .... (12)}$$

(By using equation (7))

Both kinetic and potential energy oscillate with time when the kinetic energy is maximum, the potential energy is minimum and vice versa. Both kinetic and potential energy attain their maximum value twice in one complete oscillation.

Total energy in SHM:-

Total energy= K.E+P.E

$$=\frac{1}{2}MA^2\omega^2\cos^2(\theta+\omega t)+\frac{1}{2}kA^2\sin^2(\theta+\omega t)$$

$$=\frac{1}{2}kA^2\cos^2(\theta+\omega t)+\frac{1}{2}kA^2\sin^2(\theta+\omega t)$$

Total energy $=\frac{1}{2}kA^2$

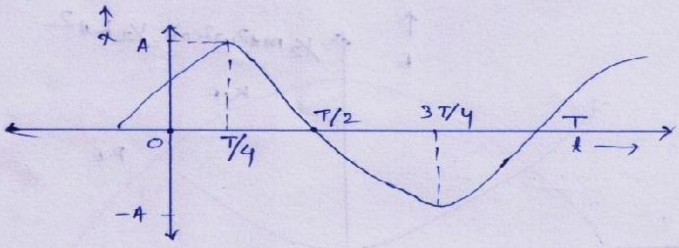
Total energy $=\frac{1}{2}MA^2\omega^2$

The total energy of an oscillatory system is constant.

Graphical relation between different characteristics in SHM.

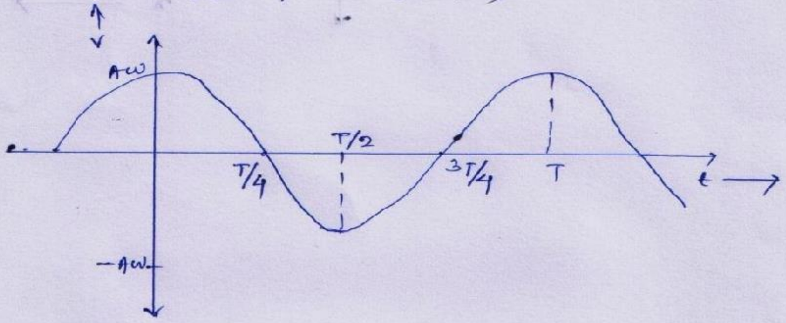
(a) Displacement ~ time

$x = A \sin(\omega t + \theta)$



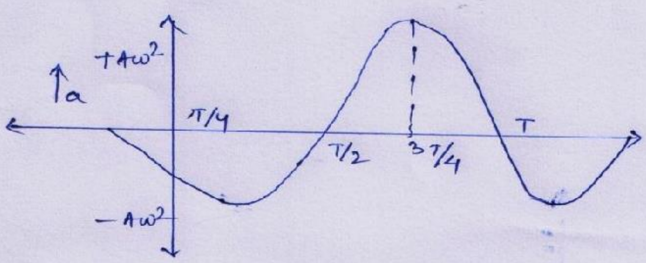
(b) Velocity ~ time

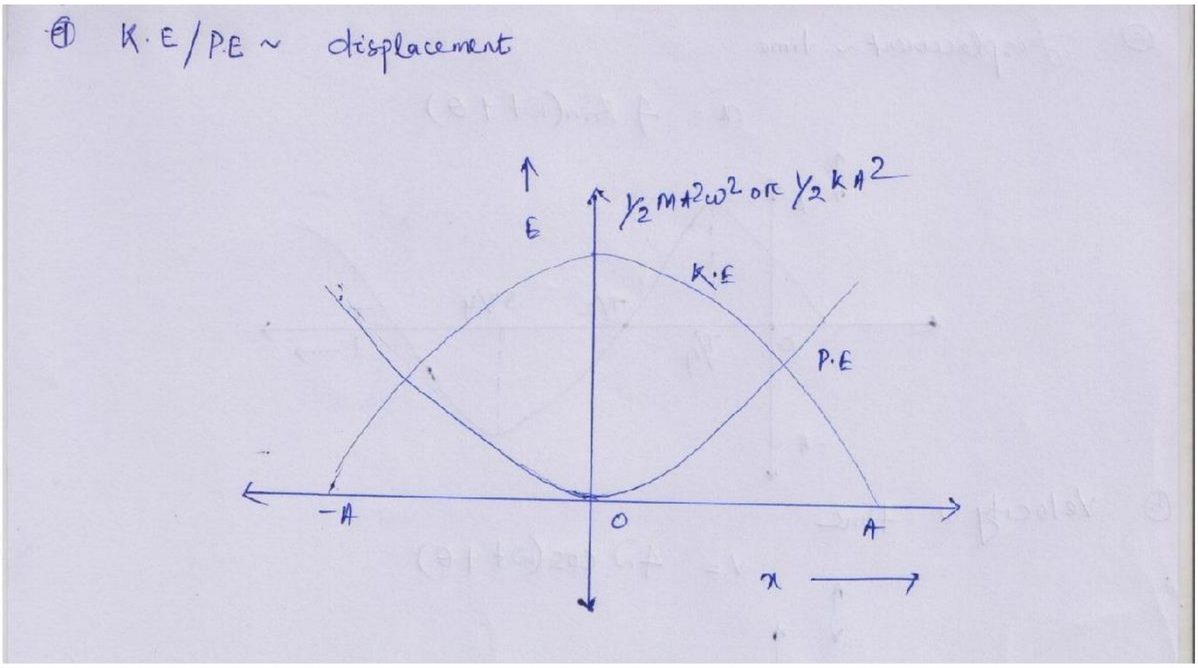
$v = A\omega \cos(\omega t + \theta)$



(c) Acceleration ~ time

$a = A\omega^2 \sin(\omega t + \theta)$



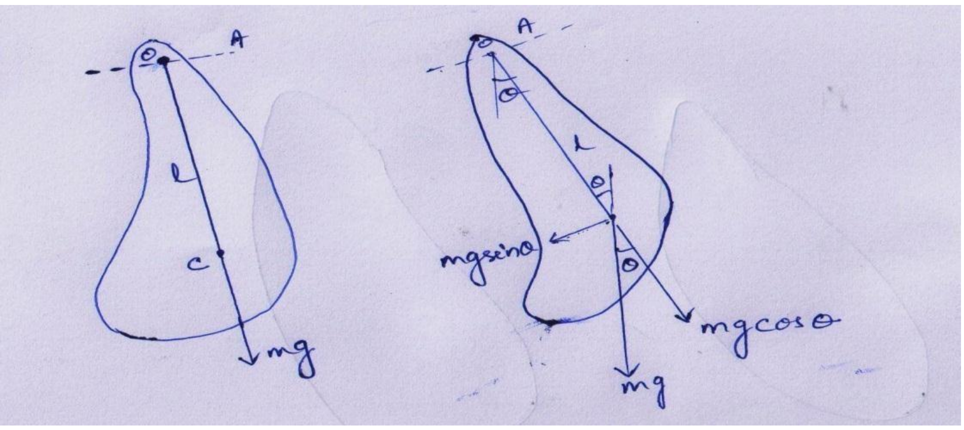


COMPOUND PENDULUM (Physical pendulum):-

Compound /physical pendulum is a rigid body of any arbitrary shape capable of rotating in a vertical plane about an axis passing through the pendulum but not through the pendulum but not through centre of gravity of pendulum.

The distance between the point of suspension the centre of gravity is called the length of length of the pendulum &denoted by

When the pendulum is displaced through a angle θ from the mean position,a restoring torque come to play which tries to bring the pendulum back to the mean position .But the oscillation continues due to the inertia of restoring force.



Here the restoring force is -mgsinθ. So the restoring torque about the point of suspension "O" is

$\tau = -mgl\sin\theta$.

If the moment of inertia of the body about "OA" is "I", the angular acceleration becomes,

$\alpha = \tau / I$

$\alpha =$ ....(1)

For very small angular displace "θ ", we assume that

$\sin\theta \sim \theta$.

So, $\alpha = -mgl\theta / I$.

$\alpha = -(mgl/I)\,\theta$ ....(2)

Also $\alpha = d^2\theta/dt^2$

Now we can write

$d^2\theta/dt^2 + (mgl/I)\,\theta = 0$.....................(3)

$d^2\theta/dt^2 + \omega^2\theta = 0$ .... (4)

Where, $\omega^2 = mgl/I$. And eq^n(4) is the general equation of simple harmonic.

$T = 2\pi(I/mgl)^{1/2}$

$T = 2\pi(M(k^2 + L^2)/Mgl)^{1/2}$.

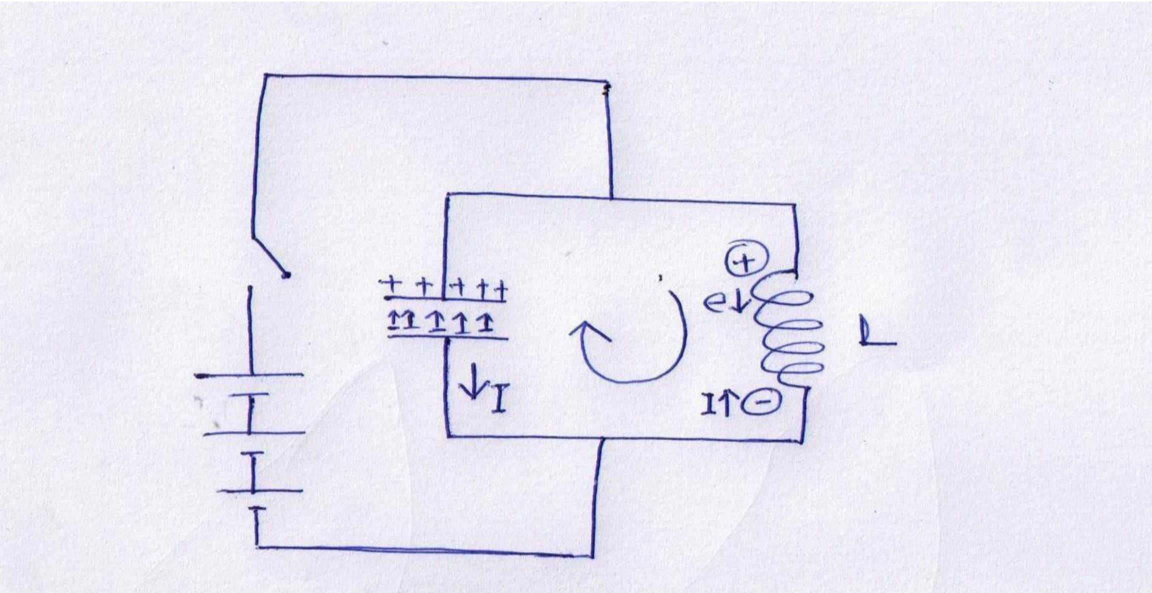
$T = 2\pi((K^2/l + l)/g)^{1/2}$ .... (5).

Here $\frac{k^2}{l} + l = L$, Called as equivalent length of pendulum..

If a line which is drawn along the line joining the point of suspension & Centre of gravity by the distance " k^2/l".we have another Point on the line called centre of Oscillation is equivalent Length of pendulum .

So,the distance between centre of suspension & centre of Oscillation is equivalent length of pendulum .If these two points are interchanged then “time period” will be constant.

L.C CIRCUIT(NON MECHANICAL OSCILLATION):-



In this region,it is combination “L” &”C” with the DC source through the key.If we Press the Key for a while then capacitor get charged & restores the charge as “+Q” and”-Q” with the potential “v=q/c” between the plates .When the switch is off the capacitor gets discharged.

As capacitor gets discharged, q also decreases. So, current at that situation is given by

I=dq/dt.

As q decreases, electric field energy (Energy stored in electric field) gradually decreases .This energy is transferred to magnetic field that appears around the inductor. At a time,all the charge on the capacitor becomes zero,the energy of capacitor is also Zero. Even though q equals to zero,the current is zero at this time.

Mathematically, Let the potential difference across the two plates of capacitor at any instance" V" is given by

$V=q/c$… .... (1)

In the inductor due to increases in the value of flow of current, the strength of magnetic field ultimately the magnetic lines of force cut/ link with inductor changes. So a back emfdevelops which is given by

$\varepsilon \quad =-L\frac{di}{dt}$..................... (2)

Now applying KVL to this LC circuit,

$+v-\varepsilon=0$

$\frac{q}{c}+L\frac{di}{dt}=0$

$\frac{q}{Lc}+\frac{d2q}{d2t}=0$

$\frac{d2q}{d2t}+\frac{q}{Lc}=0$.....................................(3).

This represents the general equation of SHM,

Here there is periodic execution of energy between electric field of capacitor & magnetic field of inductor.

Here this LC oscillation act as an source of electromagnetic wave.

Here, $\omega^2={}^{1}/_{LC}$

$\omega = 1/\sqrt{LC}$

$T = 2\pi\sqrt{LC}$

<u>Damped oscillation</u>:-

For a free oscillation the energy remains constant. Hence oscillation continues indefinitely. However in real fact, the amplitude of the oscillatory system gradually decreases due to experiences of damping force like friction and resistance of the media.

The oscillators whose amplitude, in successive oscillations goes on decreasing due to the presence of resistive forces are called damped oscillators, and oscillation called damping oscillation.

The damping force always acts in a opposite directions to that of motion of oscillatory body and velocity dependent.

$F_{dam} \propto -v$

$F_{dam} = -bv$

b= damping constant which is a positive quantity defined as damping force/velocity,

$F_{net} = F_{res} + F_{dam}$

$F_{net} = -kx - bv$

$F_{net} = -kx - b\frac{dx}{dt}$

$M\frac{d2x}{dt2} + kx + b\frac{dx}{dt} = 0$

$\frac{d2x}{dt2} + \frac{b}{M}\frac{dx}{dt} + \frac{k}{M}x = 0$

$$\frac{d2x}{dt2}+2\beta\frac{dx}{dt}+\omega_0{}^2x = 0 \;.................(2)$$

Where $\beta = \frac{b}{2M}$ is the damping co-efficient & $\omega_0 = \sqrt{\frac{k}{M}}$ is called the natural frequency of oscillating body.

The above equation is second degree linear homogeneous equation.

The general solution of above equation is found out by assuming x(t), a function which is given by

$$x(t) = Ae^{\alpha t}$$

$$\frac{dx}{dt} = A\alpha e^{\alpha t} = \alpha x$$

$$\frac{d2x}{dt2} = A\alpha^2 e^{\alpha t} = \alpha^2 x$$

Putting these values in equation

$$\alpha^2 x + 2\alpha^2\beta x + \omega_0{}^2 x = 0$$

$$\alpha^2 + 2\alpha^2\beta + \omega_0{}^2 = 0 \;................(3)$$

$\alpha = -\beta \pm \sqrt{\beta 2 -}\omega_0{}^2$, is the general solution of above quadratic equation.

As we know,

$$x(t) = A_1 e^{at} + A_2 e^{at}$$

$$x(t) = A_1 e^{(-\beta+\sqrt{\beta 2 -}\ \omega 02)t} + A_2 e^{(-\beta-\sqrt{\beta 2 -}\ \omega 02)t}$$

$$x(t) = e^{-\beta t}\,(A_1 e^{\sqrt{\beta 2 -}\ \omega 02\ t} + A_2 e^{-\sqrt{\beta 2 -}\ \omega 02\ t}) \ldots (4)$$

Depending upon the strength of damping force the quantity ($\beta^2 - \omega_0{}^2$) can be positive /negative /zero giving rise to three different cases.

Case-1:- if $\beta < \omega_0{}^2 \Rightarrow$ underdamping (oscillatory)

Case-2:- if $\beta > \omega_0{}^2 \Rightarrow$ overdamping (non-oscillatory)

Case-3:- if $\beta=\omega_0^2$ => critical damping (non-oscillatory)

Case-1: [Under damping $\omega_0^2>\beta^2$]

If $\beta^2<\omega_0^2$, then $\beta^2- \omega_0^2$= -ve

let $\beta^2-\omega_0^2=-\omega^2_1 \Rightarrow \sqrt{\beta^2-\omega_0^2} = i\,\omega_1$

where $\omega_1= \sqrt{\omega_0^2- \beta^2}$ = Real quantity

So the general equation of damped oscillation/equation (IV) becomes

$X(t) = e^{-\beta t} (A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t})$

By setting

$A_1=r/_2 e^{i\theta}$ and $A_2= r/_2 e^{-i\theta}$,

$X(t)= e^{-\beta t}[r/_2 e^{i(\theta+\omega_1 t)}+ r/_2 e^{-i(\theta+\omega_1 t)}]$

$=re^{-\beta t}[e^{i(\theta+\omega_1 t)}+ e^{-i(\theta+\omega_1 t)}]/2$

$X(t)= re^{-\beta t} \cos(\theta+ \omega_1 t)$………(v)

Here $\cos(\theta+ \omega_1 t)$ represents the motion is oscillatory having angular frequency "ω_1" .The constant "r" and " θ" are determined from initial potion & velocity of oscillatior

$T_1=2\pi/ \omega_1$

$T_1=2\pi/\sqrt{\omega_0^2- \beta^2}$……(vi) (time period of damped oscillator)

$T_1>T$ (where T= time period of undamped oscillator

Implies $f_1< f$

Frequency of damped oscillator is less than that of the undamped oscillator.

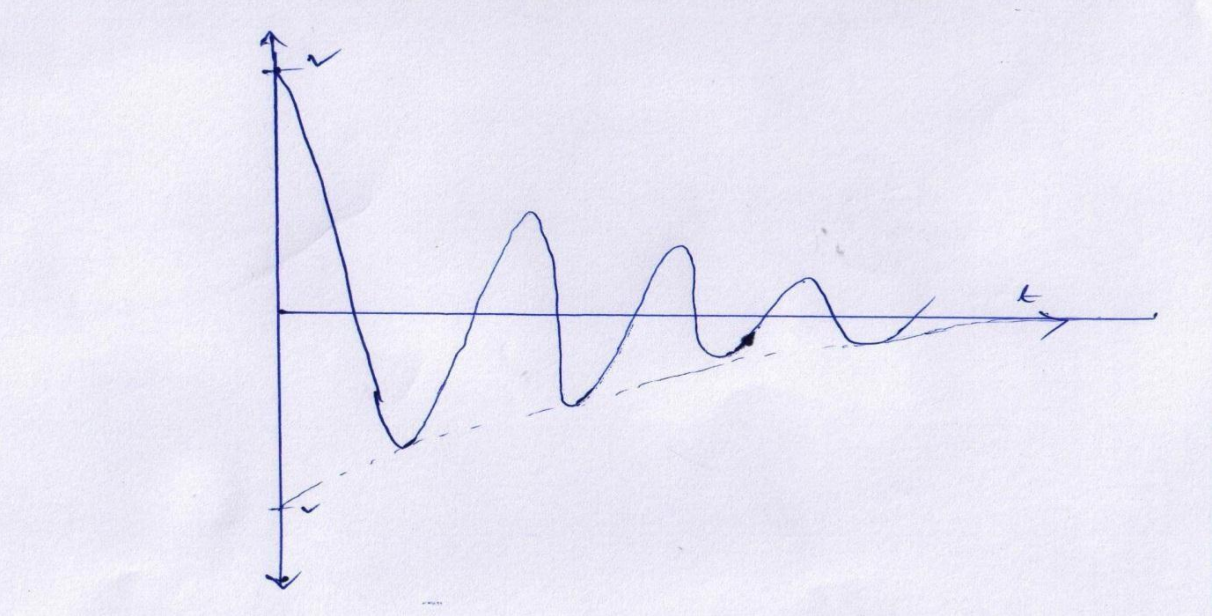
In under damped condition amplitude is no more constant and decreases exponentially with time, till the oscillation dies out.

Mean life time: The time interval in which the oscillation falls to 1/e of its initial value is called mean life time of the oscillator. (τ)

$1/_e\, a = a\, e^{-\beta\tau}m \Rightarrow e^{-\beta cm} = \frac{1}{e}$,

$\Rightarrow -\beta\, r_m = \log_e 1/e$

$\Rightarrow r_m = \frac{1}{\beta}$



Velocity of underdamped oscillation:

$X(t) = re^{-\beta t}\cos(\omega_1 t + \theta)$

$\Rightarrow \frac{dx}{dt} = r[-\beta e^{-\beta t}\cos(\omega_1 t + \theta) - e^{-\beta t}\omega_1 \sin(\omega_1 t + \theta)$

$\Rightarrow \frac{dx}{dt} = v = -re^{-\beta t}[\beta\cos(\omega_1 t + \theta) + \omega_1 \sin(\omega_1 t + \theta)$…(vi)

Now , x=0& t=0,

$X(t) = re^{-\beta t}\cos(\omega_1 t + \theta)$

$\Rightarrow 0 = re^{0}\cos(0 + \theta)$

$\Rightarrow 0 = \cos\theta$

$$\Rightarrow \theta = \frac{\pi}{2}$$

Using the value of θ & t=0 in the equation (vii) we have

$_0$= -r ω_1

Where value of V_0 in ....

Calculation of Energy(instantaneous):

$K.E = \frac{1}{2}mv^2$

$K.E = \frac{1}{2}mv^2 e^{-2\beta}[\beta^2\cos^2(\omega_1 t+\theta)+\omega_1^2\sin^2(\omega_1 t+\theta)+\beta\omega_1\sin^2(\omega_1 t+\theta)]$

Potential Enegy:

$P.E=\frac{1}{2}kx^2$

$=\frac{1}{2}kr^2e^{-2\beta t}\cos^2(\omega_1 t+\theta)$

Total Energy:

T.E=K.E+P.E

$=e^{-2\beta}[(\frac{1}{2}mv^2+\frac{1}{2}kr^2)\cos^2(\omega_1 t+\theta)+\frac{1}{2}mr^2\omega_1^2\sin^2(\omega_1 t+\theta)$

$+ \frac{1}{2}mv^2\beta\omega_1\sin 2(\omega_1 t+\theta)]$

Total average energy:

$<E>=\frac{1}{2}mr^2\omega_0^2e^{-2\beta t}$

$=E_0e^{-2\beta t}$

Where, E_0 =Total energy of free oscillation

The average energy decipated during one cycle

$<(t)>$ =Rate of energy

$$=\frac{d}{dt} < E >$$

$$=2\beta E$$

Decrement

The decrement measures the rate at which amplitude dies away.

The ratio between amplitude of two successive maxima, is the decrement of the oscillator.

$re^{-\beta t}/ re^{-\beta(t+T)} = re^{+\beta t}$

The logarithmic decrement of oscillator is „λ"

$\lambda = \log_a e^{\beta t}$

$\Rightarrow \beta T = 2\pi\beta/\sqrt{\omega_0^2 - \beta^2}$

$\Rightarrow \lambda = \log_a a_0/a_1 = \log a_1/a_2 = \ldots\ldots\ldots = e^{\beta t} = e^{\,0}$

Rate of two amplitudes of oscillation whichare separated by one period

Relaxation time(r):

It is the time taken by damped oscillation by decaying of its energy 1/e of its initial energy.

$\Rightarrow \frac{1}{e}\varepsilon_0 = \varepsilon_0 e^{-2\beta r}$

$\Rightarrow \frac{1}{e} = e^{-2\beta r}$

$\Rightarrow \text{Log}\, e^{-1} = \log e^{-2\beta r}$

$\Rightarrow -1 = -2\beta r$

$\Rightarrow r = 1/2\beta = m/b$

Case-II:(over damping oscillation)

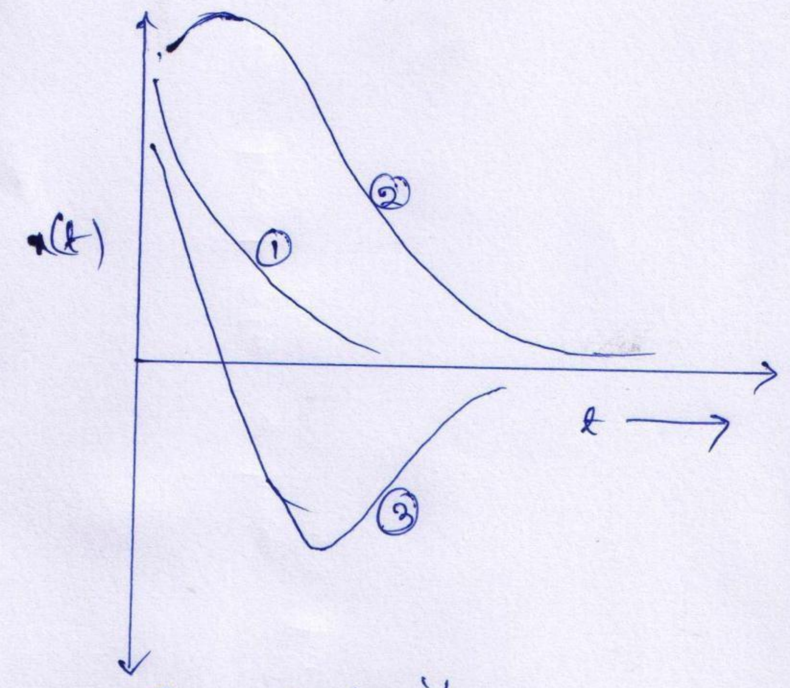
Here $\beta^2 > \omega_0^2$

$$\sqrt{\beta^2-\omega_0^2} = +ve \text{ quantity}$$

$= \alpha$ (say)

$$X(t) = e^{-\beta t}\left(A_1 e^{\alpha t} + A_2 e^{-\alpha t}\right)\ldots\ldots\ldots\ldots(viii)$$

Depending upon the relative values of α, β ,A_1 , A_2& initial position and velocity the oscillator comes back to equilibrium position.



① Displacement decreases to '0'

② Displacement first increases, attains max. & then decreases to zero.

③ Displacement changes sign i.e the oscillator over that mean position, attain max. & comes back mean position.

The motion of simple pendulum in a highly viscous medium is an example of over damped oscillation.

Quality factor:

$$Q=2\pi.\frac{\text{Energy stored in system}}{\text{Energy loss per period}}= 2\pi.\frac{<E>}{<P>T}=\frac{2}{T}.r$$

$$\Rightarrow Q = mr$$

Critical damping:

$\beta^2 = \omega_0^2$

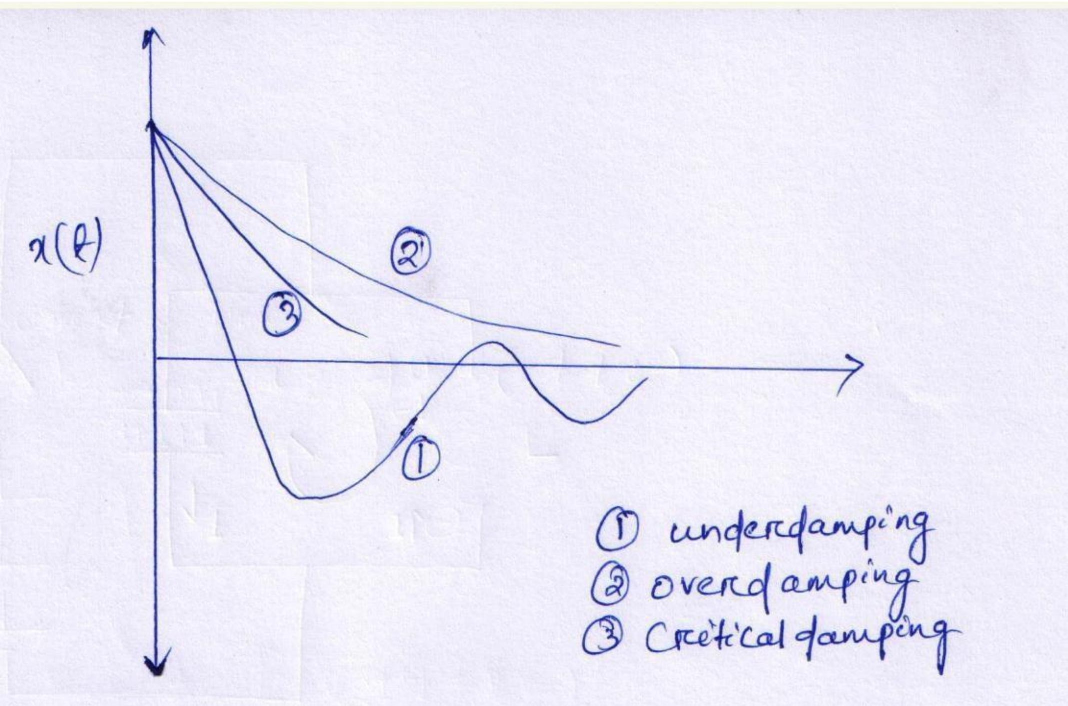
The general solution of equation (ii) in this case,

$X(t) = (Ct+D)\, e^{-\beta t}$ ……………………………………………(ix)

Here the displacement approaches to zero asymptotically for given value of initial position and velocity a critically damped oscillator approaches equilibrium position faster than other two cases.

Example: The springs of automobiles or the springs of dead beat galvanometer.

Curves of three Cases:



Forced Oscillation

The oscillation of a oscillator is said to be forced oscillator or driven oscillation if the oscillator is subjected to external periodic force.

If an external periodic sinusoidal force „Fcosωt" acts on a damped oscillator, its equation of motion is written as,

$$F_{net} = -kx - b\frac{dx}{dt} + F\cos\omega t$$

$$m\frac{d^2x}{dt^2} + = -kx - b\frac{dx}{dt} + F\cos\omega t$$

$$\frac{d^2x}{dt^2} + + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}\cos\omega t$$

$$\boxed{\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = f_0\cos\omega t}$$

-- (i)

Where $\beta=\frac{b}{2m}$, $\omega_0^2=\frac{k}{m}$ and $f_0=\frac{F}{m}$, and β and ω_0^2 respectively called as damping coefficient, natural frequency.

Equation (i) is also represented as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos\omega t$$

Equation (i) represents the general equation of forced oscillation.

Equation (i) is a non-homogenous differential equation with constant co-efficient. For weak damping ($\omega_0^2 > \beta^2$) , the general equation contains,

$x(t) = x_c(t) + x_p(t)$

Where $x_c(t)$ is called complementary solution and its value is

$x_c(t)=e^{-\beta t}\left(A_1 e^{\sqrt{\beta 2-\omega 0^2}}+A_2 e^{-\sqrt{\beta 2-\omega 0^2}}\right)$ .... (ii)

Now $x_p(t)$ is called the particular integral part.

Let us choose

$x_p(t) = P\cos(\omega t-\delta)$

$\dot{x}(t) = -P\omega\sin(\omega t-\delta)$

$\ddot{x}(t) = -P\omega^2\cos(\omega t-\delta)$..................................(iii)

Putting $x_p(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ in eq[n] (i) we get

$-P\omega^2\cos(\omega t-\delta)-2\beta P\omega\sin(\omega t-\delta)+\omega_0^2 P\cos(\omega t-\delta) = f_0\cos\omega t$

$-P\omega^2\cos(\omega t-\delta)-2\beta P\omega\sin(\omega t-\delta)+\omega_0^2 P\cos(\omega t-\delta) = f_0\cos(\omega t-\delta+\delta)$

$- P\omega^2\cos(\omega t-\delta) - 2\beta P\omega \sin(\omega t-\delta) + \omega_0^2 P\cos(\omega t-\delta) = f_0 [\cos(\omega t-\delta).\cos\delta - \sin(\omega t-\delta).\sin\delta]$

Now, compairing the coefficient of $\cos(\omega t-\delta)$ and $\sin(\omega t-\delta)$ on both sides,

$(\omega_0^2-\omega^2)P = f_0\cos\delta$ .... (iv)

$2\beta P\omega = f_0\sin\delta$ .... (v)

Squaring and adding eq^n (iv) & (v)

$\{(\omega_0^2-\omega^2)P\}^2 + 4\beta^2 P^2\omega^2 = f_0^2$

$$P = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}} \quad \text{....................(vi)}$$

Now dividing eq^n (v) by (iv)

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right) \text{ ....(vii)}$$

$$x_p = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}\cos(\omega t-\delta) \quad \text{(steady state solution)}$$

Now, $x(t) = x_c(t) + x_p(t)$

$$x(t) = e^{-\beta t}\left(A_1 e^{\sqrt{\beta^2-\omega_0^2}} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}}\right) + \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}\cos(\omega t-\delta)$$

<u>Steady state behavior:</u>

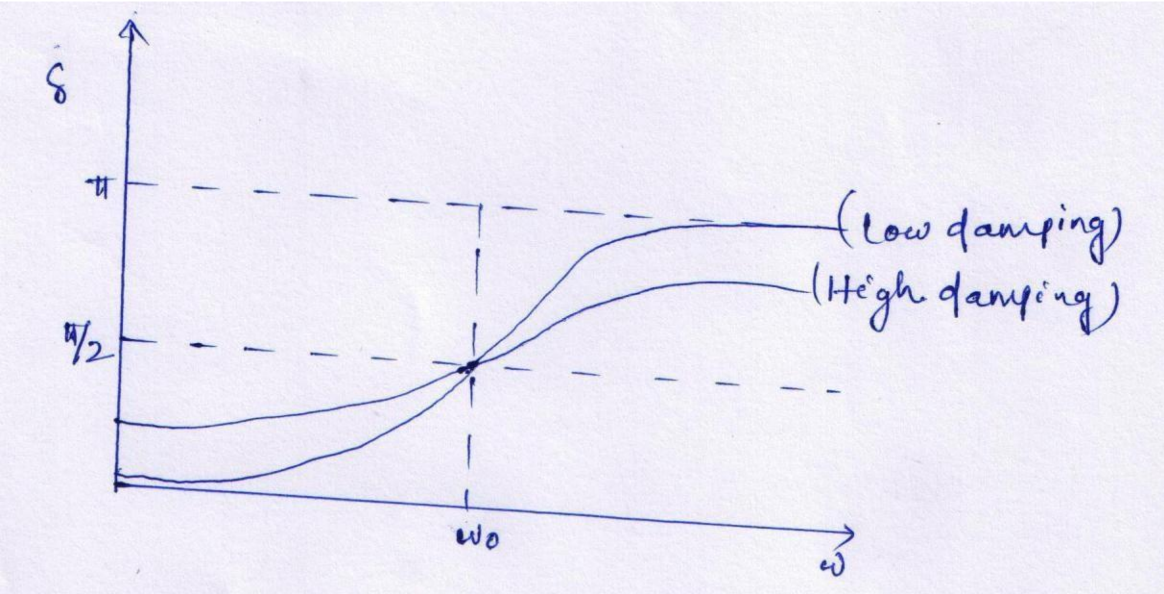
Frequency:-The Oscillator oscillates with the same frequency as that of the periodic force.

$\omega_0$and ω are very close to each other then beats will be produced and these beats are transient as it lasts as long as the steady state lasts. The duration between transient beats is determined by the damping coefficient „β‟.

Phase: The phase difference „δ" between the oscillator and the driving force or between the displacement and driving is

$$\delta=\tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$$

This shows that there is a delay between the action of the driving force and response of the oscillator.



(In the above figure $f_Q=\omega_0$ and $f_p=\omega$)

At $\omega=\omega_0$, $\varphi=\frac{\pi}{2}$, the displacement of the oscillator lags behind the driving force by $\frac{\pi}{2}$.

At $\omega<<\omega_0$ then $\delta=0\rightarrow\delta=0$

For $\omega>>\omega_0$ then $\delta=-\frac{2p}{\omega}\rightarrow -0=\pi$

Amplitude: The amplitude of driven oscillator , in the steady state , is given by

$$A=\frac{F/m}{\sqrt{(\omega_0^2-\ ^2)^2+4\ ^2\ ^2}}=\frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

It depends upon $(\omega_0^2-\omega^2)$. If it is very small, then the amplitude of forced oscillation increases.

Case-1: At very high driving force i.e $\omega >> \omega_0$ and damping is small (β is small) or ($\beta \rightarrow 0$)

$$A = \frac{f_0}{\sqrt{\omega^4+0}}$$

$$A = \frac{f_0}{\omega^2}$$

$$\boxed{A = \frac{F}{m\omega^2}}$$

Amplitude is inversely proportional to the mass of the oscillator & hence the motion is mass controlled motion.

Case-2: At very low driving force ($\omega << \omega_0$) and damping is small

($\beta \rightarrow 0$),

i.e. $\omega_0^2 - \omega^2 \cong \omega_0^2$

$$A = \frac{f_0}{\sqrt{\omega_0^4}}$$

$$A = \frac{f_0}{\omega_0^2}$$

$$\boxed{A = \frac{F}{m\omega_0^2}}$$

So, when the low driving force is applied to oscillator, the amplitude remains almost constant for low damping. The amplitude of the forced oscillator in the region $\omega << \omega_0$ and $\beta < \omega_0$ is inversely proportional to the stiffness constant (k) and hence motion is called the stiffness controlled motion.

Case:-iii (Resistance controlled motion)

When angular frequency of driving force=natural frequency of oscillator i.e.($\omega=\omega_0$)

$A=f_0/\sqrt{4\beta^2\omega^2}=f_0/2\beta\omega$

$A=f/b\omega=f/b\omega_0$

RESONANCE:-

The amplitude of vibration becomes large for small damping(β is less) and the maximum amplitude is inversely proportional to resistive term (b) hence called as resonance. It is the phenomenon of a body setting a body into vibrations with its natural frequency by the application of a periodic force of same frequency.

If the amplitude of oscillation is maximum when the driving frequency is same as natural frequency of oscillator. (I.e. $\omega=\omega_0$).

„A" will be the max. Only the denominator of the expression

$\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$ is minimum i.e.

$\frac{d}{d\omega}[\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}=0$

$=>-4\omega\omega_0^2+4\omega^3+8\beta^2\omega=0$

$=>-\omega_0^2+\omega^2+2\beta^2=0$

$=>\omega=\sqrt{(\omega_0^2-2\beta^2)}=\omega_0\sqrt{(1-2\beta^2/\omega_0^2)}$

It is the value of angular frequency, where „A" will be maximum in presence of damping force

But when damping is very small,

$\omega=\omega_0\ (\beta\rightarrow 0)$

The max value of „A „when damping is present

$A = f_0/\sqrt{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]}$

$= f_0/\sqrt{[\omega_0^2 - (\omega_0^2 - 2\beta^2)]2 + 4\beta(\omega_0^2 - 2\beta^2)}$

$= f_0/\sqrt{4\beta^4 + 4\beta^2\omega_0^2 - 8\beta^4}$

$= f_0/\sqrt{[4\beta^2\omega_0^2 - 4\beta^4]}$

$A_{max} = f_0/2\beta\sqrt{(\omega_0^2 - \beta^2)} = f/2m\beta\sqrt{(\omega_0^2 - \beta^2)}$

This is called amplitude Resonance.

Value of the frequency at which amplitude resonance occurs i.e. amplitude becomes maximum.

$B_1 < \beta_2 \quad f_r = \omega/2\pi$

$= \sqrt{(\omega_0^2 - 2\beta^2)}/2\pi$

Damping is small,

$f_r = \omega_0/2\pi$

Here, fr" is called resonant frequency.

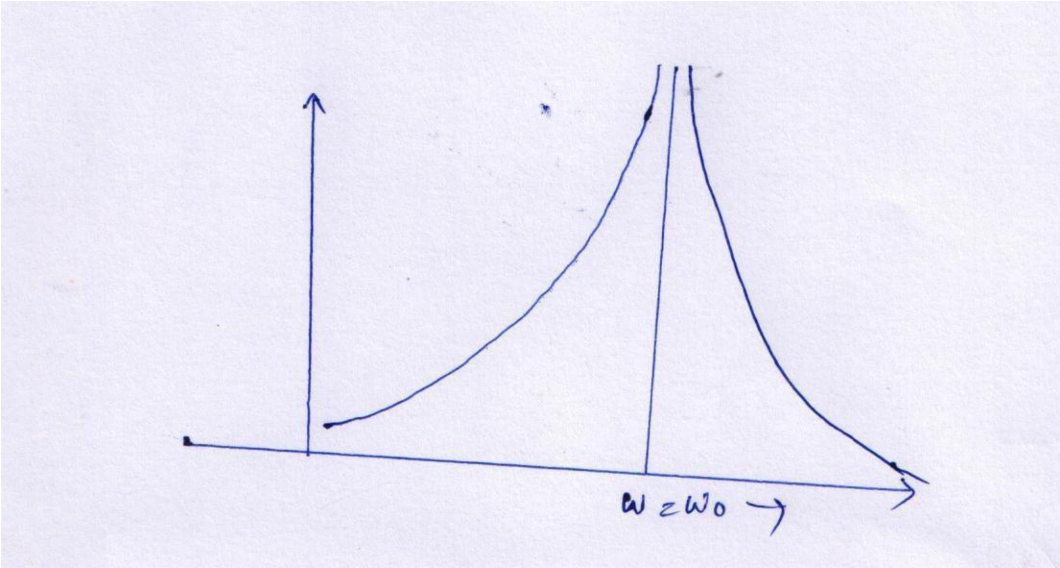
Phase at resonance:-

$\Phi = \pi/2$

Velocity of oscillator is in same phase with the driving force .Therefore, the driving force always acts in the direction of motion of oscillator. So energy transfers from driving force to oscillation are max^im.

Sharpness of resonance:-

The amplitude is maximum at resonance frequency which decreases rapidly as the frequency increases or decreases from the resonant frequency.



The rate at which the amplitude decreases with the driving frequency on either side of resonant frequency is termed as „‟sharpness of resonance‟‟‟‟.

Different condition:-

(i) For $\omega=\omega_0 \pm \beta$, the amp. Becomes $A=A_{max}/\sqrt{2}$.The width of resonance curve i.e. the range of frequency over which the amplitude remains more than $A_{max}/\sqrt{2}$.

$\Delta\omega= (\omega_0+\beta)-(\omega_0-\beta)=2\beta$

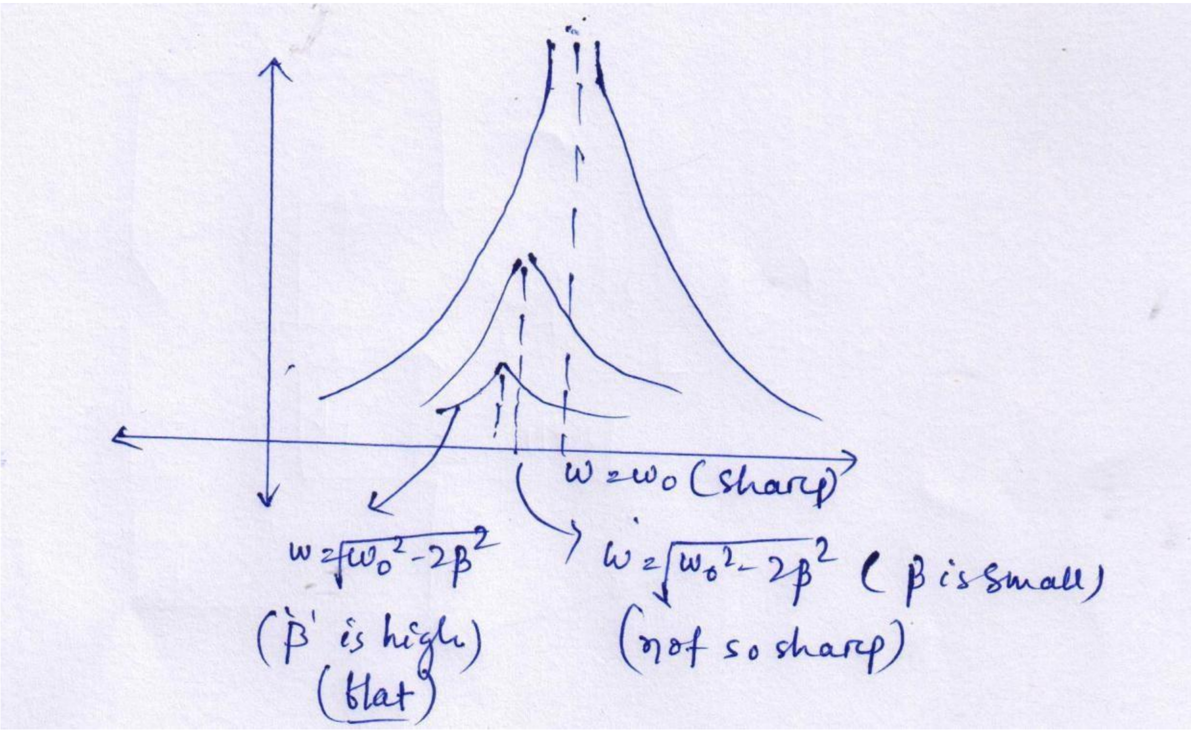
Thus if β‟ is small,$\Delta\omega$ is small.

(ii) For $\beta=0$, $A\rightarrow\infty$ at $\omega=\omega_0$.

(iii)If there is small „β‟, amp. Resonance occurs lesser value amp is max at $\omega=\omega_r$.

(iv)If β‟‟‟is high, A‟ is max but the peak moves towards left &max. amp decreases.

(v)So resonance is sharp for low „β‟ & flat for high „β‟.



Velocity:-

$X = x_p = f_0/\sqrt{[[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]}\cos(mt - \delta)$

$V = -\omega f_0/\sqrt{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]}\sin(mt - \delta)$

$$V = \omega f_0/\sqrt{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]}\sin(mt - \delta + \frac{\pi}{2})$$

$V_{max} = \omega f_0/\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$

*here also „v‟is max. When $\omega_0 = \omega$.

$(V_{max.amp}) = f/b$

Calculation of energy:-

$x = \frac{f0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}\cos(\omega t - \delta)$

Where $\frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}=A$

$V=\frac{f_0\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}\cos(pt-\delta+\pi/2)$

Average potential energy:-

$P.E=\frac{1}{2}kx^2$

$=\frac{1}{2}k\frac{f_0^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\cos^2(\omega t-\delta)$

$=\frac{1}{2}kA^2\cos^2(\omega t-\delta)$

$<P.E>=\frac{1}{4}kA^2$

$=\frac{1}{4}m\omega_0^2A^2$

(average of $\cos^2\theta=1/2$)

Average kinetic energy:-

$K.E=\frac{1}{2}mv^2$

$=\frac{1}{2}m\frac{f_0^2\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\cos^2(\omega t-\delta+\pi/2)$

$=\frac{1}{2}m\omega^2A^2\cos^2(pt-\delta+\pi/2)$

$<K.E>=\frac{1}{4}m\omega^2A^2$

Total average energy:-

$<\epsilon> = <K.E> + <P.E>$

$=\frac{1}{4}mm^2A^2+\frac{1}{4}mm_0^2A^2$

$$<\epsilon> = \frac{1}{4} mA^2 (m_0^2 + m^2)$$

POWER:-

i). Power absorption:

$$P_{ab} = F_{P_e}.v$$

$$= F \cos (mt - \delta). \frac{f0 \ \omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \cos (mt - \delta + \pi/2)$$

$$= A F \cos (mt - \delta) \sin (mt - \delta)$$

$$= \frac{1}{2} AF \sin 2 (mt - \delta)$$

$$< P_{abs} > = \frac{f_0^2 \ \omega^2}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]}$$

$$= m A^2 \beta m^2$$

$*$ P_{max}absorbed when $m_o = m$

$$\Rightarrow P_{max} = \frac{f_0^2 \beta \omega_0^2}{4\beta^2 \omega_0^2} = \frac{f_0^2}{4\beta}$$

ii). Power dissipation:

$$P_{dis} = F_{damp}.v \quad \text{Or} \quad F_{resistive} \times \text{Inst. velocity}$$

$$= + b.\frac{dx}{dt}.v$$

$$= + b v^2$$

$$P_{dis} = 2m \beta v^2 \qquad (\beta = b/_{2m})$$

$$\Rightarrow P_{dis} = 2m\beta.\frac{f0^2 \omega^2}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]}. \cos^2 (pt - \delta + \frac{}{2})$$

$\Rightarrow < P_{dis} > = 2m\beta A^2 m^2 \times {}^1/_2$

$\Rightarrow < P_{dis} > = m\beta A^2 m^2$

Thus in the steady state of forced vibration, the average rate of power supplied by the forcing system is equal to the average of work done by the forced system against the damping force.

QUALITY FACTOR:-

Quality factor is a measure of sharpness of resonance.

Q- Factor is defined as,

$$Q = 2\pi \times \frac{\text{average energy stored per cycle}}{\text{average energy dissipated per cycle}}$$

$$= 2\pi \times \frac{E_{av}}{T.P_{av}}$$

$$= 2\times \frac{\frac{1}{4} m A^2(\omega_0^2 + \omega^2)}{T \times m\,\beta\omega^2 A^2}$$

$$= \frac{(\omega_0^2 + \omega^2)}{4\beta\omega^2} = \frac{(\omega_0^2 + \omega^2)}{4(\beta\omega)}$$

At $m = m_0$, for weak damping

$$Q = \frac{2\omega_0^2}{4(\beta\omega_0)}$$

=>

$$\boxed{Q = \frac{\omega 0}{2\beta}}$$

βSmall, Q → Large, sharpness of resonance is more.

Again,

$$\mathbf{Q} = \frac{\text{Resonant frequency}}{\text{width of resonance curve}} = \frac{\omega_0}{2\beta}$$

Larger value of Quality factor (less), sharper is the resonance.

System	Q value
Earthquake	250 – 1400
Violin string	10^3
Microwave resonator	10^5
Crystalosill	10^6
Excetetation	10^8

Amplitude Resonance	Velocity Resonance

1. In amp. Resonance, the amp. of oscillator is maximumfor a particular frequency of the applied force. 2. Amplitude resonance occurs at $m_r = (m_0 - 2\beta^2)^{1/2}$ 3. At applied frequency $m = 0$, the amp. of the freq. oscillator is F/k 4. The phase of the forced oscillator with respect to that of applied force is $^{\pi}/_{2}$	1. The velocity amplitude of the forced oscillator is the maximum at a particular frequency of applied force. 2. Velocity resonance occurs at $m = m$ 3. Applied frequency $m = 0$, the velocity amplitude is zero. 4. Phase of the forced oscillator with respect to that of applied force is ….

Mechanical Impedance

The force required to produce unit velocity is called the mechanical impedance of the oscillator.

$$z = {}^{F}/_{v}$$

$$x = A\, e^{i(\omega t - \varphi)}$$

$$v = \frac{dx}{dt} = miAe^{i(\omega t - \varphi)}$$

$$\Rightarrow z = \frac{F\, e^{i\omega t}}{i\omega A e^{i(\omega t - \varphi)}} = \frac{F}{i\omega A e^{-i\varphi}} = \frac{mf}{i\omega A e^{-i\varphi}}$$

$$= \frac{mf}{i\omega \dfrac{F}{((\omega_0{}^2 - \omega^2)^2 + 2\beta\omega i)}}$$

$$= 2\beta m - \frac{im(\omega_0^2 - \omega^2)}{\omega}$$

$$z^* = 2\beta m + \frac{im(\omega_0^2 - \omega^2)}{\omega}$$

$$|z| = z^* z = m \left[4\beta^2 + \frac{1}{m^2}(m_0^2 - m^2)^2\right]^{1/2}$$

$$A = \frac{F}{m|z|}$$

For a particular m, $\boxed{A \propto \frac{1}{|z|}}$

Module-II

INTERFERENCE

Coherent Superposition:

The superposition is said to be coherent if two waves having constant phase or zero phase difference.

In this case, the resultant intensity differs from the sum of intensities of individual waves due to interfering factor.

i.e. $I \neq I_1 + I_2$

Incoherent Superposition:

The superposition is said to be incoherent if phase changes frequently or randomly.

In this case, the resultant intensity is equal to the sum of the intensities of the individual waves.

i.e. $I = I_1 + I_2$

Two Beam Superposition:

When two beam having same frequency, wavelength and different in amplitude and phase propagates in a medium, they undergo principle of superposition which is known as two beam superposition.

Let us consider two waves having different amplitude and phase are propagated in a medium is given as

$f_1 = a_1 \sin(kx - \omega t + \varphi_1)$ (1)

$f_2 = a_2 \sin(kx - \omega t + \varphi_2)$ (2)

Applying the principle of superposition

$$f = f_1 + f_2$$

$$f = a_1 \sin(kx - \omega t + \varphi_1) + a_2 \sin(kx - \omega t + \varphi_2)$$

$$= a_1 \sin(kx - \omega t) \cos \varphi_1 + a_2 \cos(kx - \omega t) \sin \varphi_1 + a_2 \sin(kx - \omega t) \cos \varphi_2 + a_2 \cos(kx - \omega t) \sin \varphi_2$$

$$= (a_1 \cos \varphi_1 + a_2 \cos \varphi_2) \sin(kx - \omega t) + (a_1 \sin \varphi_1 + a_2 \sin \varphi_2) \cos(kx - \omega t) \quad (3)$$

Let

$a_1 \cos \varphi_1 + a_2 \cos \varphi_2 = A \cos \theta$ (4)

and $a_1 \sin \varphi_1 + a_2 \sin \varphi_2 = A \sin \theta$ (5)

$$f = A \cos \theta \sin(kx - \omega t) + A \sin \theta \cos(kx - \omega t)$$

$$= [\sin(kx - \omega t) \cos \theta + \cos(kx - \omega t) \sin \theta] f = A \sin(kx - \omega t + \theta) \quad (6)$$

Squaring and adding equation (4) and (5)

$$A^2 = (a_1 \cos \varphi_1 + a_2 \cos \varphi_2)^2 + (a_1 \sin \varphi_1 + a_2 \sin \varphi_2)^2$$

$$= a_1^2 cos^2\varphi_1 + a_2^2 cos^2\varphi_2 + 2a_1a_2 \cos \varphi_1 \cos \varphi_2 + a_1^2 sin^2\varphi_1 + a_2^2 sin^2\varphi_2 + 2a_1a_2 \sin \varphi_1 \sin \varphi_2$$

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2[\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2]$$

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\varphi_2 - \varphi_1)$$

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos(\varphi_2 - \varphi_1)} \quad (7)$$

We know, $I \propto A^2$

$\Rightarrow \quad = KA^2$

$= (a_1^2 + a_2^2 + 2a_1a_2 \cos(\varphi_2 - \varphi_1)$

$\Rightarrow \quad = Ka^2_1 + Ka^2_2 + K2a_1a_2 \cos(\varphi_2 - \varphi_1)$

$\Rightarrow \quad = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2} \cos(\varphi_2 - \varphi_1)$

(8)

Dividing equation (5) by (4), we get,

$$\tan\theta = \frac{a_1 \sin\varphi_1 + a_2 \sin\varphi_2}{a_1 \cos\varphi_1 + a_2 \cos\varphi_2}$$

Coherent Superposition:

In coherent superposition, the phase difference remains constant between two beams.

$\text{i.e.} \cos(\varphi_2 - \varphi_1) = 1 or - 1$

$If\ \cos(\varphi_2 - \varphi_1) = 1$

Now equation (7) and (8) becomes,

$A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2}$

$\Rightarrow \quad = (a_1 + a_2)^2$

$\Rightarrow A_{max} = a_1 + a_2$ and $\quad I = I_1 + I_2 + 2\sqrt{I_1I_2}$

$\Rightarrow I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2$

The two beams having same amplitude,

$a_1 = a_2 = a$

$\Rightarrow \quad A_{max} = 2a$

$\Rightarrow I_{max} = 4I_0$

Again, if $\cos(\varphi_2 - \varphi_1) = -1$

$A_{min} = \sqrt{a_1^2 + a_2^2 - 2a_1a_2}$

$\Rightarrow A_{min} = a_1 - a_2$

$I = I_1 + I_2 - 2\sqrt{I_1I_2}$

$I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2$

For same amplitude,

$A_{min} = 0$

$\Rightarrow I_{min} = 0$

Incoherent Superposition:

In incoherent superposition the phase difference between the waves changes frequently or randomly, so the time average of the interfering term$(2\sqrt{I_1 I_2}\cos(\varphi_2 - \varphi_1))$ vanishes as the cos value varies from -1 to 1.

Here, $A = \sqrt{a_1^2 + a_2^2}$

$\Rightarrow \quad = I_1 + I_2$

Multiple beam superpositions:

When a number of beams having same frequency, wavelength and different amplitude and phase are undergoing the superposition, such superposition is known as multiple beam superpositions.

Let $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, \mathfrak{f}_4 \ldots\ldots\ldots\ldots \mathfrak{f}_n$ be the number of beams having same frequency, wavelength and different in amplitude and phase are propagating in a medium are given as,

$\mathfrak{f}_1 = A_1 \sin(kx - \omega t + \varphi_1)$

$\mathfrak{f}_2 = A_2 \sin(kx - \omega t + \varphi_2)$

:

:

:

$\mathfrak{f} = A_N \sin(kx - \omega t + \varphi_N)$

According to principle of superposition,

$\mathfrak{f} = \mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, \mathfrak{f}_4 \ldots\ldots\ldots\ldots \mathfrak{f}_N$

$$= \sum_{i=1}^{N} \mathfrak{f}_i$$

$$\Rightarrow \quad \mathfrak{f} = \sum_{i=1}^{N} A_i \sin(kx - \omega t + \varphi_i)$$

$$\Rightarrow \quad \mathfrak{f} = \sum_{i=1}^{N} A_i \sin(kx - \omega t + \varphi_i) \qquad (1)$$

where A_i = resultant amplitude of the i^th^ component.

φ_i =Phase of the i^th^ component.

$$A \sin \varphi = \sum_{i=1}^{N} A_i \sin \varphi_i \qquad (2)$$

$$A \cos \varphi = \sum_{i=1}^{N} A_i \cos \varphi_i \qquad (3)$$

Squaring and adding (2) and (3) we get,

$$A^2 = \sum_{i=1}^{N} A_i^2 + 2 \sum_{\substack{i=1 \\ i' j}}^{N} A_i A_j \cos(\varphi_j - \varphi_i)$$

The phase angle is given as,

$$\tan \varphi = \frac{\sum_{i=1}^{N} A_i \sin \varphi_i}{\sum_{i=1}^{N} A_i \cos \varphi_i}$$

Coherent Superposition:

In this case the phase difference between the waves remains constant i.e. $(\cos \varphi_j - \varphi_i) = +1$

$$\Rightarrow A^2 = \sum_{i=1}^{N} A_i^2 + 2 \sum_{\substack{i=1 \\ i' j}}^{N} A_i A_j$$

If all the beams having equal amplitudes.

i.e.$A_1 = A_2 = \cdots = A_N = A_1$

$$\Rightarrow A^2 = (NA_1)^2 = N^2 A_1^2$$

Now, $I = kA^2$

$$\Rightarrow \quad = kN^2 A^2{}_1$$

$$\Rightarrow I_{coherent} = N^2 I_1$$

Incoherent Superposition

In incoherent superposition, the phase difference between the beams changes frequently or randomly due to which the time average of factor $< \sum_{i=1}^{N} A_i A_j \cos(\varphi_j - \varphi_i) >$ vanishes as cos value varies from-1 to +1

$$\sum_{i=1}^{N} A_i A_j \cos(\varphi_j - \varphi_i) = 0$$

$$A^2 = N \sum_{i=1}^{N} A_i^2$$

Now , $I_{incoherent} = KA^2$

$$= KN \sum_{i=1}^{N} A_i^2$$

$$= KNA_1^2$$

$$\Rightarrow I_{incoherent} = NI_1 \qquad \Rightarrow N = \frac{I_{coherent}}{I_{icoherent}}$$

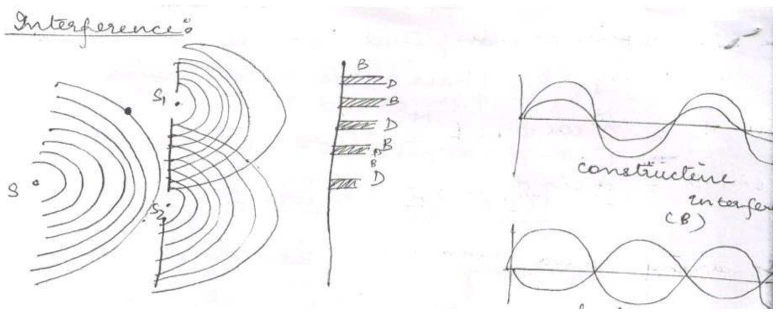
Interference:

The phenomenon of modification in distribution of energy due to superposition of two or more number of waves is known as interference.

To explain the interference, let us consider a monochromatic source of light having wavelength λ and emitting light in all possible directions.

According to Huygens‟s principle, as each point of a given wavefront will act as centre of disturbance they will emit secondary wave front on reaching slit S_1 and S_2.

As a result of which, the secondary wave front emitted from slit S1 and S2 undergo the Principle of superposition.



During the propagation, the crest or trough of one wave falls upon the crest and trough of other wave forming constructive interference, while the crest of one wave of trough of other wave producing destructive interference.

Thus, the interfering slit consisting of alternate dark and bright fringes, which explain the phenomenon of interference.

Mathematical treatment:

Let us consider two harmonic waves of same frequency and wavelength and different amplitude and phase are propagating in a medium given as

$$Y = y_1 + y_2$$

$$= \mathrm{a}\sin \omega t + b\sin(\omega t + \varphi)$$

$$= a\sin \omega t + b\sin \omega t\cos \varphi + b\cos \omega t\sin \varphi$$

$$= (a + b\cos \varphi)\sin \omega t + b\sin \varphi\cos \omega t$$

Let $a + b\cos \varphi = A\cos \theta$

$b\sin \varphi = A\sin \theta$

$y = A\cos \theta\sin \omega t + A\sin \theta\cos \omega t$

$y = A\sin(\omega t + \theta)$

Squaring and adding (2) and (3)

$A^2cos^2\theta + A^2sin^2\theta = (a + b\cos \varphi)^2 + b^2sin^2\varphi$

$A^2 = a^2 + b^2 + 2ab\cos \varphi$

$*A = \sqrt{a^2 + b^2 + 2ab\cos \varphi}$

As, $I \propto A^2$

$I = KA^2$

$= (a^2 + b^2 + 2ab\cos\varphi)$

$[I = I_1 + I_2 + \sqrt{I_1} + \sqrt{I_2}\cos\varphi]$

Dividing equation (3) by (2) we get,

$$\tan\theta = \frac{b\sin\varphi}{a + b\cos\varphi}$$

Condition for maxima:

The intensity will be maximum when the constructive interference takes place i.e.

$\cos\varphi = +1$

$\cos\varphi = \cos 2n\pi$

$\varphi = \pm 2n$, n=0, 1, 2...

$\Rightarrow \frac{2\pi}{\lambda} \times path\ difference(\Delta x) = \pm 2n\pi$

$[\Delta x = \pm n\lambda]$

$[\Delta x = 2n\frac{\lambda}{2}]$

The constructive interference is when φ difference is even multiple of π or integral multiple of 2π and path difference is an integral multiple of $\frac{}{2}$

Now, $[I = I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}]$

$[I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2]$

$[A_{max} = a + b]$

If the waves having equal amplitude,

$[A_{max} = 2a]$

$I_{max} = KA_{max}^2$

$= (2a)^2$

$= K4a^2$

$[I_{max} = 4I_0]$

Condition for minima

The intensity will be minimum destructive interference takes place

i.e. $\cos\varphi = -1$

$[\varphi = \pm(2n + 1)\pi]$ Where n = 0, 1, 2, 3...

$$\Rightarrow \frac{2\pi}{\lambda} \times (\Delta x) = \pm(2n + 1)\pi$$

$$[\Delta x = \pm(2n + 1)\frac{\lambda}{2}]$$

Thus destructive interference takes place when phase difference is odd multiple of π and path difference is odd multiple of $\frac{\,}{2}$

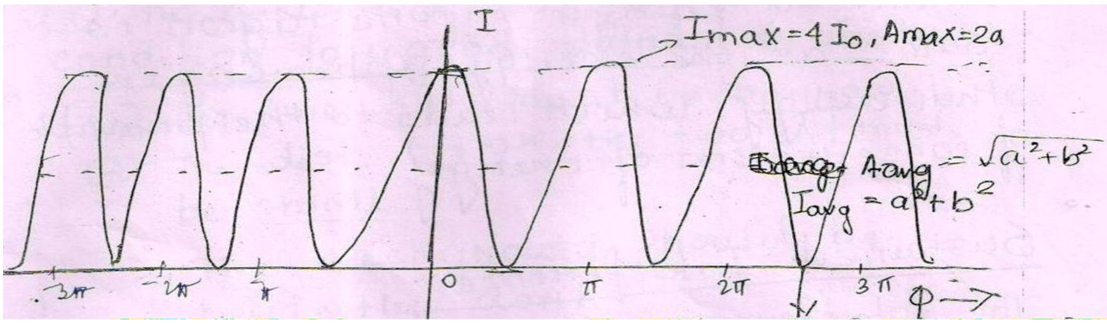
Now, $[I = I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}]$

$[I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2]$

$[A_{min} = a - b]$

Intensity distribution curve

If we plot a graph between phase difference or path difference along X-axis and intensity along Y-axis, the nature of the graph will be symmetrical on either side.



$$I_{ave} = \frac{I_{max} + I_{min}}{2} = 2I_0$$

From the graph, it is observed that,

1) The fringes are of equal width
2) Maxima having equal intensities
3) All the minima‟s are perfectly dark

The phenomenon of interference tends to conservation of energy i.e. the region where intensity is 0, actually the energy present is maxima. As the minima‟s and maxima position changes alternatively so the disappearance of energy appearing is same as the energy appearing in other energy which leads to the principle of conservation of energy.

Sustained Interference

The interference phenomenon in which position of the maxima and minima don‟t changes with time is called sustained interference.

Condition for Interference

1) The two waves must have same frequency and wavelength.
2) The two source of light should be coherent.
3) The amplitude of wave may be equal or nearly equal.

Condition for good Contrast

I. The two slit must be narrow.
II. The distance between the two slit must be small.
III. The background should be perfectly dark.
IV. The distribution between the slit and the screen should be large.
V. The two waves may have equal or nearly equal amplitude (for sharp superposition).

Coherent Sources

The two sources are said to be coherent if they have same phase difference, zero phase difference or their relative phase is constant with respect to time.

Practical resolution of Coherent

Coherent sources from a single source of light can be realised as follows

A narrow beam of light can be split into its number of component waves and multiple reflections.

Component light waves are allowed to travel different optical path so that they will suffer a path difference and hence phase difference.

$$[phase\ difference = \frac{2\pi}{\lambda} \times path\ difference]$$

Methods for producing coherent sources/Types of interferences

Coherent sources can be produced by two methods

1) Division of wave front
2) Division of amplitude

Division of Wave front

The process of coherent source or interference by dividing the wave front of a given source of light is known as division of wave front.

This can be done by method of reflection or refraction. In this case a point source is used.

Examples

1. YDSE
2. Lylord‟s single mirror method
3. Fresnel‟s bi-prism
4. Bilet splitting lens method

DIVISION OF AMPLITUDE

The process of obtaining a coherent source by splitting the amplitude of light waves is called division of amplitude which can be done by multiple reflections.

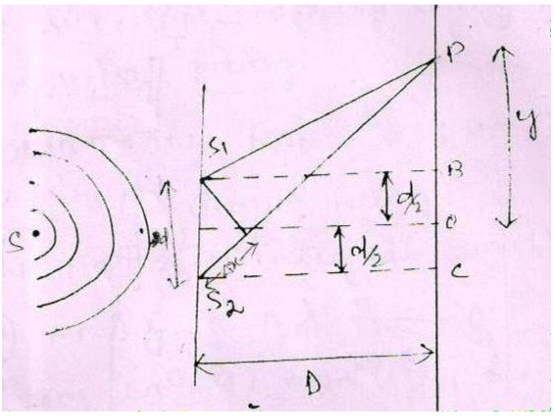
In this case, extended source of light is used.

1.Newton‟s ring method

2.Thin film method

3. Michelson‟s interferometer

Young'sDouble Slit Experiment:



In 1801 Thomas Young demonstrated the phenomenon of interference in the laboratory with a suitable arrangement. It is based on the principle of division of wavefront of interference. The experiential arrangement consists of two narrow slits, S_1 and S_2 closely spaced, illuminated by a monochromatic source of light S. A screen is placed at a distance D from the slit to observe the interference pattern.

In the figure,

$d \rightarrow$ Slit separation

$D \rightarrow$ Slit and screen separation

$\lambda \rightarrow$ Wavelength of light

$Y \rightarrow$distance of interfering point from the centre of slit

$\Delta x \rightarrow$Path difference coming from the light S_1 and S_2

Optical path difference between the rays coming through S_1 and S_2

Now the path difference,

$$\Delta x = S_2P - S_1P$$

In figure, $S_2P = [S_2C^2 + PC^2]^{1/2}$

$$= [D^2 + (y + \frac{d}{2})^2]^{1/2}$$

$$= [1 + \frac{(y+\frac{d}{2})^2}{D2^2}]^{1/2}$$

$$= D\,[1 + \frac{(y+\frac{d}{2})^2}{2D^2}]$$ (Using binomial theorem)

$$S_2P = D + \frac{(y+\frac{d}{2})^2}{2D^2}$$

Similarly,

$$S_1P = D + \frac{(y-\frac{d}{2})^2}{2D}$$

$$\Delta x = D + \frac{(y+\frac{d}{2})^2}{2D} - D - \frac{(y - d/2)^2}{2D}$$

$$= \frac{1}{2D} * (y + \frac{d}{2})^2 - (y - d/2)^2 +$$

$$= \frac{1}{2D} \times 4y\frac{d}{2}$$

$$= y\frac{d}{D}$$

$$\Delta x = y\frac{d}{D}$$

The alternative dark and bright patches obtained on the interference screen due to superposition of light waves are known as fringe.

Condition for bright fringe

The bright fringe is obtained when the path difference is integral multiple of λ i.e.

$$\Delta x = n\lambda$$

From equation (4) and (5), we get

$$y_n \frac{d}{D} = n\lambda$$

$$y_n = \frac{n\ D}{d}$$ Where n = 0, 1, 2 ……

Condition for dark fringe

It will be obtained when the path difference is an odd multiple of λ/2 i.e.

$$\Delta x = \frac{(2n+1)}{2}$$

From (4) and (6), we get

$$\frac{ynd}{D} = \frac{(2n+1)}{2}$$

$$y_n = \frac{(2n+1)}{2}$$ Where n = 0, 1, 2 ……

Fringe Width

The separation between two consecutive dark fringes and bright fringes is known as fringe width.

If y_n and y_{n-1} be the two consecutive bright fringe.

$$\beta = y_n - y_{n-1}$$

$$= \frac{n\lambda D}{d} - \frac{(n-1)D}{d}$$

$$\beta = \frac{\lambda D}{d}$$

Similarly, is y_n and y_{n-1} be the two consecutive dark fringes.

$$\beta' = (2n+1)\frac{\lambda D}{2d} - [2(n-1)+1] \times \frac{\lambda D}{2d}$$

$$= \frac{D}{2d} + \frac{D}{2d}$$

$$\beta' = \frac{\lambda D}{d}$$

It is concluded that the separation between the two consecutive bright fringes is equal to the consecutive dark fringes.

So $\beta = \beta'$

Hence bright and dark fringes are equispaced.

Discussion:

From the expression for $\beta = \beta' = \frac{D}{d}$

$$\Rightarrow \beta \propto \lambda$$

$$\Rightarrow \beta \propto D$$

$$\Rightarrow \beta \propto \frac{1}{d}$$

If young double slit apparatus is shifted from air to any medium having refractive index (μ), fringe pattern will remain unchanged and the fringe width decreases (1/μ) as λ decreases.

$$C = f\lambda_0$$

$$\mu = \frac{C}{V} = \frac{f_0}{f_m}$$

$$\Rightarrow \lambda_m = \frac{0}{\mu}$$

If YDSE is shifted from air to water, the fringe width decreases3/4 times width in air.

$$\beta_w = \frac{\lambda_w D}{d}$$

When YDSE is performed with white light instead of monochromatic light we observed,

I. Fringe pattern remains unchanged
II. Fringe width decreases gradually
III. Central fringe is white and others are coloured fringes overlapping

When YDSE is performed with red, blue and green light

$$\lambda_R > \lambda_G > \lambda_B$$

So $\beta_R > \beta_G > \beta_B$

$$\mu = \frac{C}{V} = \frac{f\lambda_0}{f\lambda_m}$$

$$\mu = \frac{\lambda_0}{\lambda_m}$$

$$\Longrightarrow [\lambda_m = \frac{\lambda_0}{\mu}]$$

Wavelength of light in any given medium, decreases to1/μ times of wavelength in vacuum.

$$\beta \propto \lambda_m$$

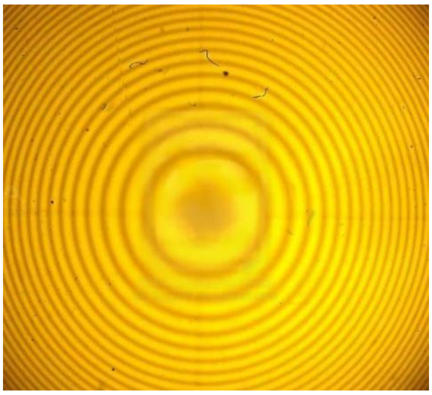
$$\beta_m = \frac{\lambda_m D}{d}$$

$$[\beta_m = \frac{\lambda_0 D}{\mu d}]$$

So, it decreases 1/μ times.

Newton’s Ring

The alternate dark and bright fringe obtained at the point of contact of a Plano convex lens with its convex side placed over a plane glass plate are known as Newton‟s ring as it was first obtained by Newton.



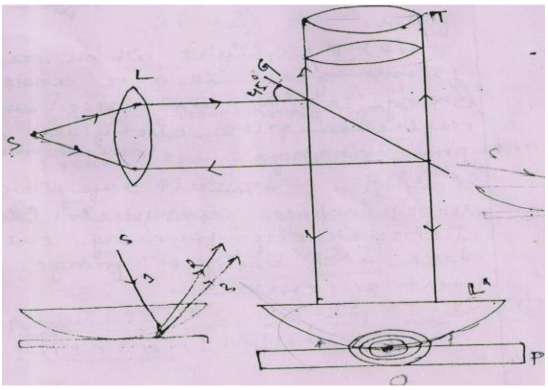
The formation of the Newton‟s ring is based on the principle of interference due to division of amplitude.

Experimental Arrangement

The experimental arrangement consist of

a) S: Monochromatic source of monochromatic light

b) P: A plane glass plate



c) L: A convex lens which is placed at its focal length to make the rays parallel after refraction

d) G: A plane glass plate inclined at on 45^0 to make the parallel rays travel vertically downwards

e) L‟: A plane convex lens of long focal length whose convex side kept in contact with plane glass plate

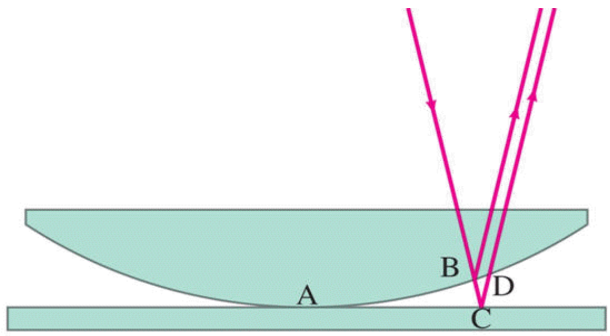
f) T: Travelling microscope mounted over the instrument to focus the Newton‟s ring.

Formation of Newton's Ring

I. To explain the formation of Newton‟s ring, let us consider a plano-convex lens with its convex side kept in contact with a plane glass plate.

II. At the point of contact air film is formed whose thickness gradually goes on increasing towards outside.

III. When a beam of monochromatic light is incident on the arrangement, a part of it get reflected from the upward surface of the air film and the part of light get reflected from the lower surface of the air film.



IV. The light which reflected from glass to air undergoes a phase change of „π‟ and those are reflected from air glass suffers no phase change.

V. As a result of which they super-impose constructively and destructively forming the alternate dark and bright fringe at the point of contact.

Condition for bright and dark fringe in Reflected light

In Newton"s ring experiment, the light travels from upper and lower part of the air film suffers a path difference of λ/2 (phase change of π). Again, as the ray of light reflected twice between the air films having thickness„t". Then the total path travelled by the light is given as $(2t + \frac{\lambda}{2})$.

Now, from the condition for bright ring, we have,

$$2t + \frac{\lambda}{2} = n\lambda$$

$$2t = n\lambda - \frac{\lambda}{2}$$

$$2t = (2n-1)\frac{\lambda}{2}$$

From the condition for the dark fringe we have,

$$2t + \frac{\lambda}{2} = (2n+1)\frac{\lambda}{2}$$

$$\Rightarrow 2t = \frac{\lambda}{2}(2n+1-1) = n\lambda$$

$$\Rightarrow 2t = n\lambda$$

Newton's ring in transmitted light

The Newton"s rings obtained in transmitted light are complementary to that of Newton"s ring obtained in reflected light i.e.

In transmitted light, the condition for bright ring is,

$$2t = n\lambda$$

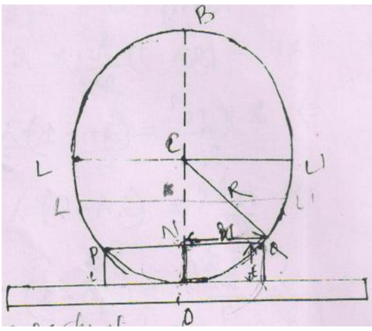
And for dark ring is,

$$2t = (2n - 1)\frac{\lambda}{2}$$

Newton’s ring in Reflected Light and Transmitted Light

In reflected light	**In transmitted light**
(a) Condition for bright ring; $2t = (2n - 1)\frac{}{2}$ (b) Condition for bright ring; $2t = n\lambda$ (c) Newton‟s rings are more intense.	(a) Condition for bright ring; $2t = n\lambda$ (b) Condition for dark ring; $2t = (2n - 1)\frac{}{2}$ (c) Newton‟s rings are less intense.

<u>DETRMINATION OF DIAMETER OF NEWTON‟S RING</u>



LOL‟ is the section of lens placed on glass plate AB. C is the centre of curvature of curved surface LOL‟. R is its radius of curvature and r is the radius of Newton‟s ring corresponding to film if thickness t.

From the property of circles,

$$PN \times NQ = ON \times NB$$

$$r \times r = t \times (2R - t)$$

t = thickness of air film

$$r^2 = 2Rt - t^2$$

$r^2 = 2R$(· t« 1)

$$\Rightarrow t = \frac{r^2}{2R}$$

From the condition for bright Newton"s ring,

$$2t = (2n - 1)\frac{\lambda}{2}$$

$$\Rightarrow 2 \times \frac{r^2}{2R} = (2n - 1)\frac{\lambda}{2}$$

$$\Rightarrow r^2 = (2n - 1)\frac{\lambda R}{2}$$

$$\Rightarrow \frac{D^2}{4} = (2n - 1)\lambda R$$

$$\Rightarrow D^2 = 2(2n - 1)R$$

$\Rightarrow D_n^2 = 2(2n - 1)R$, For the nth ring.

Q) Show that diameter of Newton"s dark or bright fringe is proportional to root of natural numbers.

$$D = \sqrt{2(2n - 1)\lambda R}$$

$$= \sqrt{2\lambda R \times (2n-1)}$$

$$= constant \times \sqrt{(2n-1)}$$

$\Rightarrow D_n \propto \sqrt{2n-1}$, n = 1, 2, 3…….

Thus the diameter of Newton‟s bright ring is proportional to square root of odd natural numbers.

Similarly from the Newton‟s dark ring,

$$2t = n\lambda$$

$$\Rightarrow 2 \times \frac{r^2}{2R} = n\lambda$$

$$\Rightarrow \frac{r^2}{R} = n\lambda$$

$$\Rightarrow r^2 = n\lambda R$$

$$\Rightarrow \frac{D^2}{4} = n\lambda R$$

$$\Rightarrow D^2 = 4n\lambda R$$

$$\Rightarrow D_n = \sqrt{4n\lambda R}$$

$$= \sqrt{4\lambda R}\sqrt{n}$$

$$= constant \times \sqrt{n}$$

$$D_n = constant \times \sqrt{n}$$

$$D_n \propto \sqrt{n}$$

Thus the diameter of Newton‟s dark ring is proportional to square root of natural numbers.

Determination of wavelength of light using Newton's ring method

To determine the wavelength of light, let us consider the arrangement which involves a travelling microscope mounted over the Newton‟s ring.

Apparatus, on focusing the microscope over the ring system and placing the crosswire of the eye piece on tangent position, the readings are noted. On taking readings on different positions of the crosswire on various rings we are able to calculate the wavelength of light used.

Let D_n and $_{(n+p)}$ be the n^th^ and (n+p)^th^ dark ring, then we have,

$$D_n^2 = 4n\lambda R$$

$$D_{(n+p)}^2 = 4(n+p)R$$

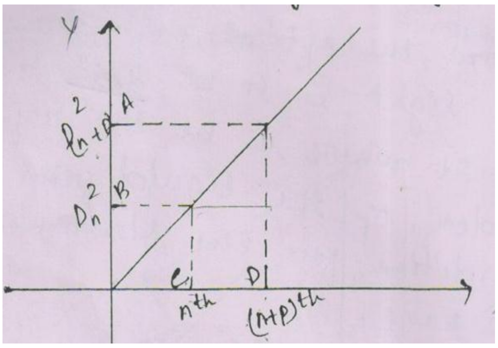
Subtracting equation (1) from (2) we get,

$$D_{(n+p)}^2 - D_n^2 = 4(n+p)R - 4n\lambda R$$

$$\frac{D_{(n+p)}^2 - D_n^2}{4PR} = \lambda$$

This is the required expression from the wavelength of light for Newton‟s ring method.

If we plot a graph between the orders of ring along X-axis and the diameter of the ring along Y-axis, the nature of the graph will be a straight line passing through origin.



From the graph the wavelength of light can be calculated the slope of the slope of the graph.

$\frac{1}{4R}$ Slope of the graph = wavelength of light

$$\Rightarrow \frac{AB}{CD} = \frac{D^2_{(n+p)} - D^2_n}{P}$$

$$Slope = \frac{D^2_{(n+p)} - D^2_n}{P}$$

Determination of refractive index of liquid by Newton's ring

The liquid whose refractive index is to be determined is to be placed between the gap focused between plane convex lens and plane glass plate. Now the optical path travelled by the light is to be 2μt, instead of 2t where μ be the refractive index of the liquid from the condition for the Newton"s ring we have,

$$2\mu t = n\lambda$$

$$\Rightarrow 2\mu \frac{r^2}{2R} = n\lambda$$

$$\Rightarrow \mu \frac{r^2}{R} = n\lambda$$

$$\Rightarrow \frac{r^2}{R} = \frac{n\lambda}{\mu}$$

$$\Rightarrow \frac{D^2}{4R} = \frac{n\lambda}{\mu}$$

$$\Rightarrow D^2 = \frac{4n\lambda R}{\mu}$$

For n^th ring, $D^2 = \frac{4n R}{\mu}$

Let D'_{n+p} and D'_n be the diameter of the (n+p)^th and n^th dark ring in presence of liquid then

$$D'^2_{n+p} = \frac{4(n+p)\lambda R}{\mu} \text{ and } D'^2_n = \frac{4n\lambda R}{\mu}$$

Now ,

$$D'^2_{n+p} - D'^2_n = \frac{4(n+p)\lambda R}{\mu} - \frac{4n\lambda R}{\mu} = \frac{4p\lambda R}{\mu} \qquad (1)$$

If the same order ring observed in air then

$$D^2_{n+p} - D^2_n = 4p\lambda R \qquad (2)$$

Dividing equation (2) by (1) ,we have

$$\mu = \frac{\left(D^2_{n+p} - D^2_n\right)_{air}}{\left(D'^2_{n+p} - D'^2_n\right)_{liquid}}$$

This is the required expression for refractive index of the liquid.

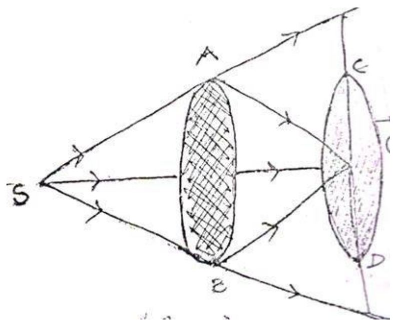
DIFFRACTION

Fundamental Idea about diffraction:

- The phenomenon of bending of light around the corner of an aperture or at the edge of an obstacle is known as diffraction
- The diffraction is possible for all types of waves
- The diffraction verifies the wave nature of light
- Diffraction takes place is due to superposition of light waves coming from two different points of a single wave front
- Diffraction takes place when the dimension of the obstacle is comparable with the wavelength of the incident light.

Explanation of diffraction:

To explain diffraction, let us consider an obstacle AB is placed on the path of an monochromatic beam of light coming from a source „S" which produces the geometrical shadow CD on the screen. This proves the rectilinear propagation of light.



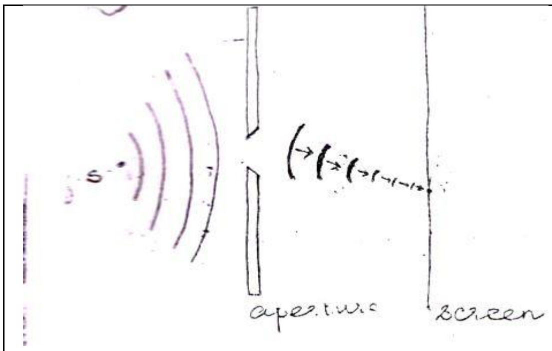
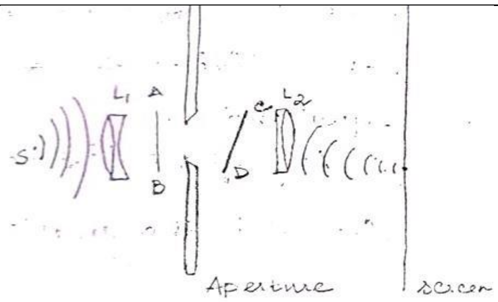
If the dimension or size of the obstacle is **comparable** with the wave length of the incident light, then light bends at the edge of the obstacle and enters in to the geometrical shadow region of the obstacle. According to Fresnel inside a well region, the destructive interference takes place for which we get brightest central maxima, which is associated with the diminishing lights on either side of the shadow as the constructive interference takes place out side the well region. This explains the diffraction phenomena.

Types of Diffraction:

Depending on the relative position of the obstacle from the source and screen, the diffraction is of 2 types.

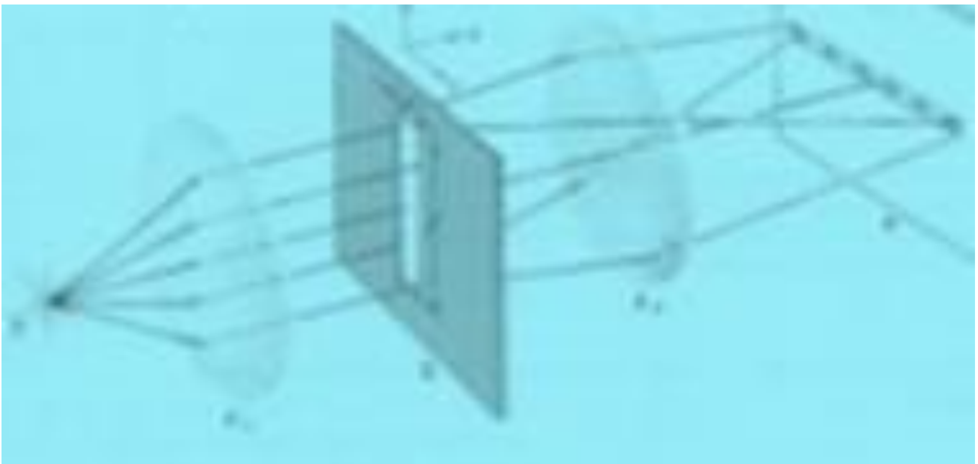
a. Fresnel Diffraction
b. Fraunhoffer Diffraction

Fresnel's Diffraction	**Fraunhoffer Diffraction**
(1) The type of diffraction in which the distance of either source or screen or both from the obstacle is **finite**, such diffraction is known as Fresnel"s diffraction. (2) No lenses are used to make the rays converge or parallel. (3) The incident wave front is either cylindrical or spherical. Ex:The diffraction at the straight edge.	(1) The type of diffraction in which the distance of either source or screen or both from the obstacle is **infinite**, such diffraction is known as Fraunhoffer diffraction. (2) Lenses are used to make the rays converge or parallel. (3) The incident wave front is plane. Ex:. The diffraction at the narrow.

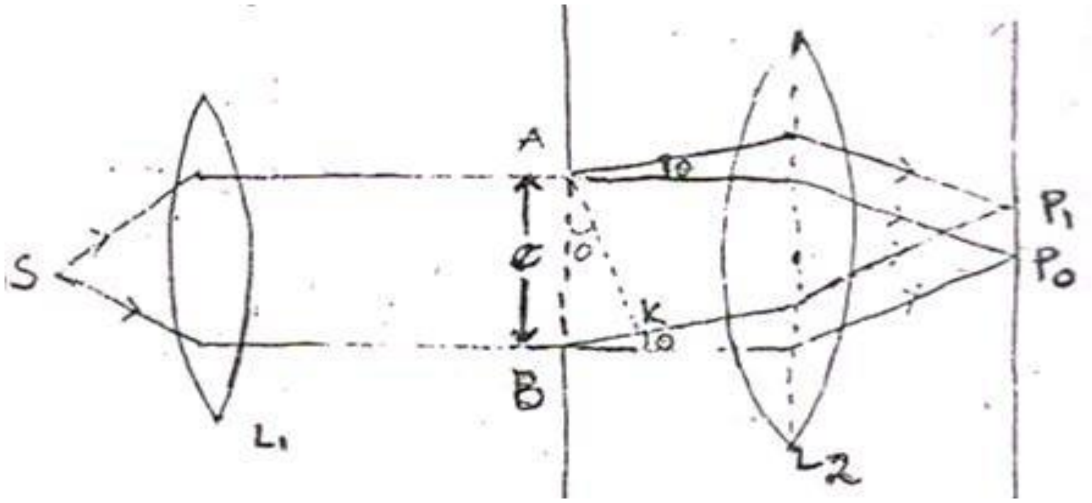
	

Fraunhoffer Diffraction due to a single slit:

Let us consider a parallel beam of monochromatic light inside on a slit „AB‟ having width „e‟. The rays of the light which are incident normally on the convex lens „L_2‟, they are converged to a point „P_0‟ on the screen forming a central bright image.



Fraunhoffer diffraction due to single slit



Schematic digram for Fraunhoffer diffraction due to single slit

The rays of light which get deviated by an angle „θ‟, they are converged to a point „P_1‟, forming an image having lens intensity.

As the rays get deviated at the slit „AB‟ they suffer a path difference. Therefore path difference, BK = AB Sinθ

= e sinθ

Therefore, Phase difference = $\frac{2\pi}{\lambda} e \sin\theta$

Let us divide the single slit into ‟n‟ no. of equal holes and a be the amplitude of the light coming from each equal holes.

Then Avg. phase difference= $\frac{1}{n} \frac{2\pi}{\lambda} e \sin\theta$

Now the resultant amplitude due to superposition of waves is given as

$$R = \frac{a\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} = \frac{a\sin\left\{\left(\frac{n}{2}\right)\frac{1}{n}\frac{2\pi}{\lambda}e\sin\theta\right\}}{\sin\left\{\frac{\left(\frac{1}{n}\frac{2\pi}{\lambda}e\sin\theta\right)}{2}\right\}} = \frac{a\sin\left(\frac{\pi}{\lambda}e\sin\theta\right)}{\sin\left(\frac{1}{n}\frac{\pi}{\lambda}e\sin\theta\right)}$$

Let $\alpha = \frac{\pi}{\lambda}e\sin\theta$,then $R = \frac{a\sin\alpha}{\sin\frac{\alpha}{n}}$

Since α is very small and n is very large so $\frac{\alpha}{n}$ is also very small.

Therefore, $\sin\frac{\alpha}{n} \approx \frac{\alpha}{n}$

Thus, R = $\frac{a\sin\alpha}{\frac{\alpha}{n}} = \frac{na\sin\alpha}{\alpha} = \frac{A\sin\alpha}{\alpha}$ $\quad where A = an$

Now the intensity is given as

Iα $R^2 \Rightarrow I = KR^2 \Rightarrow I = K\frac{A^2\sin^2\alpha}{\alpha^2} = I_0\frac{\sin^2\alpha}{\alpha^2}$ where $I_0 = KA$

Condition for Central /principal maxima:

When $\alpha = 0$,

$$\Rightarrow \frac{\pi}{\lambda}e\sin\theta = 0 \Rightarrow \sin\theta = 0$$

$$\Rightarrow \theta = 0$$

Thus, the condition for principal maxima will be obtained at $\theta = 0$ position for all the rays of light.

Position for/Condition for minima:

The minimum will be obtained when $\sin\alpha = 0 = \sin(\pm m\pi)$

$$\Rightarrow \sin\alpha = \sin(\pm m\pi)$$

$$\Rightarrow \alpha = \pm m\pi$$

$$\Rightarrow \frac{\pi}{\lambda} e\sin\theta = \pm m\pi$$

$$\Rightarrow e\sin\theta = \pm m\lambda$$ where $m = 1,2,3,4,.......$

$$\Rightarrow \theta = \pm \frac{m\lambda}{e}$$

Thus, the minimas are obtained at $\pm\frac{\lambda}{e}, \pm 2\frac{\lambda}{e}, \pm 3\frac{\lambda}{e}, \pm 4\frac{\lambda}{e}, ...$

Position/Condition for secondary maxima:

The maxima‟s occurring in between two consecutive secondary maxima is known as secondary maxima.

The positions for secondary maxima will be obtained as

$$\frac{dI}{d\alpha} = 0$$

$$\Rightarrow \frac{d}{d\alpha}\left[I_0 \frac{\sin^2\alpha}{\alpha^2} \right] = 0$$

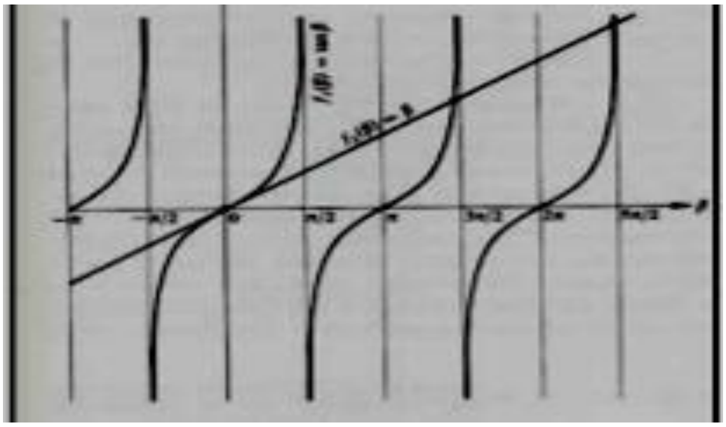
$$\Rightarrow 2I_0 \frac{\sin\alpha}{\alpha}\left[\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} \right] = 0$$

$$\Rightarrow \frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} = 0$$

$$\Rightarrow \alpha\cos\alpha - \sin\alpha = 0$$

$$\Rightarrow \alpha = \tan\alpha$$

This is a transdectional equation.It can be solved by graphical method. Taking $y = \alpha$ and $y = \tan\alpha$,where the two plots are interests, this intersection points gives the position for secondary maxima. Thus the secondary maxima‟s are obtained at $\alpha = \frac{3\pi}{2}, \alpha = \frac{5\pi}{2}, \alpha = \frac{7\pi}{2}.........$



From the expression for amplitude we have

$$R = \frac{A\sin\alpha}{\alpha} = \frac{A}{\alpha}\left[\alpha + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \frac{\alpha^7}{7!} + \dots\dots\dots\dots\right]$$

$$= \frac{A}{\alpha} x\alpha\left[1 + \frac{\alpha^2}{3!} + \frac{\alpha^4}{5!} + \dots\dots\dots\dots\dots\right] = A, \text{ since } \alpha \ll 1$$

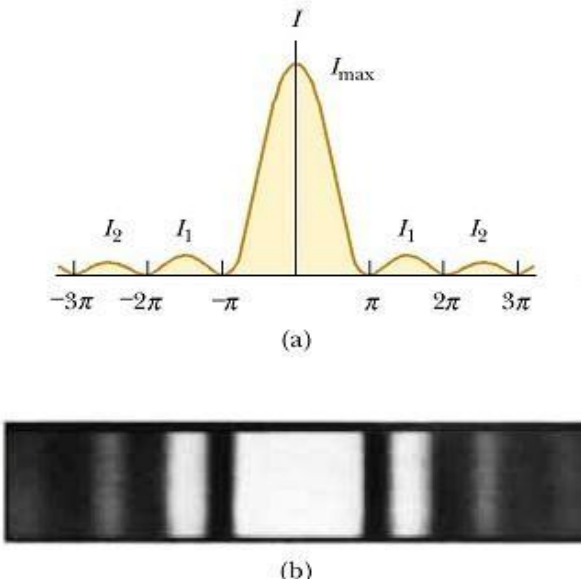
Thus the intensity at the central principal maxima is I_0

For $\alpha = \frac{3\pi}{2}$, $I_1 = I_0 \frac{Sin^2\alpha}{\alpha^2} = I0 \frac{Sin^2(\frac{3\pi}{2})}{\left(\frac{3\pi}{2}\right)^2} = \frac{I_0}{22}$

For $\alpha = \frac{5\pi}{2}$, $I_2 = I_0 \frac{Sin^2\alpha}{\alpha^2} = I0 \frac{Sin^2(\frac{5\pi}{2})}{\left(\frac{5\pi}{2}\right)^2} = \frac{I_0}{62}$ and so on ……

Intensity distribution curve:

The graph plotted between phase difference and intensity of the fringes is known as intensity distribution curve. The nature of the graph is as follows:



Intensity distribution curve

From the nature of the graph it is clear that

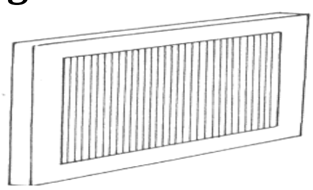
1. The graph is symmetrical about the central maximum
2. The maxima are not of equal intensity
3. The maxima are of not equal width

The minima are of not perfectly dark

PLANE TRANSMISSION GRATING:

It is an arrangement consisting of large no.of parallel slits of equal width separated by an equal opaque space is known as diffraction grating or plane transmission grating.

Diffraction grating



Construction: It can be constructed by drawing a large no. of rulings over a plane transparent material or glass plate with a fine diamond point.

Thus the space between the two lines act as slit and the opaque space will acts as obstacle.

N.B. Though the plane transmission grating and a plane glass piece looks like alike but a plane transmission grating executes rainbow colour when it exposed to sun light where as a plane glass piece does not executes so.

Grating element:

The space occurring between the midpoints of two consecutive slit in a plane transmission grating is known as **Grating element**. It can be measured by counting the no. of rulings present in a given length of grating.

Let us consider a diffraction grating having

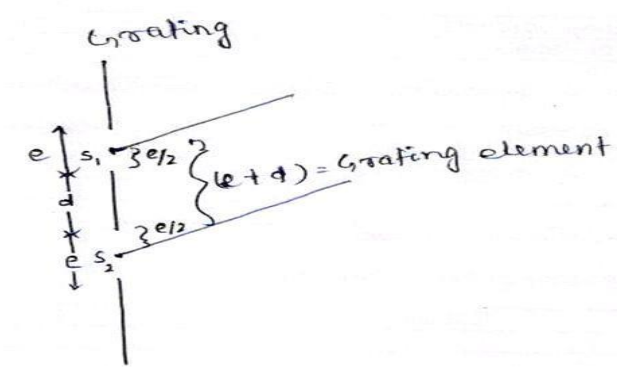
e = width of the slit

d = width of the opacity

If "N" be the no. of rulings present in a given length of grating "x" each having width (e+d), then

$$N(e+d) = x$$

$$\Rightarrow (e+d) = \frac{x}{N} = \text{Grating element}$$

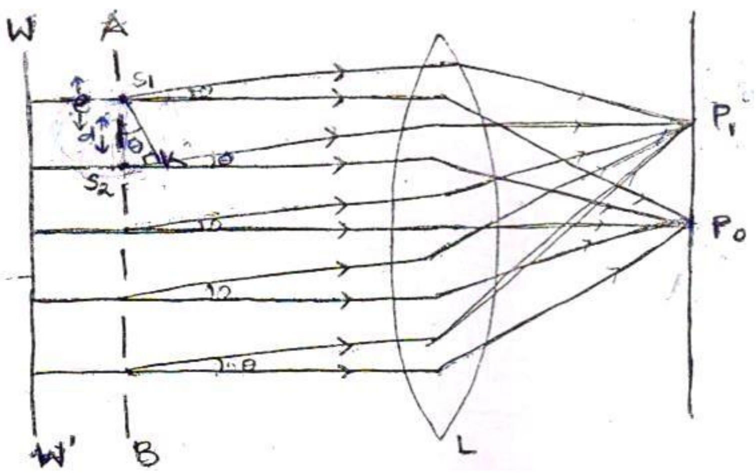


For example if a grating contain 15,000 lines per cm in a grating then the grating element of the grating

Grating element, (e+d) = $\frac{1}{15000}$ =0.00016933 cm

Diffraction due to plane transmission grating /Fraunhoffer diffraction due to N-parallel slit:

Let us consider a plane wave front coming from an infinite distance is allowed to incident on a convex lens "L" which is placed at its focal length. The rays of light which are allowed to incident normally on the lens are converged to a point "P_o" forming central principal maxima having high intensity and the rays of light which are diffracted through an angle are "θ" are converge to a point "P_1" forming a minima having less intensity as compared to central principal maxima. Again those rays of light which are diffracted through an angle "θ" are undergoes a path difference and hence a phase difference producing diffraction.



Let AB- be the transverse section of the plane transmission grating

ww'- be a plane wave front coming from infinite distance

e = width of the slit

d = width of the opacity

(e+d) = grating element of the grating

N = be the no. of rulings present in the grating

Now the path difference between the deviated light rays is

$S_2K = S_1S_2 \sin\theta = (e+d)\sin\theta$

Therefore, Phase difference = $\frac{2\pi}{\lambda}$ x $S_2K = \frac{2\pi}{\lambda}(e+d)Sin\theta = 2\beta$ (say)

where $\beta = \frac{\pi}{\lambda}(e+d)Sin\theta$

Now the resultant amplitude due to superposition of "N" no .of waves coming from "N" parallel slit is given as

$$R = A\frac{Sin\alpha}{\alpha}\frac{SinN\beta}{Sin\beta}$$

and intensity is given as $I\alpha R^2 \Rightarrow I = KR^2 = KA^2\frac{Sin^2\alpha}{\alpha^2}\frac{Sin^2 N\beta}{Sin^2\beta} = I_0\frac{Sin^2\alpha}{\alpha^2}\frac{Sin^2 N\beta}{Sin^2\beta}$

where $I_0\frac{Sin^2\alpha}{\alpha^2}$ =this is contributed due to diffraction at single slit

and $\frac{Sin^2 N\beta}{Sin^2\beta}$ = this is contributed due to interference at " N" parallel slit

Position for central principal maxima /condition for central principal maxima:

The principal maxima will be obtained when

$$Sin\beta = o = Sin(\pm m\pi)$$
$$\Rightarrow \beta = \pm m\pi$$
$$\Rightarrow \frac{\pi}{\lambda}(e+d)Sin\theta = \pm m\pi$$
$$\Rightarrow (e+d)Sin\theta = \pm m\lambda$$

where $m = 0,1,2,3......$ .This is called grating equation or condition for central principal maxima.

Position for minima /condition for minima:

The minima will be obtained when

$$Sin N\beta = o = Sin(\pm n\pi)$$
$$\Rightarrow N\beta = \pm n\pi$$
$$\Rightarrow N\frac{\pi}{\lambda}(e+d)Sin\theta = \pm n\pi$$
$$\Rightarrow N(e+d)Sin\theta = \pm n\lambda$$

Where n can take all the values except $n = 0, \pm N, \pm 2N, \pm 3N, ...$

This is the condition for minima due to diffraction at N-parallel slit.

Position/Condition for secondary maxima:

The maxima‟s occurring in between two consecutive secondary maxima is known as secondary maxima.

The positions for secondary maxima will be obtained as

$$\frac{dI}{d\alpha} = 0$$

$$\Rightarrow \frac{d}{d\alpha}\left[I_0 \frac{Sin^2\alpha}{\alpha^2}\frac{Sin^2 N\beta}{Sin^2\beta}\right] = 0$$

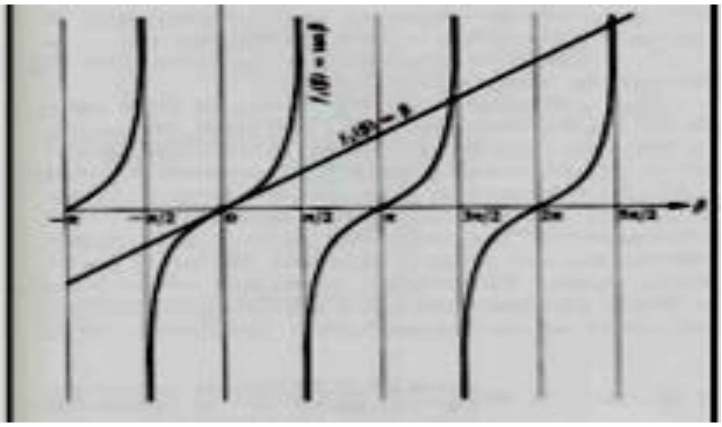
$$\Rightarrow 2I_0 \frac{Sin^2\alpha}{\alpha^2}\frac{Sin\ N\beta}{Sin\beta}\left[\frac{N\cos N\beta \sin\beta - \sin N\beta\cos\beta}{\sin^2\beta}\right] = 0$$

$$\Rightarrow \frac{N\cos N\beta \sin\beta - \sin N\beta\cos\beta}{\sin^2\beta} = 0$$

$$\Rightarrow N\cos N\beta \sin\beta - \sin N\beta\cos\beta = 0 \Rightarrow N\cos N\beta \sin\beta = \sin N\beta\cos\beta$$

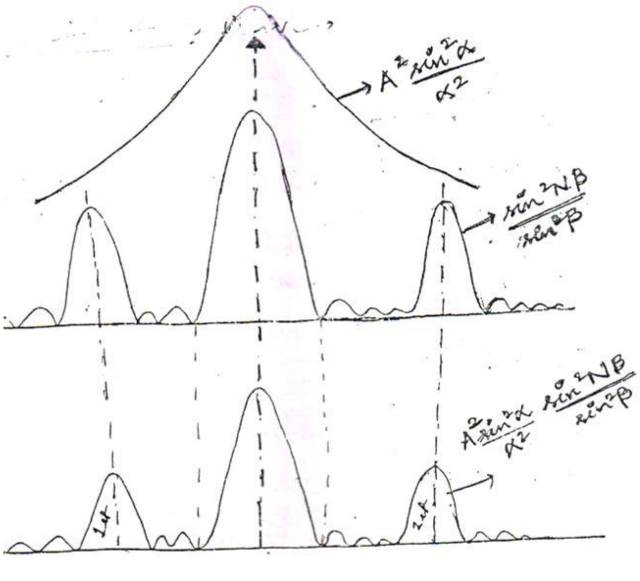
$$\Rightarrow N\tan N\beta = \tan N\beta$$

This is a transdectional equation. It can be solved by graphical method. Taking $y = \tan N\beta$ and $y = N\tan N\beta$,where the two plots are interests, this intersection points give the position for secondary maxima.Thus the secondary maxima‟s are obtained at $\beta = \frac{3\pi}{2}, \beta = \frac{5\pi}{2}, \beta = \frac{7\pi}{2}$ ....



Intensity distribution curve:

The graph plotted between phase difference and intensity of the fringes is known as intensity distribution curve. The nature of the graph is as follows:



Characteristics of the spectral lines or grating spectra:

1. The spectra of different order are situated on either side of central principal maximum
2. Spectral lines are straight and sharp
3. The spectra lines are more dispersed as we go to the higher orders.

4. The central maxima is the brightest and the intensity decreases with the increase of the order of spectra.

Missing spectra or Absent spectra:

When the conditions for minima due to diffraction at single slit and condition for central principal maxima due to diffraction at N-parallel slit is satisfied simultaneously for a particular angle of diffraction then, certain order maxima are found to be absent or missed on the resulting diffraction pattern which are known as missing spectra or absent spectra.

Condition for Missing spectra:

We have,

The condition for central principal maxima due diffraction at N-parallel slit

$$(e+d)Sin\theta = \pm m\lambda$$

$$e\sin\theta = \pm n\lambda$$

$$\Rightarrow \frac{(e+d)Sin\theta}{e\sin\theta} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

Special case:

1. If d = e, $\Rightarrow \frac{m}{n} = 2 \Rightarrow m = 2n$ where $n = 1,2,3,.....$

i.e second order or multiple of 2 order spectra will found to be missed or absent on the resulting diffraction pattern.

2. If $d = \frac{e}{2}, \Rightarrow \frac{m}{n} = \frac{3}{2} \Rightarrow m = 1.5n \approx 1$

i.e First order spectra will found to be missed or absent on the resulting diffraction pattern.

3. If $e = \frac{d}{2}, \Rightarrow \frac{m}{n} = 3 \Rightarrow m = 3n$

i.e Third order spectra or multiple of 3 spectra will found to be missed or absent on the resulting diffraction pattern.

Dispersion:

The phenomenon of splitting of light wave into different order of spectra is known as dispersion.

Dispersive power:

The variation of angle of diffraction with the wave length of light is known as dispersive power. It is expressed as $\frac{d\theta}{d\lambda}$

Where $d\theta = \theta_1 - \theta_2$ = difference in angle of diffraction and $d\lambda = \lambda_1 - \lambda_2$=difference in wave length of light

Expression for dispersive power:

We have

$$(e+d)Sin\theta = \pm m\lambda$$

$$\frac{d}{d\lambda}\left[(e+d)Sin\theta = \pm m\lambda\right] = \frac{d}{d\lambda}(m\lambda)$$

$$\Rightarrow (e+d)\frac{d}{d\lambda}(Sin\theta) = m\frac{d\lambda}{d\lambda}$$

$$\Rightarrow (e+d)\cos\theta\frac{d\theta}{d\lambda} = m$$

$$\Rightarrow \frac{d\theta}{d\lambda} = \frac{m}{(e+d)\cos\theta}$$

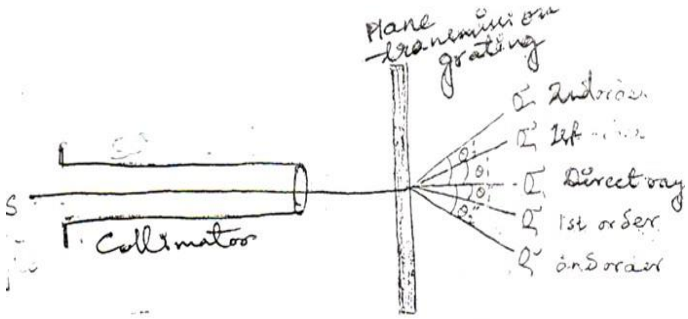
$$\Rightarrow \frac{d\theta}{d\lambda} \quad \alpha \ m$$

$$\alpha \frac{1}{(e+d)}$$

$$\alpha \frac{1}{\cos\theta}$$

Determination of wave length of light using plane transmission grating:

To determine the wave length of light let us consider a plane transmission grating with its rulled surface facing towards the source of light perpendicular to the axis of the spectrometer. The parallel beam of monochromatic light coming from source is allowed to incident on the transmission grating which are now defracted by different angle of diffraction.Rotating the telescope for different positions of the defracted ray the angles are measured.



Using the grating equation ,

$$(e+d)Sin\theta = \pm m\lambda$$

$$\Rightarrow \lambda = \frac{(e+d)Sin\theta}{m}$$

We can calculate the wave length of the monochromatic light.

MODULE-III

VECTOR CALCULUS

The electric field $(\vec{E})$, magnetic induction $(\vec{B})$, magnetic intensity $(\vec{H})$, electric displacement $(\vec{D})$, electrical current density $(\vec{J})$, magnetic

vector potential $(\vec{A})$ etc. are, in general, functions of position and time. These are vector fields.

Scalar quantities such as electric potential, electric charge density, electromagnetic energy density etc. are also function of position and time. They are known of as fields.

Time Derivative of a Vector Field

If $\vec{A}(t) \rightarrow$ time dependent vector field, then the Cartesian coordinates

$$\vec{A}(t) = \hat{i}\vec{A}_x(t) + \hat{j}\vec{A}_y(t) + \hat{k}\vec{A}_z(t)$$

$$\frac{d\vec{A}}{dt} = \hat{i}\frac{\partial\vec{A}_x(t)}{\partial t} + \hat{j}\frac{\partial\vec{A}_y(t)}{\partial t} + \hat{k}\frac{\partial\vec{A}_z(t)}{\partial t}$$

Notes: $\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dt} + (\frac{d\vec{A}}{dt})\times\vec{B}$

Gradient of a Scalar Field

The change of a scalar field with position is described in terms of gradient operator.

$$grad(V) = \vec{\nabla}V = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$$

Where $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ is **del operator or nabla**

$\vec{\nabla}V$ is a vector. The gradient of a scalar is a vector.

Divergence of a Vector Field

The divergence of a vector field $\vec{A}$ is given by

$$\vec{\nabla}.\vec{A} = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}).(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) = (\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \hat{k}\frac{\partial A_z}{\partial z})$$

Divergence of a vector field is a scalar.

Notes:

- $\vec{\nabla}.(\vec{A}+\vec{B}) = \vec{\nabla}.\vec{A} + \vec{\nabla}.\vec{B}$
- $\vec{\nabla}.(V\vec{A}) = (\vec{\nabla}V).\vec{A} + V(\vec{\nabla}.\vec{A})$ where V is a scalar field
- If the divergence of a vector field vanishes everywhere, it is called a solenoidal field.
- Divergence of a vector field is defined as the net outward flux of that field per unit volume at that point.

<u>Curl of a Vector Field</u>

The curl of a vector field is given by

$$\text{Curl A} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \end{vmatrix} = \hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$$

- Curl of a vector field is a vector
- If V is a scalar field, $\vec{A}$ and $\vec{B}$ are two vector fields, then

  $\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$

  $\vec{\nabla}\times(V\vec{A}) = (\vec{\nabla}V)\times\vec{A} + V(\vec{\nabla}\times\vec{A})$
- If curl of a vector field vanishes, then it is called an irrotational field.

<u>Successive Operation of the $\vec{\nabla}$ operator</u>

(i) <u>Laplacian</u>

$$\vec{\nabla}\times\vec{\nabla} = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}).(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Rightarrow \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This is called Laplacian Operator

(ii) <u>Curl of gradient of ascalar</u>

$$\nabla V = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$$

Where V is a scalar field

$$\underset{\to}{\nabla}\times(\underset{\to}{\nabla}V) = \hat{i}[(\frac{\partial}{\partial y})(\frac{\partial V}{\partial z}) - (\frac{\partial}{\partial z})(\frac{\partial V}{\partial y})] + \hat{j}[(\frac{\partial}{\partial z})(\frac{\partial V}{\partial x}) - (\frac{\partial}{\partial x})(\frac{\partial V}{\partial z})] + \hat{k}[(\frac{\partial}{\partial x})(\frac{\partial V}{\partial y}) - (\frac{\partial}{\partial y})(\frac{\partial V}{\partial x})]$$

$$\because \vec{\nabla}\times(\vec{\nabla}V) = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial x} & \frac{\partial V}{\partial x} \end{vmatrix}$$

$$\Rightarrow \underset{\to}{\nabla}\times\underset{\to}{(\nabla V)} = \hat{i}(\frac{\partial^2 V}{\partial y\partial z} - \frac{\partial^2 V}{\partial z\partial y}) + \hat{j}(\frac{\partial^2 V}{\partial z\partial x} - \frac{\partial^2 V}{\partial x\partial z}) + \hat{k}(\frac{\partial^2 V}{\partial x\partial y} - \frac{\partial^2 V}{\partial y\partial x}) = 0$$

Thus Curl of gradient of a scalar field is zero.

<u>Note:</u>

- If $\vec{\nabla}\times\vec{A} = 0$, then $\vec{A}$ can be expressed as gradient of a scalar field i.e. $\vec{A} = \vec{\nabla}V$
- Conversely if a vector field is gradient of a scalar then its curl vanishes.

(iii) <u>Divergence of Curl of a Vector Field</u>

$$\vec{\nabla}\times\vec{A} = \hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$$

$$\vec{\nabla}.\vec{\nabla}\times\vec{A} = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}).[(\hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})]$$

$$\Rightarrow \vec{\nabla}.\vec{\nabla}\times\vec{A} = \frac{\partial}{\partial x}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$$

$$\Rightarrow \vec{\nabla}.\vec{\nabla}\times\vec{A} = \frac{\partial^2 A_z}{\partial x\partial y} - \frac{\partial^2 A_y}{\partial x\partial z} + \frac{\partial^2 A_x}{\partial y\partial z} - \frac{\partial^2 A_z}{\partial y\partial x} + \frac{\partial^2 A_y}{\partial z\partial x} - \frac{\partial^2 A_x}{\partial z\partial y}$$

$$\Rightarrow \vec{\nabla}.\vec{\nabla}\times\vec{A} = 0$$

i.e. divergence of curl of a vector is zero.

Conversely, if the divergence of a vector field is zero, then the vector field can be expressed as the curl of a vector.

(iv)(iv) $\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2 A$

(v)(v) $\vec{\nabla}.(\vec{A}\times\vec{B}) = \vec{B}.(\vec{\nabla}\times\vec{A}) - \vec{A}.(\vec{\nabla}\times\vec{B})$

Line Integral of a Vector

The line integral of a vector field between two points a and b, along a given path is

$$I_L = \int_a^b \vec{A}.\vec{dl}$$

$\vec{dl} \rightarrow$ elemental length along the given path between a and b.

The line integral of a vector field is a scalar quantity.

$$I_L = \int_a^b (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}dx + \hat{j}dy + \hat{k}dz) = \int_a^b (A_x dx + A_y dy + A_z dz)$$

Notes:

- If the integral is independent of path of integration between a and b, then the vector field is conservative field.
- The line integral of a conservative field $\vec{A}$ along a closed path vanishes

  i.e. $\oint \vec{A}.\vec{dl} = 0$
- In general, the line integral depends upon the path between a and b.

Surface integral of a Vector

The surface integral of a vector field $\vec{A}$, over a given surface $\vec{s}$ is

$$I_s = \int_S \vec{A}.\vec{ds}$$

Where $\vec{ds} \rightarrow$ elemental area of surface S

The direction of $\vec{ds}$ is along the outward normal to the surface.

Writting $\vec{ds} = \hat{n} ds$, where $\hat{n}$ is unit vector normal to the surface at a given point.

So $I_s = \int_S \vec{A}.\vec{ds} = \int_S \vec{A}.\hat{n} ds = \int_n A_n ds$

where $A_n \to \vec{A}.\hat{n}$, normal to the component of the vector at the area element.

So, surface integral of a vector field over a given area is equal to the integral of its normal component over the area.

Surface area of a vector field is a scalar.

Example: $\phi_E = \int_S \vec{E}.\vec{ds}$

Volume integral of a Vector

The volume integral of a vector field $\vec{A}$ over a given volume V is

$I_V = \int_V \vec{A} dV$

Where dV is the elemental volume (a scalar)

Volume integral of a vector field is a vector.

Gradient, Divergence and Curl in terms of Integrals

The gradient of a scalar field φ is the limiting value of its surface integral per unit volume, as volume tends to zero

i.e. $\nabla\phi = \lim_{\Delta V \to 0} \frac{\oint \phi ds}{\Delta V}$

The divergence of a vector field A is the limiting value of its surface integral per unit volume, over an area enclosing the volume, as volume tends to zero.

$$\vec{\Delta}.\vec{A} = \lim_{\Delta V \to 0} \frac{\oint_{\Delta S} \vec{A}.\vec{ds}}{\Delta V}$$

The curl of a vector field is the limiting value of its line integral along a closed path per unit area bounded by the path, as the area tends to zero,

$$\vec{\Delta} \times \vec{A} = \lim_{\Delta S \to 0} \frac{\oint \vec{A}.\vec{dl}}{\Delta S}$$

where $\hat{n}$ is the unit vector normal to the area enclosed.

Gauss Divergence Theorem

The volume integral of divergence of a vector $\vec{A}$ over a given volume V is equal to the surface integral of the vector over a closed area enclosing the volume.

$$\int_V \nabla.\vec{A}dV = \oint_S \vec{A}.d\vec{S}$$

This theorem relates volume integral to surface integral.

Stokes Theorem

The surface integral of the curl of a vector field $\vec{A}$ over a given surface area S is equal to the line integral of the vector along the boundary C of the area

$$\int_S (\vec{\nabla} \times \vec{A}).d\vec{S} = \oint_C \vec{A}.d\vec{l}$$

For a closed surface C=0. Hence surface integral of the curl of a vector over a closed surface vanishes.

Green's Theorem

If there are two scalar functions of space f and g, then Green"s theorem is used to change the volume integral into surface integral. This theorem is expressed as

$$\int_V (f\nabla^2 g - g\nabla^2 f)dV = \int_S (f\nabla g - g\nabla f)dS$$

V- volume enclosed by surface S.

Electric Polarization ($\vec{P}$)

Electric polarization $\vec{P}$ is defined as the net dipole moment ($\vec{p}$) induced in a specimen per unit volume.

$$\vec{P} = \frac{\vec{p}}{V}$$

Unit is 1 coul/m^2

The dipole moment is proportional to the applied electric field.

So $\vec{p} = \alpha \vec{E}$, $\alpha \rightarrow proportionality constant, known as polarizability$

If N is the number of molecules per unit volume then polarization is given by

$$\vec{P} = N\alpha \vec{E}$$

Electric Displacement Vector $\vec{D}$

The electric displacement vector $\vec{D}$ is given by

$$\vec{D} = \vec{P} + \varepsilon_0 \vec{E} \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } (1)$$

where is the $\vec{P} \rightarrow polarization vector$

Unit of $\vec{D} \rightarrow 1 \frac{ampere \times \sec}{m^2}$

In linear and isotropic dielectric,

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} \text{ -------------------- } (2)$$

Comparing equations (1) and (2), we get

$$\varepsilon_0 \varepsilon_r \vec{E} = \vec{P} + \varepsilon_0 \vec{E}$$

$$\Rightarrow \vec{P} = \varepsilon_0 (\varepsilon_r - 1) \vec{E}$$

Electric Flux (φ_E)

The number of lines of force passing through a given area is known as electric flux.

It is given by

$$\phi_E = \int_S \vec{E}.\vec{dS}$$

Unit of flux-$1 \frac{N \times m^2}{Coul}$

Gauss' Law in Electrostatic:

Statement: The total electric flux (φ_E) over a closed surface is equal to $\frac{1}{\varepsilon_0}$ times the net charge enclosed by the surface.

$$\phi_E = \int_S \vec{E}.\vec{dS} = \frac{q_{net}}{\varepsilon_0}$$

Here S is known as Gaussian surface.

In a dielectric medium Gauss‟ law is given by

$$\phi_E = \int_S \vec{E}.\vec{dS} = \frac{q_{net}}{\varepsilon}$$

ε - Permittivity of the medium.

In terms of displacement vector Gauss‟ law is given by

$$\phi_E = \int_S \vec{D}.\vec{dS} = q_{net}$$

Notes:

- The charges enclosed by the surface may be point charges or continuous charge distribution.
- The net electric flux may be outward or inward depending upon the sign of charges.
- Electric flux is independent of shape & size of Gaussian surface.
- The Gaussian surface can be chosen to have a suitable geometrical shape for evaluation of flux.
- Limitation of Gauss‟ Law
  - (a) Since flux is a scalar quantity Gauss‟ law enables us to find the magnitude of electric field only.
  - (b) The applicability of the law is limited to situations with simple geometrical symmetry.

Gauss' Law in Differential form

Gauss‟ law is given by

$$\int_S \vec{E}.\vec{dS} = \frac{q_{net}}{\varepsilon}$$

For a charge distribution

$$q_{net} = \int_V \rho\, dV \qquad where\ \rho \rightarrow volumech\arg edensity$$

Using Gauss divergence theorem

$$\oint_S \vec{E}.\vec{dS} = \int_V \vec{\nabla}.\vec{E}\, dV$$

So

$$\frac{1}{\varepsilon_0}\int_V \rho\, dV = \int_V \vec{\nabla}.\vec{E}\, dV$$

Or $$\int_V (\vec{\nabla}.\vec{E} - \frac{\rho}{\varepsilon_0})\, dV = 0$$

$$\Rightarrow \quad \vec{\nabla}.\vec{E} - \frac{\rho}{\varepsilon_0} = 0$$

$$\Rightarrow \quad \vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

This is the differential form of Gauss‟ law.

Magnetic Intensity (H) and Magnetic Induction $(\vec{B})$

The magnetic intensity $(\vec{H})$ is related to the magnetic field induction $(\vec{B})$ by

$$(\vec{H}) = \frac{(\vec{B})}{\mu_0}$$

Unit: in SI system $(\vec{H})$ is in amp/m and $(\vec{B})$ in tesla.

Magnetic Flux (ϕ_m)

The magnetic flux over a given surface area S is given by

$$(\phi_m) = \int_S \vec{B}.\vec{dS} = \int_S BdS \cos\alpha$$

where $\alpha \rightarrow$ *angle between magnetic field* $\vec{B}$ *and normal to the surface*

Unit of flux: 1 weber in SI

1 maxwell in cgs(emu)

So 1T= 1 weber/m^2

1 gauss= 1maxwell/cm^2

Gauss' Law in magnetism

Since isolated magnetic pole does not exist, by analogy with Gauss‟ law of electrostatics, Gauss‟ law of magnetism is given by

$$\oint_S \vec{B}.\vec{dS} = 0$$

Using Gauss divergence theorem

$$\oint_S \vec{B}.\vec{dS} = \int_V \vec{\nabla}.\vec{B}\, dV = 0$$

$$\Rightarrow \quad \vec{\nabla}.\vec{B} = 0$$

This is the differential form of Gauss‟ law of magnetism.

Ampere's Circuital law

Statement:-The line integral of magnetic field along a closed loop is equal to $\mu_0$timesthenetelectriccurrentenclosedbyloop.

$$\oint_C \vec{B}.\vec{dl} = \mu_0 I$$

Where $I \rightarrow$ *net current enclosed bythe loop*

$C \rightarrow$*closed path enclosing the current (called ampere loop).*

In terms of magnetic intensity

$$\oint_C \vec{H}.\vec{dl} = I$$

Ampere's Law in Differential form

Ampere"s law is

$$\oint_C \vec{B}.\vec{dl} = \mu_0 I \text{ ——————————— (i)}$$

Using Stoke"s theorem, we have

$$\oint_C \vec{B}.\vec{dl} = \int_S (\nabla \times \vec{B}).\vec{ds} \text{ ——————————— (ii)}$$

In terms of current density J

$$\mu_0 I = \mu_0 \int_S \vec{J}.\vec{ds} \text{ ——————————— (iii)}$$

Using (ii) and (iii) in equation (i) we have

$$\int_S (\nabla \times \vec{B}).\vec{ds} = \mu_o \int_S \vec{J}.\vec{ds} = \int_S (\mu_o \vec{J}).\vec{ds}$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

This is Ampere"s circuital law in differential form.

Faraday's Law of electromagnetic induction

Statement :-The emf induced in a conducting loop is equal to the negative of rate of change of magnetic flux through the surface enclosed by the loop.

$$\varepsilon = -\frac{\partial \phi_m}{\partial t} \text{———————(i)}$$

The induced emf is the line integral of electric field along the loop.

$$\varepsilon = \oint_C \vec{E}.\vec{dl}$$

The magnetic flux is

$$\phi_m = \int_S \vec{B}.\vec{ds}$$

So from the above

$$\oint_C \vec{E}.\vec{dl} = -\frac{\partial}{\partial t}\int_S \vec{B}.\vec{ds}$$

This is Faraday"s law of electromagnetic induction in terms of $\vec{E}$ and $\vec{B}$

Differential form of Faraday's Law

Now using Stokes" theorem

$$\oint_C \vec{E}.\vec{dl} = \int_S (\nabla\times \vec{E}).\vec{ds}$$

But $\oint_C \vec{E}.\vec{dl} = -\frac{\partial \phi_m}{\partial t} = -\frac{\partial}{\partial t}\int_S \vec{B}.\vec{ds}$

From above two equations

$$\int_S (\vec{\nabla}\times \vec{E}).\vec{ds} = -\frac{\partial}{\partial t}\int_S \vec{B}.\vec{ds} = -\int_S \frac{\partial \vec{B}}{\partial t}.\vec{ds}$$

Or $\int_S \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right).\vec{ds} = 0$

$$\Rightarrow \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

This is differential form of Faraday‟s law electromagnetic induction.

Equation of Continuity

The electric current through a closed surface S is

$$I = \oint_S \vec{J}.\vec{ds}$$

Using Gauss divergence theorem

$$I = \oint_S \vec{J}.\vec{ds} = \int_V \vec{\nabla}.\vec{J}\, dV \text{ --------------------- (i)}$$

Where S is boundary of volume V.

Now $I = -\frac{\partial q}{\partial t} \rightarrow$ *rate of decrease of charge from the volume through surface S*

$$\Rightarrow \quad I = -\frac{\partial}{\partial t}\int_V \rho\, dV = -\int_V \frac{\partial \rho}{\partial t} dV \text{ ---------------------(ii)}$$

From (i) and (ii)

$$\int_V \vec{\nabla}.\vec{J}\, dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

$$\Rightarrow \quad \int_V \left(\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t}\right) dV = 0$$

$$\Rightarrow \quad \vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

This is equation of continuity.

Displacement Current

Maxwell associated a current (known as displacement current) with the time varying electric field.

A parallel plate capacitor connected to a cell is considered. During charging field $\vec{E}$ between varies.

Let $q \rightarrow$ instantaneous charge on capacitor plates.

$A \rightarrow$ area of each plate

We know that the electric field between the capacitor plates is

$$E = \frac{q}{\varepsilon_0 A}$$

$$\Rightarrow \frac{dE}{dt} = \frac{1}{\varepsilon_0 A}\frac{dq}{dt}$$

$$\Rightarrow \varepsilon_0 A\frac{dE}{dt} = \frac{dq}{dt}$$

$$\Rightarrow I_d = \varepsilon_0 A\frac{dE}{dt}$$ where $I_d \rightarrow$ displacement current between the plates

I_d exists till $\vec{E}$ varies with time.

In general, whenever there is a time-varying electric field, a displacement current exists,

$$I_d = \varepsilon_0 \frac{\partial}{\partial} \int_S \vec{E}.\vec{ds} = \varepsilon_0 \frac{\partial \phi_E}{\partial t}$$

Where ϕ_E is electric flux.

Modification of Ampere's circuital law

Taking displacement current into account Ampere"s Circuital law is modified as

$$\oint_C \vec{B}.\vec{dl} = \mu_o\left(I + I_d\right)$$

This law is sometimes referred as Ampere- Maxwell law.

The corresponding differential form is given as,

$$\vec{\nabla}\times\vec{B}=\mu_o\left[\vec{J}+\varepsilon_o\frac{\partial\vec{E}}{\partial t}\right]$$

Or

$$\underset{\rightarrow}{\nabla}\times\underset{\rightarrow}{H}=\left[\underset{\rightarrow}{J}+\frac{\partial\underset{\rightarrow}{D}}{\partial t}\right]$$

By using

$$\varepsilon_0\vec{E}=D,\ \frac{B}{\mu_0}=H$$

Here $\varepsilon_0\frac{\partial\vec{E}}{\partial t}=J_d\rightarrow \textit{displacement current density}$

Distinction between displacement current and conduction current

Conduction current	Displacement current
(i) Due to actual flow of charge in conducting medium. (ii) It obeys ohm"s law. (iii) Depends upon V and R	(i) Exists in vacuum or any medium even in absence of free charge carriers. (ii) Does not obey ohm"s law. (iii) Depend upon c and $\frac{6E}{6t}$

Relative magnitudes of displacement current and conduction current

Let $E=E_0 Sin\omega t\rightarrow alternating\ \ field$

Then current density

$$J=\sigma E=\sigma E_0 Sin\omega t ---------(i)$$

Displacement current density

$$J_d=\varepsilon_0\frac{\partial E}{\partial t}=\varepsilon_0\frac{\partial}{\partial t}\left(E_0\ Sin\omega t\right)=\omega\varepsilon_0 E_0\ Cos\omega t ---------(ii)$$

Thus there is a phase difference of $\frac{\Pi}{2}$ between current density and displacement current density.

The ratio of their peak values

$$\frac{(J)_{max}}{(J_d)_{max}} = \frac{\sigma E_0}{\omega\varepsilon_0 E_0} = \frac{\sigma}{\omega\varepsilon_0}$$

It means this ratio depends upon frequency of alternating field.

Notes:

- For copper conductor the ratio is $\approx \frac{10^{19}}{\omega}$
- For f>10^{20} Hz, displacement current is dominant. So normal conductors behave as dielectric at extremely high frequencies.

Maxwell's Equations

The Maxwell"s electromagnetic equations are

$$\vec{\nabla}.\vec{D} = \rho \qquad \text{--------------(1)}$$

$$\vec{\nabla}.\vec{B} = 0 \qquad \text{-----------------(2)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{------------------(3)}$$

$$\vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \qquad \text{-------------------(4)}$$

Notes:

- ➢ Equation (1) is the differential form of Gauss" law of electrostatics.
- ➢ Equation (2) is the differential form of Gauss" law of magnetism.
- ➢ Equation (3) is the differential form of Faraday"s law of electromagnetic induction.

- Equation (4) is the generalized form of Ampere‟s circuital law.
- Equations (2) and (3) have the same form in vacuum and medium. They are also unaffected by the presence of free charges or currents. They are usually called the constraint equation for electric and magnetic fields.
- Equations (1) and (4) depend upon the presence of free charges and currents and also the medium.
- Equations (1) and (2) are called steady state equations as they do not involve time dependent fields.

<u>Maxwell's Equations in terms of E and B</u>

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{--------------(1)}$$

$$\vec{\nabla}.\vec{B} = 0 \quad \text{-----------------(2)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{------------------(3)}$$

$$\vec{\nabla}\times\vec{B} - \mu\varepsilon\frac{\partial \vec{E}}{\partial t} = \mu\vec{J} \quad \text{-------------------(4)}$$

In absence of charges

$$\vec{\nabla}.\vec{E} = 0 \quad \text{--------------(1)}$$

$$\vec{\nabla}.\vec{B} = 0 \quad \text{-----------------(2)}$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \text{------------------(3)}$$

$$\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} = 0 \quad \text{-------------------(4)}$$

<u>Maxwell's Equations in Integral Form</u>

$$\oint_S \vec{E}.d\vec{S} = \frac{1}{\varepsilon_0}\int_V \rho\, dV \quad \text{--------------(1)}$$

$$\oint_S \vec{B}.\vec{dS} = 0 \qquad \text{--------------(2)}$$

$$\oint_C \vec{E}.\vec{dl} = -\frac{\partial}{\partial t}\oint_S \vec{B}.\vec{dS} \qquad \text{--------------(3)}$$

$$\oint_C \vec{B}.\vec{dl} = \mu_0 \int_S (\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}).\vec{dS} \qquad \text{--------------(4)}$$

Physical Significance of Maxwell's Equation

(i) Maxwell equations incorporate all the laws of electromagnetism.

(ii) Maxwell equations lead to the existence of electromagnetic waves.

(iii) Maxwell equations are consistent with the special theory of relativity.

(iv) Maxwell equations are used to describe the classical electromagnetic field as well as the quantum theory of interaction of charged particles electromagnetic field.

(v) Maxwell equations provided a unified description of the electric and magnetic phenomena which were treated independently.

Electromagnetic Waves

Wave Equation of electromagnetic wave in free space

In vacuum, in absence of charges, Maxwell"s equations are

$$\vec{\nabla}.\vec{E} = 0 \qquad \text{--------------(1)}$$

$$\vec{\nabla}.\vec{B} = 0 \qquad \text{-----------------(2)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{-------------------(3)}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} \qquad \text{--------------------(4)}$$

Taking curl of equation (3)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

Using equation (4)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}) = -\varepsilon_0\mu_0\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \quad \vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E} = -\varepsilon_0\mu_0\frac{\partial^2 \vec{E}}{\partial t^2}$$

Since $\vec{\nabla}(\vec{\nabla}.\vec{E}) = 0$, $\quad \nabla^2\vec{E} = \varepsilon_0\mu_0\frac{\partial^2 \vec{E}}{\partial t^2}$

Taking $\quad \varepsilon_0\mu_0 = \frac{1}{c^2}$, *where* $\quad c \rightarrow$ *velocity of light*

We have $\quad \nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$

This is the wave equation for $\vec{E}$.

Now taking curl of equation (4)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

Using equation (3)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\frac{\partial}{\partial t}(-\frac{\partial \vec{B}}{\partial t}) = -\varepsilon_0\mu_0\frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\Rightarrow \quad \vec{\nabla}(\vec{\nabla}.\vec{B}) - \nabla^2\vec{B} = -\varepsilon_0\mu_0\frac{\partial^2 \vec{B}}{\partial t^2}$$

Since $\vec{\nabla}(\vec{\nabla}.\vec{B}) = 0, \quad \nabla^2 \vec{B} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}$

Taking $\varepsilon_0 \mu_0 = \frac{1}{c^2}$, *where* $c \rightarrow$ *velocity of light*

We have $\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$

This is the wave equation for $\vec{B}$.

The general wave equation in vacuum can be written as

$$\nabla^2 \vec{\Psi} = \frac{1}{c^2} \frac{\partial^2 \vec{\Psi}}{\partial t^2}$$

Where $\vec{\Psi} = \vec{E}$ *or* $\vec{B}$

For charge free non-conducting medium, the general equation will be

$$\nabla^2 \vec{\Psi} = \frac{1}{v^2} \frac{\partial^2 \vec{\Psi}}{\partial t^2}$$

$\varepsilon\mu = \frac{1}{v^2}$, *where* $v \rightarrow$ *velocity of light in medium* s

Magnetic Vector Potential

The vector potential in a vector field is defined as when the divergence of a vector field is zero the vector can be expressed as the curl of a potential called vector potential ($\vec{A}$).

We know that $\vec{\nabla}.\vec{B} = 0$ (Maxwell equation)

Then $\vec{B} = \vec{\nabla} \times \vec{A}$ (*as div. of curl of a vector is zero*)

The vector $\vec{A}$ is called magnetic vector potential. The vector $\vec{A}$ can be chosen arbitrarily as addition of a constant vector or gradient of a scalar do not change the result.

Scalar Potential

The scalar potential in a scalar field is defined as when the curl of a field is zero the vector can be expressed as the negative gradient of a potential called scalar potential (ϕ).

We have $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Maxwell"s equation 3)

Putting $\vec{B} = \vec{\nabla}\times\vec{A}$ in above we get

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$\Rightarrow \vec{\nabla}\times\left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right] = 0$$

We know that curl of grad of a scalar is zero. So we can write

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$$ *where ϕ is a scalar function called the scalar potential* **.**

So $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$

For a time independent field $\frac{\partial \vec{A}}{\partial t} = 0$; *so*

$\vec{E} = -\vec{\nabla}\phi$ *here $\phi \rightarrow$ electrostatic potential*

Wave equation in terms of scalar & vector potential

Let us consider the Maxwell"s equations,

In free space and absence of charge

$$\vec{\nabla}.\vec{E} = 0 \quad -------------(1)$$

$$\vec{\nabla}\times\vec{B} = -\mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} \quad -------------(2)$$

Writing $E = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$; $\vec{A} \rightarrow$ *vector potential*

We have

$$\vec{\nabla}.\vec{E} = \vec{\nabla}.\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$or\ \nabla^2\phi + \frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{A}\right) = 0$$

Using Lorentz gauge condition

$$\vec{\nabla}.\vec{A} = -\frac{1}{c^2}\frac{\partial^2\phi}{\partial^2 t} = 0$$

$$We\,have \quad \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial^2 t} = 0$$

This is the wave equation in terms of scalar potential.

$$Putting \quad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad in\ equation\ (2)\ we\ get$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \varepsilon_0\mu_0\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\Rightarrow \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A} = -\varepsilon_0\mu_0\frac{\partial}{\partial t}(\nabla\phi) - \varepsilon_0\mu_0\frac{\partial^2 A}{\partial t^2}$$

$$\Rightarrow \vec{\nabla}\left(\vec{\nabla}.\vec{A} + \varepsilon_0\mu_0\frac{\partial\phi}{\partial t}\right) = \nabla^2\vec{A} - \varepsilon_0\mu_0\frac{\partial^2 A}{\partial t^2}$$

The LHS vanishes by Lorentz gauge condition.

$$So \quad \nabla^2\vec{A} - \varepsilon_0\mu_0\frac{\partial^2 A}{\partial t^2} = 0$$

This is the wave equation in terms of vector potential.

Lorentz gauge potential

$$\nabla^2.\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \qquad (Lorentz\ gauge\ condition)$$

$$\vec{\nabla}.\vec{A} = 0 \quad (Coulomb\ gauge\ condition)$$

Transverse nature of elecromagnetic wave

The plane wave solution of wave equation for $\vec{E}$ *and* $\vec{B}$ *are*

$$\vec{E}\left(\vec{r},t\right)=\hat{e}E_0e^{i(\vec{k}.\vec{r}-\omega t)} \quad ---------(1)$$

$$\vec{B}\left(\vec{r},t\right)=\hat{b}B_0e^{i(\vec{k}.\vec{r}-\omega t)} \quad ---------(2)$$

where $\hat{e},\hat{b}\rightarrow$ *unit vector along* $\vec{E}$ *and* $\vec{B}$ *respectively.*

E_0, $B_0\rightarrow$ *amplitudes of* $\vec{E}$ *and* $\vec{B}$ *respectively.*

$\vec{k}\rightarrow$ *wave propagation vector*

$\omega\rightarrow$ *angular frequency*

Using $\vec{\nabla}.\vec{E}=0$ in equation (1) we have

$$\vec{\nabla}.\left[\hat{e}E_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

$$\Rightarrow \hat{e}.\vec{\nabla}\left[E_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]+\vec{\nabla}.\hat{e}\left[E_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

$$\left\{as\ \vec{\nabla}.\left(V\vec{A}\right)=\left(\vec{\nabla}V\right).\vec{A}+V\vec{\nabla}.\vec{A}\right\}$$

Since $\hat{e}\rightarrow$ constant, $\vec{\nabla}.\hat{e}=0$

$$\hat{e}.\vec{\nabla}\left[E_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

$$or\quad \hat{e}.i\vec{k}E_0e^{i(\vec{k}.\vec{r}-\omega t)}=0$$

Since $E_0\neq 0,\quad e^{i(\vec{k}.\vec{r}-\omega t)}\neq 0,$

$$\hat{e}.\vec{k}=0 \quad -----------(3)$$

This shows the transverse nature of electric field.

Similarly, from Maxwell‟s equation

$$\vec{\nabla}.\vec{B}=0$$

We have

$$\vec{\nabla}.\left[\hat{b}B_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

$$\Rightarrow \hat{b}.\nabla\left[B_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]+\vec{\nabla}.\hat{b}\left[B_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

Since $\hat{b}\rightarrow$ constant, $\vec{\nabla}.\hat{b}=0$

$$\hat{b}.\nabla\left[B_0 e^{i(\vec{k}.\vec{r}-\omega t)}\right] = 0$$

$$or \quad \hat{b}.i\vec{k}B_0 e^{i(\vec{k}.\vec{r}-\omega t)} = 0$$

$$Since\, B_0 \neq 0, \quad e^{i(\vec{k}.\vec{r}-\omega t)} \neq 0,$$

$$\hat{b}.\vec{k} = 0 \quad ----------(4)$$

This shows the transverse nature of magnetic field.

Mutual orthogonality of **E**, **B** and **k**

Now from Maxwell‟s 3rd equation we have

$$\vec{\nabla}\times[\hat{e}E_0 e^{i(\vec{k}.\vec{r}-\omega t)}] = -\frac{\partial}{\partial t}[\hat{b}B_0 e^{i(\vec{k}.\vec{r}-\omega t)}] \quad -----------(5)$$

Using $\vec{\nabla}\times(\vec{A}V) = V(\vec{\nabla}\times\vec{A}) + (\vec{\nabla}V)\times\vec{A}$, we have

$$\vec{\nabla}\times[\hat{e}E_0 e^{i(\vec{k}.\vec{r}-\omega t)}] = E_0 e^{i(\vec{k}.\vec{r}-\omega t)}(\vec{\nabla}\times\hat{e}) + [\vec{\nabla}(E_0 e^{i(\vec{k}.\vec{r}-\omega t)})]\times\hat{e}$$

Since $\hat{e}$ is a constant unit vector, $(\vec{\nabla}\times\hat{e}) = 0$ and

we get $\vec{\nabla}(E_0 e^{i(\vec{k}.\vec{r}-\omega t)}) = E_0 i\vec{k}e^{i(\vec{k}.\vec{r}-\omega t)}$

$$\vec{\nabla}\times[\hat{e}E_0 e^{i(\vec{k}.\vec{r}-\omega t)}] = E_0 i\vec{k}e^{i(\vec{k}.\vec{r}-\omega t)}\times\hat{e} = E_0 i e^{i(\vec{k}.\vec{r}-\omega t)}(\vec{k}\times\hat{e})$$

Now $\frac{\partial}{\partial t}[\hat{b}B_0 e^{i(\vec{k}.\vec{r}-\omega t)}] = \hat{b}B_0 \frac{\partial}{\partial t}\{e^{i(\vec{k}.\vec{r}-\omega t)}\} = \hat{b}B_0 e^{i(\vec{k}.\vec{r}-\omega t)}(-i\omega)$

Then from eqn. 5

$$E_0 i e^{i(\vec{k}.\vec{r}-\omega t)}(\vec{k}\times\hat{e}) = -\hat{b}B_0 e^{i(\vec{k}.\vec{r}-\omega t)}(-i\omega) = \hat{b}B_0 i\omega e^{i(\vec{k}.\vec{r}-\omega t)}$$

$$\Rightarrow E_0(\vec{k}\times\hat{e}) = \hat{b}B_0\omega$$

$$\Rightarrow (\vec{k}\times\hat{e}) = \frac{B_0\omega}{E_0}\hat{b}$$

So $\hat{b}$ is perpendicular to both $\vec{k}$ *and* $\hat{e}$.

Thus electric field, magnetic field and propagation vector are mutually orthogonal.

Relative magnitudes of $\vec{E}$ *and* $\vec{B}$

Now taking magnitudes

$$\left|(\bar{k}\times\hat{e})\right| = \left|\frac{B_0\omega}{E_0}\hat{b}\right|$$

$$\Rightarrow k = \frac{\omega B_0}{E_0}$$

$$\Rightarrow \frac{E_0}{B_0} = \frac{\omega}{k} = c, \quad where\, c \rightarrow velocity\ of\ light$$

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Now using $B_0 = \mu_0 H_0$

$$\frac{E_0}{H_0} = \mu_0 c = \mu_0 \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{\mu_0}{\varepsilon_0} = Z_0$$

The quantity Z_0 has the dimension of electrical resistance and it is called the impedance of vacuum.

Phase relation between $\vec{E}$ *and* $\vec{B}$

In an electromagnetic wave electric and magnetic field are in phase.

Either electric field or magnetic field can be used to describe the electromagnetic wave.

Electromagnetic Energy Density

The electric energy per unit volume is

$$u_E = \frac{1}{2}\vec{E}.\vec{D} = \frac{1}{2}\varepsilon E^2 \quad --------(1)$$

The magnetic energy per unit volume is

$$u_B = \frac{1}{2}\vec{B}.\vec{H} = \frac{1}{2}\mu H^2 \quad --------(2)$$

The electromagnetic energy densityis givenby

$$u_{EM} = \frac{1}{2}(\varepsilon E^2 + \mu H^2)$$

Invacuum

$$u_{EM} = \frac{1}{2}(\varepsilon_0 E^2 + \mu_0 H^2)$$

Poynting Vector

The rate of energy transport per unit area in electromagnetic wave is described by a vector known as Poynting vector ($\vec{s}$) which is given as

$$\vec{S} = \vec{E} \times \vec{H} = \frac{\vec{E} \times \vec{B}}{\mu}$$

Poynting vector measures the flow of electromagnetic energy per unit time per unit area normal to the direction of wave propagation.

Unit of $\vec{S} \rightarrow 1 \frac{watt}{m^2}$ *in SI.*

Poynting Theorem

We have the Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ----------(i)$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad ----------(ii)$$

Taking dot product $\vec{H}$ with (i) and $\vec{E}$ with (ii) and subtracting

$$\vec{H}.\vec{\nabla} \times \vec{E} - \vec{E}.\vec{\nabla} \times \vec{H} = -\vec{H}.\frac{\partial \vec{B}}{\partial t} - \vec{E}.\frac{\partial \vec{D}}{\partial t} - \vec{E}.\vec{J} \quad -------(iii)$$

$$LHS = \vec{\nabla}.(\vec{E} \times \vec{H})$$

$$\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} = \vec{H}\cdot\frac{\partial(\mu \vec{H})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2}\right)$$

Similarly

$$\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} = \vec{E}\cdot\frac{\partial(\varepsilon \vec{E})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2}\right)$$

Then from (iii)

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) - \vec{E}\cdot\vec{J}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{S} = -\frac{\partial u_{EM}}{\partial t} - \vec{E}\cdot\vec{J}$$ as $\vec{E}\times\vec{H} = \vec{S}$ *and* $u_{EM} = \frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}$

This is sometimes called differential form of Poynting theorem.

Taking the volume integral of above

$$\int_V \vec{\nabla}\cdot\vec{S}\, dV = -\int_V \frac{\partial u_{EM}}{\partial t} dV - \int_V \vec{E}\cdot\vec{J}\, dV$$

Using Gauss divergence theorem to LHS we have

$$\oint_A \vec{S}\cdot d\vec{A} = -\int_V \frac{\partial u_{EM}}{\partial t} dV - \int_V \vec{E}\cdot\vec{J}\, dV$$

This represents Poynting theorem.

LHS of the equation $\rightarrow$ rate of flow of electromagnetic energy through the closed area enclosing the given volume

1st term of RHS $\rightarrow$ rate of change of electromagnetic energy in volume

1st term of RHS $\rightarrow$ work done by the electromagnetic field on the source of current.

Thus Poynting theorem is a statement of conservation of energy in electromagnetic field.

In absence of any source, J=o

$$then \quad \vec{\nabla}.\vec{S} + \frac{\partial u_{EM}}{\partial t} = 0$$

This is called equation of continuity of electromagnetic wave.

Poynting Vector & Intensity of electromagnetic wave

$$Since \ \vec{E} \ and \ \vec{H} \ are mutually \ perpendicular$$

$$\left|\vec{S}\right| = EH = \frac{EB}{\mu}$$

Here E and H are instantaneous values.

$$Since \ \vec{E} \ and \ \vec{H} \ are in \ phase$$

$$\frac{E}{H} = \frac{E_0}{H_0} = \mu c$$

$$or \quad S = \frac{E^2}{\mu c}$$

$$If \quad E = E_0 \sin \omega t, \ then average value of \ Poynting \ vector \ is$$

$$\langle S \rangle = \frac{\langle E_0^2 \sin^2 \omega t \rangle}{\mu c} = \frac{E_0^2}{2\mu c} \quad as \ \langle \sin^2 \omega t \rangle = \frac{1}{2}$$

$$\Rightarrow \quad \langle S \rangle = \frac{c\varepsilon E_0^2}{2} = c\varepsilon E_{rms}^2 \qquad as \ E_{rms} = \frac{E_0}{\sqrt{2}}$$

The average value of Poynting vector is the intensity (I) of the electromagnetic wave,

$$I = \langle S \rangle = c\varepsilon E^2_{rms}$$

MODULE-4

QUANTUM PHYSICS

Need for quantum physics: Historical overview

- About the end of 19^{th} century, classical physics had attained near perfection and successfully explains most of the observed physical phenomenon like motion of particles, rigid bodies, fluid dynamics etc under the influence of appropriate forces and leads to conclusion that there is no more development at conceptual level.

- But some new phenomenon observed during the last decade of 19^{th} century which is not explained by classical physics. Thus to explain their phenomena a new revolutionary concept was born which is known as Quantum physics developed by many outstanding physicists such as Planck, Einstein, Bohr, De Broglie, Heisenberg, Schrodinger, Born, Dirac and others.
- The quantum idea was 1^{st} introduced by Max Planck in 1900 to explain the observed energy distribution in the spectrum of black body radiation which is later used successfully by Einstein to explain Photoelectric Effect.
- Neils Bohr used a similar quantum concept to formulate a model for H-atom and explain the observed spectra successfully.
- The concept of dual nature of radiation was extended to Louis De Broglie who suggested that particles should have wave nature under certain circumstances. Thus the wave particle duality is regarded as basic ingredient of nature.
- The concept of Uncertainty Principle was introduced by Heisenberg which explains that all the physical properties of a system cannot even in principle, be determined simultaneously with unlimited accuracy.
- In classical physics, any system can be described in any deterministic way where as in quantum physics it is described by probabilistic description.
- Every system is characterized by a wave function ψ which describes the state of the system completely and developed by Max Born.
- The wave function satisfies a partial differential equation called Schrodinger equation formulated by Heisenberg.
- The relativistic quantum mechanics was formulated by P.A.M. Dirac to incorporate the effect of special theory of relativity in quantum mechanics.

In this way, this leads to the development of quantum field theory which successfully describes the interaction of radiation with matter and describes most of the phenomena in Atomic physics, nuclear physics, Particle physics, Solid state physics and Astrophysics.

The Quantum Physics deals with microscopic phenomena where as the classical physics deals with macroscopic bodies. All the laws of quantum physics reduces to the laws of classical physics under certain circumstances of quantum physics are a super set then classical physics is a subset.

i.e., $\lim_{n \to 0} Quantum\ physics = Classical\ physics$

$$\lim_{n \to \infty} Classical\ physics = Quantum\ physics$$

PARTICLE ASPECTS OF RADIATION

The particle nature of radiation includes/are exhibited in the phenomena of black body radiation, Photoelectric effect, Compton scattering and pair production.

Matter waves and De-Broglie Hypothesis

The waves associated with all material particles are called Matter waves.

According to De-Broglie hypothesis, the wavelength λ of matter wave associated with a moving particle of linear momentum P is given by

$$\boxed{\lambda = \frac{h}{P}}$$

or, $$\boxed{\lambda = \frac{h}{mv}}$$

For a non-relativistic free particle of kinetic energy E, we have

$$E = \frac{P^2}{2m}$$
$$\Rightarrow P = \sqrt{2mE}$$
$$\therefore \lambda = \frac{h}{\sqrt{2mE}}$$

If q=charge of a particle

m=mass of the particle

V=potential difference

Then, $\frac{P^2}{2m} = qv \Rightarrow P = \sqrt{2mqv}$

$$\therefore \boxed{\lambda = \frac{h}{\sqrt{2mqv}}}$$

If T=absolute temperature, then

$$\frac{P^2}{2m} = \frac{3}{2}kT \Rightarrow P = \sqrt{3mkT}$$
$$\boxed{\Rightarrow \lambda = \frac{h}{\sqrt{3mkT}}}$$

For a free relativistic particle,

$$E = \sqrt{P^2c^2 + m_0^2c^4}$$

$$\Rightarrow P = \frac{\sqrt{E^2 - m_0^2c^4}}{c}$$

$$\therefore \boxed{\lambda = \frac{hc}{\sqrt{E^2 - m_0^2c^4}}}$$

* Experimental confirmation of matter wave was demonstrated by Davision-Germer experiment.
* The wave nature of electron was demonstrated by division and Germer.

Transition from deterministic to probabilistic

In classical physics, the physical properties of a system can be specified exactly in principle. If the initial conditions of a system are known, its subsequent configurations can be determined by using the relevant laws of physics applicable to the system. Thus classical physics is deterministic in nature. But this deterministic description is inconsistent with observation. In quantum mechanics every physical system is characterized by a wave function which contains all the information‟s for the probabilistic description of a system. This probabilistic description is the basic characteristic of quantum physics and is achieved by the wave function.

Wave function

- The state function which contains all information‟s about a physical system is called wave function $\psi\left(\vec{r},t\right)$.
- It describes all information‟s like amplitude, frequency, wavelength etc.
- It is not a directly measurable quantity.
- It is a mathematical entity by which the observable physical properties of a system can be determined.

Characteristics

- It is a function of both space and time co-ordinate.
  i.e. $\psi\left(\vec{r},t\right) = \psi\left(x,y,z;t\right)$
- It is a complex function having both real and imaginary part.

- It is a single valued function of its arguments.

- The wave function ψ and its first derivative $\frac{\partial\Psi}{\partial x}$ are continuous at all places including boundaries.
- It is a square integrable function i.e. $\int|\psi|^2 dv = 1$.
- The quantity $|\psi|^2$ represents the probability density.
- It satisfies the Schrodinger"s equation.

Superposition principle

This principle states that "Any well behaved state of a system can be expressed as a linear superposition of different possible allowed states in which the system can exists."

If $\psi_1, \psi_2, \psi_3 ......$ be the wave functions representing the allowed states, then the state of the system can be expressed as

$$\psi = \psi_1 + \psi_2 + \psi_3 + \ ... + \psi_n = \sum c_n \psi_n$$

Probability density

The probability per unit volume of a system being in the state ψ is called probability density.

i.e. $\rho = |\psi|^2$

As the probability density is proportional to square of the wave function, so the wavefunction is called "probability amplitude".

The total probability is,

$$\int_v \rho dv = \int_v |\psi|^2 dv = 1$$

As the total probability is a dimensional quantity, so it has dimension $\left[L^{-3}\right]$ and the wavefunction has dimension $\left[L^{-3/2}\right]$.

- Dimension of 1-D wave function is $\left[L^{-1/2}\right]$.

- Dimension of 2-D wave function is $[L^{-1}]$.

Observables

The physical properties associated with the wave function provides the complete description of the system state or configuration are called observables.

Ex: energy, angular momentum, position etc.

Operators

The tools used for obtaining new function from a given function are called operators.

If $\hat{A}$ be an operator and f(x) be a function, then $\hat{A}f(x)=g(x)$; g(x)=new function

Ex: energy operator, momentum operator, velocity operator etc.

Physical Quantity	Operator
Energy-E	$i\hbar\frac{\partial}{\partial t}$
Momentum- $\vec{p}$	$-i\hbar\vec{\nabla}$
Potential Energy(V)	V
Kinetic Energy($p^2/2m$)	$\frac{-\hbar^2}{2m}\nabla^2$

Eigen States:

The number of definite allowed states for the system are called eigen states.

Eigen Values:

The set of allowed values of a physical quantity for a given system is called eigen values of the

quantity.

For any operator $\hat{A}$ having eigen values α_i corresponding to the eigen states ψ_i the eigen value

equation is $\boxed{\hat{A}\psi_i = \alpha_i\psi_i}$

Expectation Values:

The expectation values of a variable is the weighted average of the eigen values with their relative

probabilities.

If $q_1, q_2, q_3, \ldots\ldots$ are the eigen values of a physical quantity Q and they occur with probabilities

$p_1, p_2, p_3, \ldots.$ for a given state of the system then weighted average of Q is

$$\langle Q \rangle = \frac{p_1 q_1 + p_2 q_2 + \ldots.}{p_1 + p_2 + \ldots.} = \frac{\sum p_n q_n}{\sum p_n}$$

Since the total probability is 1, so $p_1 + p_2 + p_3 + \ldots\ldots = 1$

$$\therefore \langle Q \rangle = p_1 q_1 + p_2 q_2 + \ldots. = \sum p_n q_n$$

In general if A be a physical quantity, then

$$\langle A \rangle = \int \alpha |\psi|^2 \, dV$$

$$= \int \psi^* \alpha \psi \, dv$$

$$\boxed{\langle A \rangle = \int \psi^* \hat{A} \psi \, dV} \qquad \because \hat{A}\psi = \alpha\psi$$

For normalized wave function.

* For any function to be normalized is given as

$$\int \left|\psi\left(\vec{r},t\right)\right|^2 dV = 1$$

* The expectation value of energy,

$$\langle E\rangle = \int \psi^* \hat{H} \psi \, dV = \int \psi^* \left(i\hbar \frac{\partial}{\partial t} \right) \psi \, dV$$

$$= i\hbar \int \psi^* \frac{\partial \psi}{\partial t} \, dV$$

Schrodinger‟s Equation:-

The partial differential equation of a wave function involving the derivatives of space and time coordinates is called Schrodinger equation.

Time-dependent Schrodinger equation

Let the wave function be represented by

$$\psi\left(x,t\right) = Ae^{i(kx-\omega t)}$$

$$\left.\begin{aligned} &\Rightarrow \frac{\partial \psi}{\partial x} = ik\psi, \quad \frac{\partial \psi}{\partial t} = -i\omega\psi \\ &\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -k^2\psi \end{aligned}\right\} \quad (1)$$

The energy and momentum are given as

$$E = h\nu = \hbar\omega$$

$$p = \frac{h}{\lambda} = \hbar k$$

We have $$E = \frac{p^2}{2m}$$

$$\Rightarrow \hbar\omega = \frac{\hbar^2 k^2}{2m} \quad (2)$$

Using $eq^n(1)$ in $eq^n(2)$,

$$\Rightarrow \boxed{i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar}{2m}\frac{\partial^2\psi}{\partial x^2}} \qquad (3)$$

This is the time-dependent Schrodinger equation for a free particle in 1-dimension.

If the particle is in a potential V(x), then

$$E = \frac{p^2}{2m} + V$$

$$\Rightarrow \boxed{i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi} \qquad (4)$$

Similarly along Y and Z-axis is given as

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2} + V\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} + V\psi$$

<u>Time-dependent Schrodinger equation in 3-D:</u>

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi + V\psi$$

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi} \qquad (5)$$

<u>Time-independent Schrodinger equation:</u>

If the energy of the system does not change with time then

$E = \hbar\omega$ remains constant

Now from eq^n (1),

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar(-i\omega\psi) = \hbar\omega\psi = E\psi$$

$$\therefore E\psi = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi \qquad [\text{from } eq^n\ (5)]$$

$$\Rightarrow \boxed{\nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0}$$

This is time-independent Schrodinger equation in 3-D.

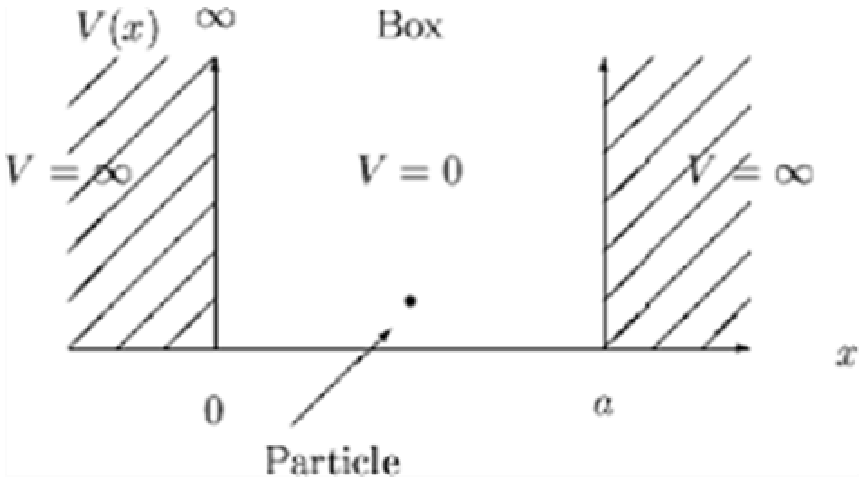
Particle in a one dimensional box:

The physical situation in which the potential between the boundary wall is zero and is infinite at the rigid walls is called one dimensional box or one dimensional infinite potential well.

The potential function for the situation is given as

$$V(x) = 0, 0 \le x < a$$

$$= \infty, x < 0 \text{ and } x > a$$



Now Schrodinger equation inside the well is given as

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}E\psi = 0$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + k^2\psi = 0 \qquad (1)$$

The general solution of eq^n(1) is given as

$$\psi(x) = C_1 e^{ikx} + C_2 e^{-ikx}$$

$$= A\sin kx + B\cos kx \qquad (2)$$

Where A and B are to be determined from the boundary condition at x=0 and x=a.

Thus eq^n(2) becomes, $0 = [A\sin kx + B\cos kx]_{x=0} = 0 + B$

$$\Rightarrow \boxed{B = 0}$$

Thus the wave function inside the well is given as

$$\boxed{\psi(x) = A\sin kx} \qquad (0 \le x \le a) \qquad (3)$$

<u>Energy eigen Values:-</u>

From eq^n(3),

$0 = [A\sin kx]_{x=0} = A\sin ka$ at x=a

$$\Rightarrow \boxed{ka = n\pi} \qquad n=1,2,3\ldots.$$

$$\Rightarrow \boxed{a = n\frac{\lambda}{2}}$$

Thus allowed bound states are possible for those energies for which the width of the potential well is equal to integral multiple of half wave length.

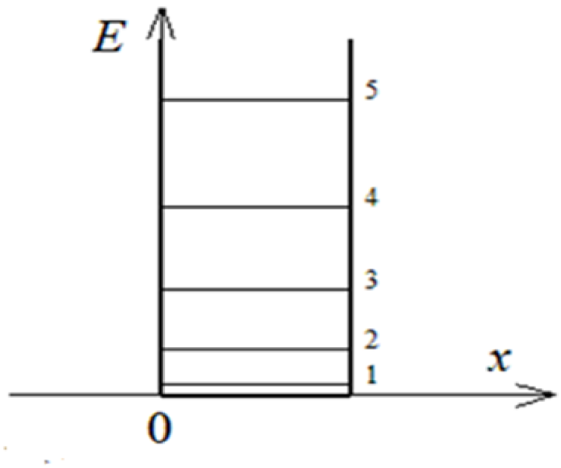
Since $k^2 = \frac{2mE}{\hbar^2}$

$$\Rightarrow k^2 a^2 = \frac{2mEa^2}{\hbar^2} = n^2\pi^2$$

$$\Rightarrow \boxed{E_n = \frac{\hbar^2\pi^2}{2ma^2} n^2}$$

Thus the energy of the particle in the infinite well is quantized.

- The ground state energy is $E_1 = \frac{\hbar^2\pi^2}{2ma^2}$ which is the minimum energy of the particle and is called the zero point energy.
- The energy of the higher allowed levels are multiple of E_1 and proportional to square of natural numbers.
- The energy levels are not equispaced.



<u>Eigen Functions</u>

The eigen functions of the allowed states can be obtained as

$$\int_{-\infty}^{+\infty} |\psi|^2 \, dx = 1$$

$$\Rightarrow \int_0^a |A|^2 \sin^2 kx dx = 1 \qquad (0 \le x \le a)$$

$$\Rightarrow |A|^2 \int_0^a \frac{1-\cos 2kx}{2} dx = 1$$

$$\Rightarrow \frac{2}{A^2} = [x]_0^a - \left[\frac{\sin 2kx}{2k}\right]_0^a = a$$

$$\Rightarrow A^2 = \frac{2}{a}$$

$$\Rightarrow A = \sqrt{\frac{2}{a}}$$

$$\therefore \boxed{\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x}$$

Thus the eigen function for each quantum state are obtained by

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi}{a} x$$

$$\psi_2(x) = \sqrt{\frac{2}{a}} \sin \frac{2\pi}{a} x$$

$$\psi_3(x) = \sqrt{\frac{2}{a}} \sin \frac{3\pi}{a} x \text{ etc.}$$

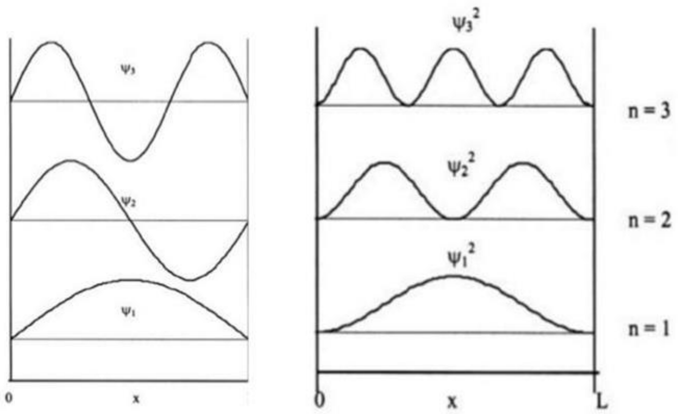


Fig. 3.1. The wave functions for the One-dimensional particle-in-a-box

Fig. 3.2. The probability densities in One-dimensional particle-in-a-box

LASER(MODULE V)

Laser' is an acronym for 'light amplification by the stimulation emission of radiation'. Its theoretical basis was postulated by Albert Einstein. The first tooth exposed to laser light was in 1960. Lasers can be applied to almost any clinical situation.

Lasers are light beams that are powerful enough to travel miles into the sky and cut through lumps of metal. Although they seem like a recent invention, they have been with us for half a century. The first practical laser was built by Theodore H. Maiman at Hughes Research Laboratories in 1960. At the time, lasers were an example of cutting-edge technology. Today, we have lasers at our homes, offices and shopping centres. Whether or not we realise it, all of us use lasers all day. But how many of us understand what they are and how they work?

How does a laser work?

The output of a laser is a coherent electromagnetic field. In a coherent beam of electromagnetic energy, all the waves have the same frequency and phase.

A basic laser consists of a chamber known as the cavity which is designed to reflect infrared, visible or ultraviolet waves so that they reinforce each other. The cavity can contain either solids, liquids or gases. The choice of the cavity material determines the wavelength of the output. Mirrors are placed at each end of the cavity. One of the mirrors is totally reflective, not allowing any of the energy to pass through them. The other mirror is partially reflective, allowing 5% percent of the energy to pass through them. Through a process known as pumping, energy is introduced into the cavity through an external source.

Due to pumping activity, an electromagnetic field appears inside the laser cavity at the natural frequency of the atoms of the material that fills the cavity. The waves are reflected back and forth between the mirrors. The length of the cavity is such that the reflected waves reinforce each other. The electromagnetic waves in phase with each other emerge from the end of the cavity having a partially reflective mirror. The output is a continuous beam, or a series of brief, intense pulses.

Characteristics of Lasers

We can separate the characteristics of laser beam into four major categories as:

- Superior Monochromatism
- Superior Directivity
- Superior Coherence
- High Output

Using these characteristics of lasers, they are applied in various fields such as optical communication and defence. In the next section, let us look at the various applications of lasers.

Uses of Laser

When lasers were first invented, they were called “a solution looking for a problem”. Since then they have become ubiquitous finding utility in various applications of modern society ranging from consumer electronics to the military.

Tools

- Cutting tools that employ CO2 lasers are widely used in industries. They are precise, easy-to-automate and don’t need sharpening, unlike knives.
- We use robot-guided lasers to cut pieces of cloth to make things such as denim jeans than using our bare hands. They are faster, more accurate and improve efficiency and productivity.
- The same precision is of utmost importance in the field of medicine. Doctors use lasers for everything from blasting cancerous tumours to correcting defective eyesight.

Communication

- The barcode scanner in a grocery store uses a laser to convert a printed barcode into a number that a checkout computer can understand.
- Every time you play a CD or a DVD, a semiconductor laser beam bounces off the spinning disc to convert its printed pattern of data into numbers; a computer chip converts these numbers into movies, music, and sound.

- Lasers are used in fibre optic cables and a technology known as photonics which uses photons of light to communicate.

Defence

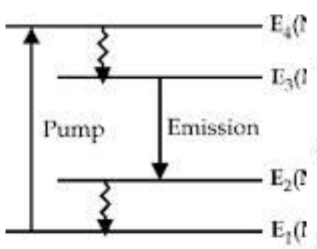
The military uses laser guided weapons and missiles.

Difference between a Flashlight and Laser

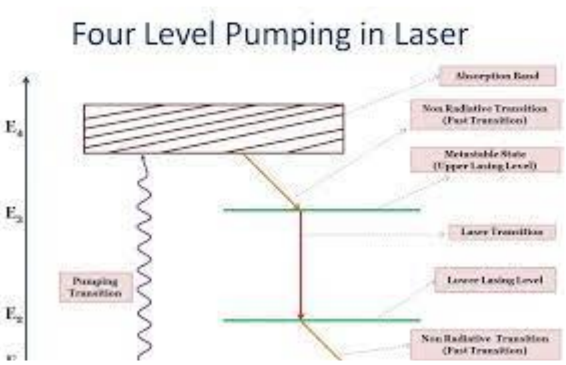
Flash Light	**Laser Light**
Flash light produces a white light which is a mixture of different colours of different frequencies	Laser produces a monochromatic light of single colour and frequency
Flash light spreads out through a lens into a short fuzzy cone	A laser shoots a much tighter, narrower beam over a much longer distance
Light waves in a flashlight beam are all jumbled up (the crests of some beams mixed with the troughs of others.)	Light waves in a laser beam are aligned (the crest of every wave is lined up with the crest of every other wave.)

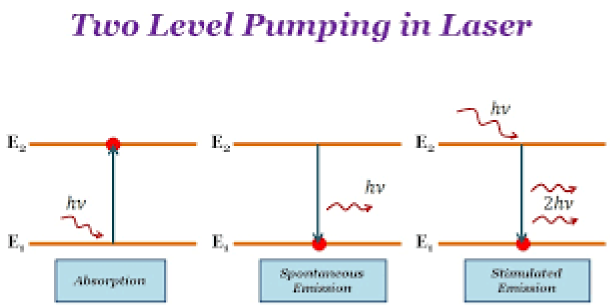
PUMPING

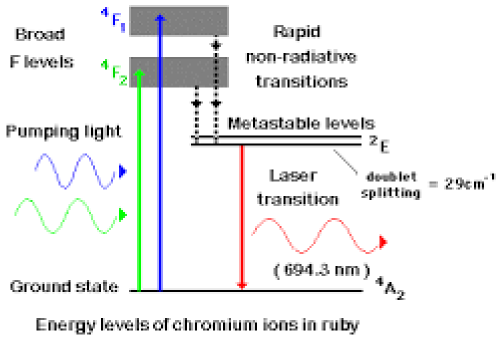
Depending on the laser type, pumping can be achieved through various methods, including optical pumping, electrical pumping, and chemical pumping. Regardless of the pumping method used, the key to achieving laser action is to produce a population inversion in the gain medium.



(a)



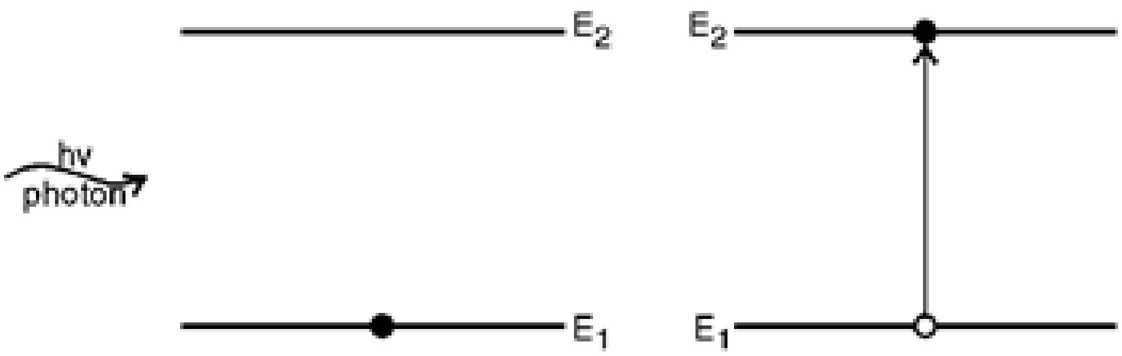




Einstein coefficients

Einstein Coefficient Relation derivation and discussion: Einstein showed the interaction of radiation with the matter with the help of three processes called stimulated absorption, spontaneous emission, and stimulated emission. He showed in 1917 that for a proper description of radiation with matter, the process of stimulated emission is essential. Let us first derive the Einstein coefficient relation on the basis of the above theory:

Let R_1 be the rate of absorption of light by E_1 -> E_2 transitions by the process called ~~stimulated absorption~~



Stimulated Absorption

This rate of absorption R_1 is proportional to the number of atoms N_1 per unit volume in the ground state and proportional to the energy density E of radiation

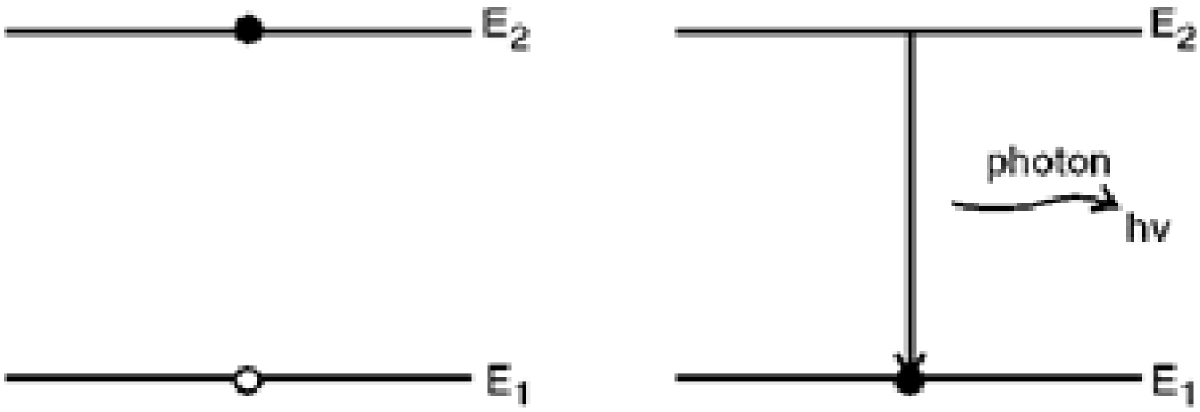
That is $R_1 \propto N_1 E$

Or $$R_1 = B_{12} N_1 E \quad (1)$$

Where B_{12} is known as the Einstein's coefficient of stimulated absorption and it represents the probability of absorption of radiation. Energy density e is defined as the incident energy on an atom as per unit volume in a state.

Einstein Coefficient for S*pontaneous Emission:*

Now atoms in the higher energy level E_2 can fall to the ground state E_1 automatically after 10^{-8} sec by the process called spontaneous emission



Spontaneous Emission

The rate R_2 of spontaneous emission E_2-> E_1 is independent of energy density E of the radiation field.

R_2 is proportional to number of atoms N_2 in the excited state E_2 thus

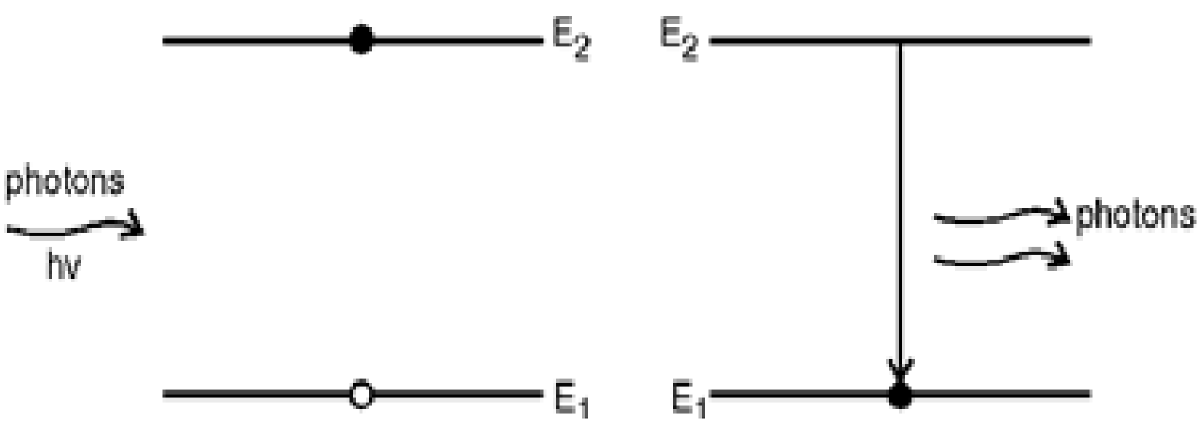
$R_2 \infty N_2$

$R_2 = A_{21} N_2$ (2)

Where A_{21} is known as Einstein's coefficient for spontaneous emission and it represents the probability of spontaneous emission.

Einstein Coefficient for *Stimulated Emission:*

Atoms can also fall back to the ground state E_1 under the influence of the electromagnetic field of an incident photon of energy E_2-E_1 =hv by the process called stimulated emission (Refer below Figure):



Stimulated Emission

Rate R_3 for stimulated emission E_2-> E_1 is proportional to energy density E of the radiation field and proportional to the number of atoms N_2 in the excited state,thus

$R_3 \alpha N_2 E$

Or $R_3=B_{21}N_2 E$ (3)

Where B_{21} is known as the Einstein coefficient for stimulated emission and it represents the probability of stimulated emission.

Einstein Coefficient *Relation Derivation:*

In steady-state (at thermal equilibrium), the two emission rates (spontaneous and stimulated) must balance the rate of absorption.

Thus $R_1=R_2+R_3$

Using equations (1,2, and 3) ,we get

$N_1B_{12}E=N_2A_{21}+N_2B_{21}E$

Or $N_1B_{12}E - N_2B_{21}E=N_2A_{21}$

Or $(N_1B_{12}-N_2B_{21})\ E = N_2A_{21}$

Or $E = N_2A_{21}/N_1B_{12}-N_2B_{21}$

$= N_2A_{21}/N_2B_{21}[N_1B_{12}/N_2B_{21} -1]$

[by taking out common $N_2B_{21}$from the denominator]

Or $E=A_{21}/B_{21}\ \{1/N_1/N_2(B_{12}/B_{21}-1))$ (4)

Einstein proved thermodynamically, that the probability of stimulated absorption is equal to the probability of stimulated emission. thus

$B_{12}=B_{21}$

Then equation(4) becomes

$E=A_{21}/B_{21}(1/N_1/N_2-1)$ (5)

From Boltzman's distribution law, the ratio of populations of two levels at temperature T is expressed as

$N_1/N_2=e^{(E_2-E_1)/KT}$

$N_1/N_2=e^{hv/KT}$

Where K is the Boltzman's constant and h is Planck's constant.

Substituting value of N_1/N_2in equation (5) we get

$E= A_{21}/B_{21}(1/e^{hv/KT}-1)$ (6)

Now according to Planck's radiation law, the energy density of the black body radiation of frequency v at temperature T is given as

$E = \quad 8\pi h v^3/c^3(1/e^{hv/KT})$ (7)

By comparing equations (6 and 7),we get

$A_{21}/B_{21}=8\pi h v^3/c^3$

This is the relation between Einstein’s coefficients in laser.

Significance of Einstein coefficient relation: This shows that the ratio of Einstein’s coefficient of spontaneous emission to the Einstein’s coefficient of stimulated absorption is proportional to the cube of frequency v. It means that at thermal equilibrium, the probability of spontaneous emission increases rapidly with the energy difference between two states.

Construction and working of Ruby laser

Construction

Ruby is a crystal of aluminum oxide (Al2O3) in which some of the aluminum ions (Al^3+) are replaced by chromium ions (Cr^3+). This is done by doping small amounts of chromium oxide (Cr_2O_3) in the melt of purified Al_2O_3.

These chromium ions give the crystal a pink or red color depending upon the concentration of chromium ions. Laser rods are prepared from a single crystal of pink ruby which contains 0.05% (by weight) chromium. Al_2O_3 does not participate in the laser action. It only acts as the host.

The ruby crystal is in the form of a cylinder. The length of ruby crystal is usually 2 cm to 30 cm and diameter 0.5 cm to 2 cm. As a very high temperature is produced during the operation of the laser, the rod is surrounded by liquid nitrogen to cool the apparatus.

Active medium or active center: Chromium ions act as active centers in ruby crystals. So it is the chromium ions that produce the laser.

Pumping source: A helical flash lamp filled with xenon is used as a pumping source. The ruby crystal is placed inside a xenon flash lamp. Thus, optical pumping is used to achieve population inversion in ruby laser.

Optical resonator system: The ends of ruby crystal are polished, grounded, and made flat. One of the ends is completely silvered while the other one is partially silvered to get the output. Thus the two polished ends act as an optical resonator system.

Working

Let us now discuss the working of ruby laser.

Ruby is a three-level laser system. Suppose there are three levels E1, E2, and (E3 & E4). E1 is the ground level, E2 is the ~~metastable level~~, E3 and E4 are the bands. E3 & E4 are considered as only one level because they are very closed to each other.

Pumping: The ruby crystal is placed inside a xenon flash lamp and the flash lamp is connected to a capacitor which discharges a few thousand joules of energy in a few milliseconds. A part of this energy is absorbed by chromium ions in the ground state. Thus optical pumping raises the chromium ions to energy levels inside the bands E3 and E4. This process is called stimulated absorption. The transition to bands E3 and E4 are caused by absorption of radiations corresponding to wavelengths approximately 6600 angstroms and 4000 angstroms respectively. The levels inside the bands E3 and E4 are also known as pumping levels.

Achievement of population inversion: Cr^{3+} ions in the excited state lose a part of their energy during interaction with crystal lattice and decay to the metastable state E2. Thus, the transition from excited states to metastable states is a non-radiative transition or in other words, there is no emission of photons. As E2 is a metastable state, chromium ions will stay there for a longer time. Hence, the number of chromium ions goes on increasing in the E2 state, while due to pumping, the number in the ground state E1 goes on decreasing. As a result, the number of chromium ions becomes more in an excited state(metastable state) as compared to ground state E1. Hence, the ~~population inversion~~ is achieved between states E2 and E1.

Achievement of laser: Few of the chromium ions will come back from E2 to E1 by the process of spontaneous emission by emitting photons. The wavelength of a photon is 6943 Å. This photon travels through the ruby rod and if it is moving in a direction parallel to the axis of the crystal, then it is reflected to and fro by the silvered ends of the ruby rod until it stimulates the other excited ions and causes it to emit a fresh photon in phase with the stimulating photon. Thus, the reflections will result in stimulated emission and it will result in the amplification of the stimulated emitting photons. This stimulated emission is the laser transition.

The two stimulated emitted photons will knock out more photons by stimulating the chromium ions and their total number will be four and so on. This process is repeated again and again, thus photons multiply. When the photon beam becomes sufficiently intense, then a very powerful and narrow beam of red light of wavelength 6943 Å emerges through the partially silvered end of the ruby crystal.

In the energy level diagram, E2 is the upper laser level and E1 is the lower laser level because the laser beam is achieved in between these levels. Thus, the ruby laser fits into the definition of three-level laser system.

Output: The output wavelength of the ruby laser is 6943 Å and output power is 10 raise to power 4 to 10 raised to power 6 watts and it is in the form of pulses.

Spiking in Ruby laser:

As we have discussed in the working of ruby laser that the terminus of laser action is the ground state E1 in ruby laser. Therefore it is difficult to maintain the population inversion. This will result in the depletion of the upper laser level E2 population (due to stimulated emission) more rapidly than it can be restored by the flashlight that is an optical pumping source. The laser emission is made up of spikes of high-intensity emissions. This phenomenon is called the spiking of the laser.

After the depletion of the E2 state, the laser action ceases for a few microseconds. Since the flash lamp is still active, it again pumps the ground state chromium ions to the upper level and again laser action begins. A series of such pulses is produced until the intensity of the flashlight has fallen to such a level that it can no longer rebuild the necessary population inversion. So the output laser will be in the form of pulse in ruby laser or in other words, it will not be continuous.

Drawbacks of ruby laser

1. As the terminus of laser action is the ground state, it is difficult to maintain the population inversion. This fact results in ruby laser's low efficiency.

2. The ruby laser requires a high power pumping source.

3. The laser output is not continuous but occurs in the form of pulses of microsecond duration.

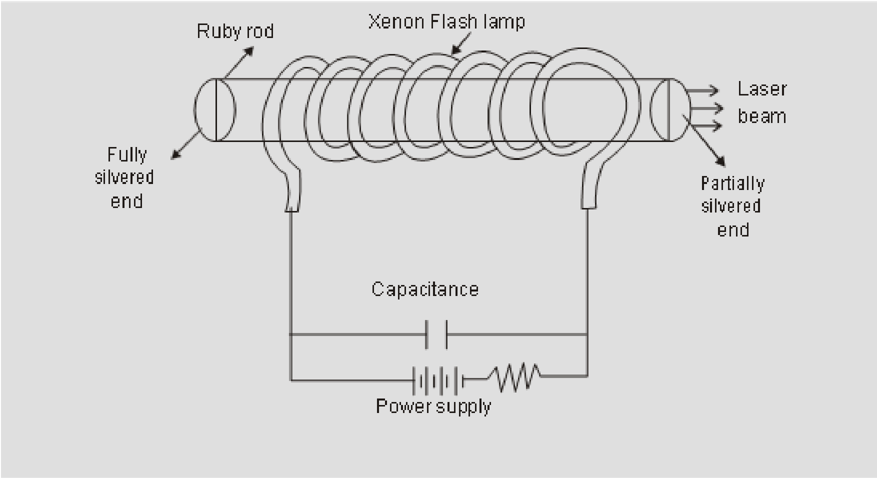
4. The defects due to crystalline imperfection are also present in ruby laser.

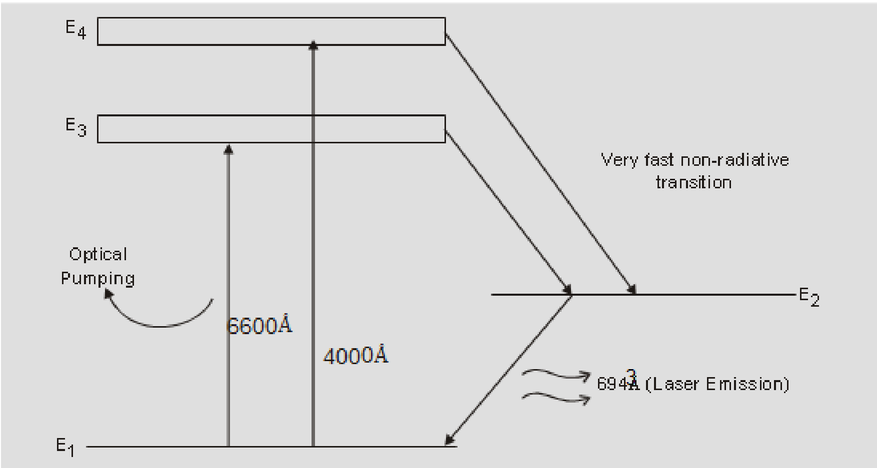
Uses of ruby laser

1. Ruby laser has very high output power of the order of $10^4 – 10^6$ watts. It has a wavelength of 6943 Angstroms.

2. Ruby lasers are used for holography, industrial cutting, and welding.

This is all about the construction and working of Ruby Laser. We have also learned about spiking in Ruby Laser, output, disadvantages, and uses of Ruby Laser. In my other articles, I have discussed the first gas laser that is construction and working of Helium-Neon laser.





Helium - Neon Laser (Introduction}

This was the first gas laser to be operated successfully.} It was invented by Ali Javan and his co-workers at Bell} Telephone Laboratories in the USA in 1961. Vivekananda College of Arts and Science (Women), Sirkali Its usual operation wavelength is 6328Å in the red portionof the visible spectrum.} He-Ne laser is a four-level l This consists of a mixture of helium and neon gases in a ratioof about 10:1.}Construction of a Helium Construction of a Helium - Neon laserNeon laser The setup consists of a long and narrow discharge tube of} length 80 cm and diameter of 1 cm. The pressure inside the tube is about 1mm of Hg.} The energy or pump source of the laser is provided by anVivekananda College of Arts and Science (Women), Sirkali} The energy or pump source of the laser is provided by anelectrical discharge of around 1000 volts through an anodeand cathode at each end of the glass tube} The optical cavity of the laser typically consists of a plane, highreflecting mirror at one end of the laser tube, and a partiallytransparent mirror of approximately 1% transmission at the other end.}aser}

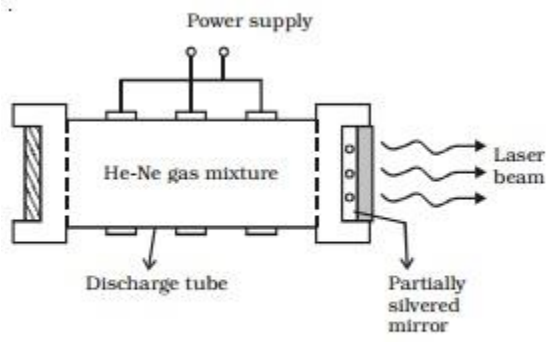


Fig He – Ne laser

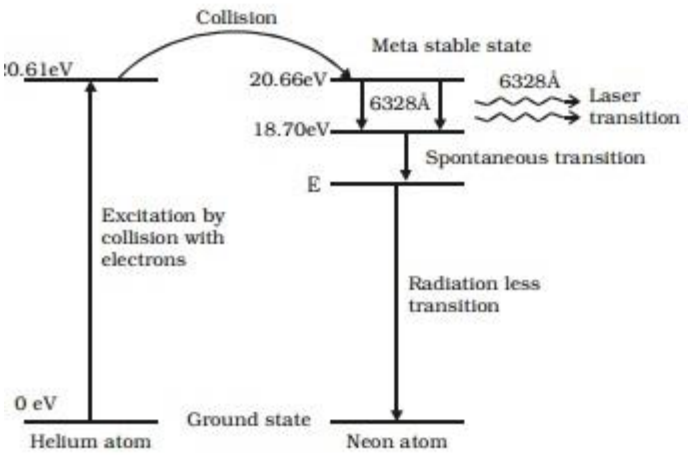


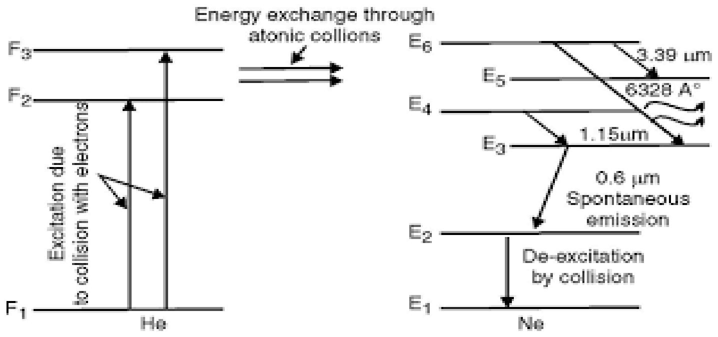
Fig Energy level diagram for He-Ne laser system

<u>WORKING OF HE-NE LASER</u>

Working of a Helium Working of a Helium - Neon laser Neon laser Electric discharge is passed through the gas. As electrons have a} smaller mass than ions, they acquire a higher velocity. The He atoms are more readily excitable than Neon as they are} in higher concentration. The role of He atoms is to assist in pumping Ne atoms to higher energy levels via inter atomic collisions} Vivekananda College of Arts and Science (Women), Sirkali Electrons collides with the He atoms, excite them to the} metastable states F2(19.81eV) and F3(20.61eV) stay for a sufficiently long time. The excited He atoms losses energy through collision with} unexcited Ne atoms, Ne atoms are excited to the metastablestates E4(18.7eV) & E6(20.66eV) which have nearly the same energy as the levels of F2 & F3 of He.

The probability of energy transfer from He atoms to Neon atoms is more as there are 10 He atoms to 1Neon atoms in the medium. Population inversion is achieved between E6} & E5, E6 &E3, E4 &E3. E6 E3 transition generates a laser beam of red colour of} Vivekananda College of Arts and Science (Women), Sirkali wavelength 6328Å. E4 E3 transition produces laser beam of wavelength 1.15µm} (not in visible region). E6 E5 transition results in a laser beam of 3.39µm (not in} visible region). E3 E2 transition generates incoherent light due to} spontaneous emission (~6000Å)

From the level E2 , the Ne atoms are brought back to the ground state through collisions with the walls. Also since E2 level is a metastable state , it can decrease the} population inversion by exciting atoms from E2 to E3 . Hence the tube is made narrow so that Ne atoms in level E2 deVivekananda College of Arts and Science (Women), Sirkali excite by collision with the walls of the tube. By a proper design of resonator , laser action in Ne is obtained} in the visible region (6328Å)



APPLICATION

The Narrow red beam of He-Ne laser is used in supermarkets to read bar codes. The He- Ne Laser is used in Holography in producing the 3D} images of objects. Vivekananda College of Arts and Science (Women), Sirkali He-Ne lasers have many industrial and scientific uses, and are} often used in laboratory demonstrations of optics.

ADVANTAGE /DIS ADVANTAGES

Following are the benefits or advantages of Helium - Neon Laser: He-Ne laser tube has very small length approximately from 10 to} 100cm. Cost of He-Ne laser is less from most of other lasers.} Construction of He-Ne laser is also not very complex.} He-Ne laser provide inherent safety due to lower power output} Vivekananda College of Arts and Science (Women), Sirkali Following are the drawbacks or disadvantages of Helium -} Neon Laser: He-Ne laser is low gain system / device.} High voltage requirement.} Escaping of gas from laser plasma tube}