Numerical Methods

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NUMERICAL METHOD

We use numerical method to find approximate solution of problems by numerical calculations with aid of calculator. For better accuracy we have to minimize the error.

Error = Exact value – Approximate value

Absolute error = modulus of error

Relative error = Absolute error / (Exact value)

Percentage error = 100 X Relative error

The error obtained due to rounding or chopping is called rounding error.

For example π = 3.14159 is approximated as 3.141 for chopping (deleting all decimal)

or 3.142 for rounding up to 3 decimal places.

Significant digit:

It is defined as the digits to the left of the first non-zero digit to fix the position of decimal point.

For example each of following numbers has 5 significant digits.

0.00025610, 25.610, 25601, 25610

Solution of Equations by Iteration:

Intermediate value Theorem: If a function f(x) is continuous in closed interval [a,b] and satisfies $f(a)f(b) < 0$ then there exists atleast one real root of the equation f(x) = 0 in open interval (a,b).

Algebraic equations are equations containing algebraic terms (different powers of x). For example $x^2-7x+6=0$

Transcendental equations are equations containing non-algebraic terms like trigonometric, exponential, logarithmic terms. For example $\sin x - e^x = 0$

A. **Fixed point iteration method for solving equation f(x) = 0**
   Procedure

   Step-I We rewrite the equation f(x) = 0 of the form x = h(x), x=g(x), x = D(x)

   We find the interval (a,b) containing the solution (called root).

   Step-II We choose that form say x = h(x) which satisfies $| h'(x) | < 1$ in interval (a,b) containing the solution (called root).

   Step-III We take $x_{n+1} = h(x_n)$ as the successive formula to find approximate solution (root) of the equation f(x) = 0

   Step-III Let $x=x_0$ be initial guess or initial approximation to the equation f(x) = 0

Then $x_1=h(x_1)$, $x_2=h(x_2)$, $x_3=h(x_3)$ and so on.We will continue this process till we get solution (root) of the equation $f(x) = 0$ up to desired accuracy.

Convergence condition for Fixed point iteration method

If x=a is a root of the equation $f(x) = 0$ and the root is in interval (a, b). The function $h'(x)$ and $h(x)$ defined by $x = h(x)$ Is continuous in (a,b) .Then the approximations $x_1=h(x_1)$, $x_2=h(x_2)$, $x_3=h(x_3)$ .... converges to the root x=a provided $|h'(x)| < 1$ in interval (a,b) containing the root for all values of x.

Problems

1. **Solve x^3 - sin x -1 =0 correct to two significant figures by fixed point iteration method correct up to 2 decimal places.**

Solution: $x^3 - \sin x - 1 = 0$.. (1)

Let $f(x) = x^3 - \sin x - 1$

$f(0) = -1$, $f(1)= - 0.8415$, $f(2)=6.0907$

As $f(1)f(2)< 0$ by Intermediate value Theorem the root of real root of the equation $f(x) = 0$ lies between 1 and 2

Let us rewrite the equation $f(x) = 0$ of the form $x = h(x)$

$x= (1 + \text{Sin } x)^{1/3} = h_1(x)$ and $x = \text{Sin}^{-1}(x^3 - 1)= h_2(x)$

We see that $|h_1'(x)| < 1$ in interval (1,2) containing the root for all values of x.

We use $x_{n+1}= (1 + \text{Sin } x_n)^{1/3}$ as the successive formula to find approximate solution (root) of the equation (1).

Let $x_0 =1.5$ be initial guess to the equation (1).

Then $x_1= (1 + \text{Sin } x_0)^{1/3} = (1 + \text{Sin } 1.5)^{1/3} = 1.963154$

$x_2= (1 + \text{Sin } x_1)^{1/3} = (1 + \text{Sin } 1.963154)^{1/3} = 1.460827$

$x_3= (1 + \text{Sin } x_2)^{1/3} = (1 + \text{Sin } 1.460827)^{1/3} = 1.440751$

$x_4= (1 + \text{Sin } x_3)^{1/3} = (1 + \text{Sin } 1.440751)^{1/3} = 1.441289$

which is the root of equation (1) correct to two decimal places.

Newton Raphson Method

Procedure

Step-I We find the interval (a,b) containing the solution (called root) of the equation $f(x) = 0$.

Step-II Let $x=x_0$ be initial guess or initial approximation to the equation $f(x) = 0$

Step-III We use $x_{n+1} = x_n$ - $[f(x_n) / f'(x_n)]$ as the successive formula to find approximate solution (root) of the equation f(x) = 0

Step-III Then x_1 , x_2 , x_3 .... and so on are calculated and we will continue this process till we get root of the equation f(x) = 0 up to desired accuracy.

2. Solve x - 2sin x - 3 = 0 correct to two significant figures by Newton Raphson method correct up to 5 significant digits.

Solution: x - 2sin x - 3 = 0 ....(2)

Let f(x) = x-2sin x - 3

f(0) = -3, f(1)= -2 - 2 Sin 1 , f(2)= -1 - 2 Sin 2 ,f(3)= - 2 Sin 3, f(4)= 1- 2 Sin 4

f(-2)= -5 + 2 Sin 2 ,f(-1)= -4 + 2 sin 1

As f(3)f(4)< 0 by Intermediate value Theorem the root of real root of the equation f(x) = 0 lies between 3 and 4

Let Let x_0 =4 be the initial guess to the equation (2).

Then $x_1 = x_0 - [f(x_0) / f'(x_0)]$ = 2- f(2)/ f′(2) = 3.09900

$x_2 = x_1 - [f(x_1) / f'(x_1)]$ = - 1.099- f(- 1.099)/ f′(- 1.099) = 3.10448

$x_3 = x_2 - [f(x_2) / f'(x_2)]$ = 3.10450

$x_4 = x_3 - [f(x_3) / f'(x_3)]$ = 3.10451

which is the root of equation (2) correct to five significant digits.

Secant Method

Procedure

Step-I We find the interval (a,b) containing the solution (called root) of the equation f(x) = 0 .

Step-II Let $x=x_0$ be initial guess or initial approximation to the equation f(x) = 0

Step-III We use $x_{n+1} = x_n - [(x_n - x_{n-1})f(x_n)] / [f(x_n) - f(x_{n-1})]$ as the successive formula to find approximate solution (root) of the equation f(x) = 0

Step-III Then x_1 , x_2 , x_3 .... and so on are calculated and we will continue this process till we get root of the equation f(x) = 0 up to desired accuracy.

3 . Solve Cos x = x e^x correct to two significant figures by Secant method correct up to 2 decimal places.

Solution: Cos x = x e^x .... (3)

Let f(x) = Cos x − x e^x

f(0) = 1, f(1)= Cos 1 – e = - 2 .178

As f(0)f(1)< 0 by Intermediate value Theorem the root of real root of the equation f(x) = 0 lies between 0 and 1

Let Let $x_0 = 0$ and $x_1 = 1$ be two initial guesses to the equation (3).

Then

$$x_2 = x_1 - \frac{(x_1 - x_0)f(x_1)}{f(x_1) - f(x_0)} = 1 - \frac{(1-0)f(1)}{f(1) - f(0)} = 1 - \frac{2.178}{-3.178} = 0.31465$$

$f(x_2)$= f (0.31465)= Cos (0.31465) - 0.31465 $e^{0.31465}$ = 0.51987

$$x_3 = x_2 - \frac{(x_2 - x_1)f(x_2)}{f(x_2) - f(x_1)} = 0.31465 - \frac{(0.31465-1)f(0.31465)}{f(0.31465) - f(1)} = 0.44672$$

$$x_4 = x_3 - \frac{(x_3 - x_2)f(x_3)}{f(x_3) - f(x_2)} = 0.64748$$

$$x_5 = x_4 - \frac{(x_4 - x_3)f(x_4)}{f(x_4) - f(x_3)} = 0.44545$$

which is the root of equation (3) correct to two decimal places.

4. Solve x^4 - x - 7 = 0 correct to two significant figures by Newton- Raphson method correct up to 6 significant digits.

Solution: x^4 - x - 7 = 0 ....(4)

Let f(x) = x^4 - x - 7

f(0) = -7, f(1)= -7 , f(2)= 5

As f(1)f(2)< 0 by Intermediate value Theorem the root of real root of the equation f(x) = 0 lies between 1 and 2

Let Let x_0 = 1.5 be the initial guess to the equation (2).

Then x_1= x_0 - [$f(x_0)$ / $f'(x_0)$]   = 1.5 - f(1.5)/ f′(1.5) =   1.78541

x_2= x_1 - [$f(x_1)$ / $f'(x_1)$] = 1.7854- f1.7854)/ f′(1.7854) = 1.85876

x_3= x_2 - [$f(x_2)$ / $f'(x_2)$] = 1.85643

x_4= x_3 - [$f(x_3)$ / $f'(x_3)$]   = 1.85632

which is the root of equation (2) correct to 6S.

INTERPOLATION

Interpolation is the method of finding value of the dependent variable y at any point x using the following given data.

x	x_0	x_1	x_2	x_3	..	..	..	x_n
y	y_0	y_1	y_2	y_3	..	..	..	y_n

This means that for the function y = f(x) the known values at x = $x_0, x_1, x_2, \ldots\ldots, x_n$ are respectively

y = $y_0, y_1, y_2, \ldots\ldots, y_n$ and we want to find value of y at any point x.

For this purpose we fit a polynomial to these datas called interpolating polynomial. After getting the polynomial p(x) which is an approximation to f(x), we can find the value of y at any point x.

Finite difference operators

Let us take equispaced points $x_0, x_1, x_2, \ldots\ldots, x_n$

i.e. $x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

Forward difference operator $\Delta y_n = y_{n+1} - y_n$

Backward difference operator $\nabla y_n = y_n - y_{n-1}$

Central difference operator $\delta y_i = y_{i+1/2} - y_{i-½}$

Shift Operator $E\, y_i = y_{i+1}$

Newton's Forward difference Interpolation formula

Let us take the equi-spaced points x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

Then $\Delta y_n = y_{n+1} - y_n$ is called the first Forward difference

i.e. $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$ and so on.

$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$ is called the second Forward difference

i.e. $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$, $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ and so on.

Newton's Forward difference Interpolation formula is

$$Pn(x) = y_0 + p\Delta y_0 + [p(p-1)/2!]\Delta^2 y_0 + [p(p-1)(p-2)/3!]\Delta^3 y_0$$

$$+ \ldots\ldots\ldots + [p(p-1)(p-2)\ldots\ldots(p-n-1)/n!]\Delta^n y_0$$

Where $p = (x - x_0)/h$

Problems

5. **Using following data find the Newton's interpolating polynomial and also find the value of y at x=5**

x	0	10	20	30	40
y	7	18	32	48	85

Solution

Here $x_0 = 0, x_1 = 10, x_2 = 20, x_3 = 30, x_4 = 40,$

$x_1 - x_0 = 10 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$

The given data is equispaced.

As x= 5 lies between 0 and 10 and at the start of the table and data is equispaced, we have to use Newton's forward difference Interpolation.

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	**7**				
		11			
10	18		**03**		
		14		**02**	
20	32		05		**10**
		19		12	
30	51		17		
		36			
40	87				

Here $x_0 = 0$, $y_0 = 7$, $h = x_1 - x_0 = 10 - 0 = 10$

$\Delta y_0 = 11$, $\Delta^2 y_0 = 3$,

$\Delta^3 y_0 = 2$, $\Delta^4 y_0 = 10$

$p = (x - x_0)/h = (x - 0)/10 = 0.1x$

$$P_n(x) = y_0 + p\Delta y_0 + [p(p-1)/2!]\Delta^2 y_0 + [p(p-1)(p-2)/3!]\Delta^3 y_0$$

$$+ [p(p-1)(p-2)(p-3)/4!]\Delta^4 y_0$$

$= 7 + 0.1x (11) + [0.1x(0.1x - 1)/2!] (3) + [0.1x(0.1x - 1) (0.1x - 2)/3!] (2)$

$+ [0.1x(0.1x - 1) (0.1x - 2) (0.1x - 3)/4!] (10)$

$= 7 + 1.1x + (0.01x^2 - 0.1x)1.5 + (0.001x^3 - 0.03x^2 +0. 2x)/3$

$+ 0.416 (0.0001x^4 - 0.006x^3 +0. 11x^2 -0.6x)$

$P_n (x) = 0.0000416 x^4 - 0.0022 x^3 +0.05x^2 + 1.26 x +7$

Is the **Newton's** interpolating polynomial

To find the approximate value of y at x=5 we put x=5 in the interpolating polynomial to get

$y(5)=P_n (5) = 0.0000416 (5)^4 - 0.0022 (5)^3 +0.05(5)^2 + 1.26 (5) +7 = 14.301$

6. **Using following data find the Newton's interpolating polynomial and also find the value of y at x=24**

x	20	35	50	65	80
y	3	11	24	50	98

Solution

Here $x_0 = 20, x_1 = 35, x_2 = 50, x_3 = 65, x_4 = 80,$

$x_1 - x_0= 15 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$

The given data is equispaced.

As x= 24 lies between 20 and 35 and at the start of the table and data is equispaced, we have to use Newton's forward difference Interpolation.

Here $x_0 = 20$, $y_0 = 3$, $h= x_1 - x_0 = 35 - 20 = 15$

$\Delta y_0 = 8$, $\Delta^2 y_0 = 5$,

$\Delta_3 y_0 = 8$, $\Delta^4 y_0 = 1$

$p = (x - x_0)/h = (x - 20)/15 = 0.0666 x - 1.333333$

Forward difference table

x	y	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
20	**3**				
		8			
35	11		**05**		
		13		**08**	
50	24		13		0**1**
		26		9	
65	50		22		
		48			
80	98				

$P_n(x) = y_0 + p\,\Delta y_0 + [\,p(p-1)/2!\,]\,\Delta^2 y_0 + [\,p(p-1)(p-2)/3!\,]\,\Delta^3 y_0$

$+ [\,p(p-1)(p-2)(p-3)/4!\,]\,\Delta^4 y_0$

$= 3 + 8\,(0.0666\,x - 1.333333) + 5[(0.0666\,x - 1.333333)(0.0666\,x - 2.333333)/2!\,]$

$+ 8[\,(0.0666\,x - 1.333333)(0.0666\,x - 2.333333)(0.0666\,x - 3.333333)/3!\,]$

$+ [\,(0.0666\,x - 1.333333)(0.0666\,x - 2.333333)(0.0666\,x - 3.333333)(0.0666\,x - 4.333333)/4!\,]$

$= 3 + 0.53333333\,x - 10.666666 + 0.01111x^2 - 0.16666666\,x + 7.777777$

$+ [\,(0.5333333\,x - 10.66666)(0.0666\,x - 2.333333)(0.011111\,x - 0.5555555)\,]$

$+ [\,(0.0666\,x - 1.333333)(0.0666\,x - 2.333333)(0.011111\,x - 0.5555555)(0.01666\,x - 1.083333)]$

Is the **Newton's** interpolating polynomial

To find the approximate value of y at x = 24 we put x = 24 in the interpolating polynomial to get

$y(24) = P_n(24) = 3 + (0.53333333)24 - 10.666666 + 0.01111(24^2) - (0.16666666)24 + 7.777777$

$+ [\,(0.5333333(24) - 10.66666)(0.0666\,(24) - 2.333333)(0.011111\,(24) - 0.5555555)\,]$

$+ [\,(1.59999 - 1.333333)(1.59999 - 2.333333)(0.266666 - 0.5555555)(0.399999 - 1.083333)]$

Newton's Backward difference Interpolation formula

Let us take the equi-spaced points x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

Then $\nabla y_n = y_n - y_{n-1}$ is called the first backward difference

i.e. $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$ and so on.

$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$ is called the second backward difference

i.e. $\nabla^2 y_1 = \nabla y_1 - \nabla y_0$, $\nabla^2 y_2 = \nabla y_2 - \nabla y_1$ and so on.

Newton's backward difference Interpolation formula is

$Pn(x) = y_n + p\nabla y_n + [p(p+1)/2!]\nabla^2 y_n + [p(p+1)(p+2)/3!]\nabla^3 y_n$

$+ .... + [p(p+1)(p+2).......(p+n-1)/n!]\nabla^n y_n$

Where $p = (x - x_n)/h$

7. Using following data to find the value of y at x = 35

x	0	10	20	30	40
y	7	18	32	48	85

Solution :

Here $x_0 = 0$, $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, $x_4 = 40$,

$x_1 - x_0 = 10 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$

The given data is equispaced.

As x= 35 lies between 3 0 and 40 and at the end of the table and given data is equispaced ,we have to use Newton's Backward difference Interpolation.

Here $x = 35$, $x_n = 40$, $y_n = 87$, $h = x_1 - x_0 = 10-0 = 10$

$\nabla y_n = 36$, $\nabla^2 y_n = 17$,

$\nabla^3 y_n = 12$, $\nabla^4 y_n = 10$

$p = (x - x_n)/h = (35 - 40)/10 = -0.5$

Backward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	7				
		11			
10	18		03		
		14		02	
20	32		05		**10**
		19		**12**	
30	51		**17**		
		36			
40	**87**				

$P_n(x) = y_n + p \nabla y_n + [p(p+1)/2!] \nabla^2 y_n + [p(p+1)(p+2)/3!] \nabla^3 y_n$

$+ [p(p+1)(p+2)(p+3)/4!] \nabla^4 y_n$

$= 87 + (-0.5)(36) + (-0.5)(-0.5+1)(17)/2! + (-0.5)(-0.5+1)(-0.5+2)(12)/3!$

$+ (-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(10)/4!$

$= 87 - 18 - 0.25(8.5) - 0.25(18)/6 - 0.25(15)(2.5)/24$

$= 65.734375$

This is the approximate value of y at x=35

$y(35) = P_n(35) = 65.734375$

Inverse Interpolation

The process of finding the independent variable x for given values of f(x) is called Inverse Interpolation .

8. Solve ln x = 1.3 by inverse Interpolation using x= G(y) with G(1)=2.718 ,G(1.5)= 4.481 , G(2)= 7.387 ,G(2.5)= 12.179 and find value of x

Forward difference table

y	x	Δy	$\Delta^2 y$	$\Delta^3 y$
1	**2.718**			
		1.763		
1.5	4.481		**1.143**	
		2.906		**0.743**
2	7.387		1.886	
		4.792		
2.5	12.179			

Here $y_0 = 1$, $h = y_1 - y_0 = 1.5 - 1 = 0.5$

$x_0 = 2.718$, $\Delta x_0 = 1.763$, $\Delta^2 x_0 = 1.143$,

$\Delta^3 x_0 = 0.743$

$p = (y - y_0)/h = (1.3 - 1)/0.5 = 0.6$

Newton's Forward difference Interpolation formula is

$Pn(y) = x_0 + p\Delta x_0 + [p(p-1)/2!]\Delta^2 x_0 + [p(p-1)(p-2)/3!]\Delta^3 x_0$

$= 2.718 + 0.6(1.763) + 0.6(0.6-1)1.143/2 + 0.6(0.6-1)(0.6-2)0.743/6$

$= 3.680248$

Lagrange Interpolation (data may not be equispaced)

Lagrange Interpolation can be applied to arbitrary spaced data.

Linear interpolation is interpolation by the line through points (x_1,y_1) and (x_0,y_0)

Linear interpolation is $P_1(x) = l_0 y_0 + l_1 y_1$

Where $l_0 = (x - x_1)/(x_0 - x_1)$ and $l_1 = (x - x_0)/(x_1 - x_0)$

Quadratic Lagrange Interpolation is the Interpolation through three given points (x_2,y_2) , (x_1,y_1) and (x_0,y_0) given by the formula

$P_2(x)= l_0 y_0 + l_1 y_1 + l_2 y_2$

Where $l_0 = \frac{(x-x_2)(x-x_1)}{(x_0-x_2)(x_0-x_1)}$, $l_1 = \frac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)}$ $and\ l_2 = \frac{(x-x_1)(x-x_0)}{(x_2-x_1)(x_2-x_0)}$

9. Using quadratic Lagrange Interpolation find the Lagrange interpolating polynomial $P_2(x)$ and hence find value of y at x=2 Given y(0) = 15, y(1) = 48, y(5) = 85

Solution :

Here $x_0 = 0, x_1 = 1, x_2 = 5$ and $y_0 = 15, y_1 = 48, y_2 = 85$

$x_1 - x_0 = 1 \neq x_2 - x_1 = 4$

The given data is not equispaced.

$$l_0 = \frac{(x-x_2)(x-x_1)}{(x_0-x_2)(x_0-x_1)} = \frac{(x-5)(x-1)}{(0-5)(0-1)} = \frac{(x^2-6x+5)}{5}$$

$$l_1 = \frac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} = \frac{(x-5)(x-0)}{(1-5)(1-0)} = \frac{(x^2-5x)}{(-4)}$$

$$and\ l_2 = \frac{(x-x_1)(x-x_0)}{(x_2-x_1)(x_2-x_0)} = \frac{(x-1)(x-0)}{(5-1)(5-0)} = \frac{(x^2-x)}{20}$$

$$y = l_0 y_0 + l_1 y_1 + l_2 y_2 = \frac{(x^2-6x+5)}{5}15 + \frac{(x^2-5x)}{(-4)}48 + \frac{(x^2-x)}{20}85$$

$$= -4.75x^2 + 37.75x + 15$$

Which is the Lagrange interpolating polynomial $P_2(x)$

Hence at x=2 the value is $P_2(2) = -4.75(2^2)+37.75(2)+15 = 71.5$

General Lagrange Interpolation is the Interpolation through n given points (x_0,y_0), (x_1,y_1) , (x_2,y_2)...................... , (x_n,y_n) given by the formula

$P_n(x)= l_0 y_0 + l_1 y_1 + l_2 y_2 + \ldots\ldots\ldots + l_n y_n$

Where $l_0 = \frac{(x-x_n)\ldots\ldots\ldots(x-x_2)(x-x_1)}{(x_0-x_n)\ldots\ldots\ldots(x_0-x_2)(x_0-x_1)}$

$$l_1 = \frac{(x - x_n)\ldots\ldots\ldots\ldots(x - x_2)(x - x_0)}{(x_1 - x_n)\ldots\ldots\ldots\ldots(x_1 - x_2)(x_1 - x_0)}$$

$$l_2 = \frac{(x - x_n)\ldots\ldots\ldots\ldots(x - x_1)(x - x_0)}{(x_2 - x_n)\ldots\ldots\ldots\ldots(x_2 - x_1)(x_2 - x_0)}$$

....

....

$$and\, l_n = \frac{(x - x_{n-1})\ldots\ldots\ldots\ldots(x - x_1)(x - x_0)}{(x_n - x_{n-1})\ldots\ldots\ldots\ldots(x_2 - x_1)(x_2 - x_0)}$$

10. Using Lagrange Interpolation find the value of y at x=8

Given y(0) = 18, y(1) = 42, y(7) = 57 and y(9) = 90

Solution :

Here $x_0 = 0, x_1 = 1, x_2 = 7, x_3 = 9$ and $y_0 = 26, y_1 = 40, y_2 = 75, y_3 = 90$

$x_1 - x_0 = 1 \neq x_2 - x_1 = 6$

The given data is not equispaced.

$$l_0 = \frac{(x - x_3)(x - x_2)(x - x_1)}{(x_0 - x_3)(x_0 - x_2)(x_0 - x_1)} = \frac{(8-9)(8-7)(8-1)}{(0-9)(0-7)(0-1)} = \frac{-7}{-63} = \frac{1}{9}$$

$$l_1 = \frac{(x - x_3)(x - x_2)(x - x_0)}{(x_1 - x_3)(x_1 - x_2)(x_1 - x_0)} = \frac{(8-9)(8-7)(8-0)}{(1-9)(1-7)(1-0)} = \frac{(-8)}{(48)} = \frac{1}{6}$$

$$l_2 = \frac{(x - x_3)(x - x_1)(x - x_0)}{(x_2 - x_3)(x_2 - x_1)(x_2 - x_0)} = \frac{(8-9)(8-1)(8-0)}{(7-9)(7-1)(7-0)} = \frac{-56}{-84} = \frac{2}{3}$$

$$and\ l_3 = \frac{(x - x_2)(x - x_1)(x - x_0)}{(x_3 - x_2)(x_3 - x_1)(x_3 - x_0)} = \frac{(8-7)(8-1)(8-0)}{(9-7)(9-1)(9-0)} = \frac{56}{144} = \frac{7}{18}$$

$$y = l_0\, y_0 + l_1\, y_1 + l_2\, y_2 + l_3\, y_3 = \frac{1}{9}(18) + \frac{1}{6}(42) + \frac{2}{3}(57) + \frac{7}{18}(90)$$

$$= 2 + 7 + 38 + 35 = 82$$

Which is the value of y at x=8

Newton divided difference Interpolation (data may not be equispaced)

Newton divided difference Interpolation can be applied to arbitrary spaced data.

The first divided difference is $f[x_0, x_1] = (y_1 - y_0)/(x_1 - x_0)$

$f[x_1, x_2] = (y_2 - y_1)/(x_2 - x_1)$

The second divided difference is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

The third divided difference is

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

The nth divided difference is

$$f[x_0, x_1, x_2, x_3, \ldots\ldots, x_n] = \frac{f[x_1, x_2, x_3, \ldots\ldots, x_n] - f[x_0, x_1, x_2, \ldots\ldots, x_{n-1}]}{x_n - x_0}$$

Newton divided difference Interpolation formula is

$$Y = y_0 + (x-x_0)\, f[x_0, x_1] + (x-x_0)(x-x_1)\, f[x_0, x_1, x_2] + \ldots\ldots\ldots$$

$$+ (x-x_0)(x-x_1) \ldots\ldots\ldots (x-x_{n-1})\, f[x_0, x_1, x_2, \ldots\ldots\ldots, x_n]$$

Problems

11. Using following data find the Newton's divided difference interpolating polynomial and also find the value of y at x= 15

x	0	6	20	45
y	30	48	88	238

Newton's divided difference table

x	y	First divided difference	Second divided difference	Third divided difference
0	**30**			
		(48-30)/6=**3**		
6	48		(8-3)/11=**0.45**	
		(88-48)/5=8		(0.1 -0.45)/26 = **-0.0136**
11	88		(10-8)/20=0.1	
		(238-88)/15=10		
26	238			

$$Y = y_0 + (x-x_0)\ f[x_0, x_1] + (x-x_0)(x-x_1)\ f[x_0, x_1, x_2]$$

$$+ (x-x_0)(x-x_1)(x-x_2)\ f[x_0, x_1, x_2, x_3]$$

$$= 30 + 3x + x(x-6)(0.45) + x(x-6)(x-11)(-0.0136)$$

The value of y at x= 15

$= 30 + 3(15) + 15(9)(0.45) + 15(9)(4)(-0.0136) = 128.406$

NUMERICAL DIFFERENTIATION

When a function y = f(x) is unknown but its values are given at some points like (x_0, y_0), (x_1, y_1),

.... (x_n, y_n) or in form of a table, then we can differentiate using numerical differentiation.

Sometimes it is difficult to differentiate a composite or complicated function which can be done easily in less time and less number of steps by numerical differentiation.

We use following methods for numerical differentiation.

(i) Method based on finite difference operators

(ii) Method based on Interpolation

(i) Method based on finite difference operators

Newton's forward difference Interpolation formula is

$Pn(x) = y_0 + p\Delta y_0 + [p(p-1)/2!]\,\Delta^2 y_0 + [p(p-1)(p-2)/3!]\,\Delta^3 y_0 + \ldots\ldots\ldots\ldots$

where $p = (x - x_0)/h$

Newton's backward difference Interpolation formula is

$Pn(x) = y_n + p\nabla y_n + [p(p+1)/2!]\,\nabla^2 y_n + [p(p+1)(p+2)/3!]\,\nabla^3 y_n$

$+ \ldots\ldots\ldots\ldots + [p(p+1)(p+2)\ldots\ldots(p+n-1)/n!]\,\nabla^n y_n$

where $p = (x - x_n)/h$

Using forward difference the formula for numerical differentiation is

$y'(x_0) = (1/h)\,[\Delta y_0 - \Delta^2 y_0/2 + \Delta^3 y_0/3 + \ldots\ldots\ldots\ldots]$

$y''(x_0) = (1/h^2)\,[\Delta^2 y_0 - \Delta^3 y_0 + (11/12)\,\Delta^4 y_0 \ldots\ldots\ldots\ldots]$

Using backward difference the formula for numerical differentiation is

$y'(x_n) = (1/h)\,[\nabla y_n + \nabla^2 y_n/2 + \nabla^3 y_n/3 + \ldots\ldots\ldots\ldots]$

$y''(x_n) = (1/h^2)\,[\nabla^2 y_n + \nabla^3 y_n + (11/12)\,\nabla^4 y_n \ldots\ldots\ldots\ldots]$

If we consider the first term only the formula becomes

$y'(x_0) = (1/h)\,[\Delta y_0] = (y_1 - y_0)/h$

$y''(x_0) = (1/h^2)\,[\Delta^2 y_0] = (\Delta y_1 - \Delta y_0)/h^2$

$= [(y_2 - y_1) - (y_1 - y_0)]/h^2 = [y_2 - 2y_1 + y_0]/h^2$

12. Using following data find the first and second derivative of y at x=0

x	0	10	20	30	40
y	7	18	32	48	85

Solution

Here $x_0 = 0$, $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, $x_4 = 40$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	**7**				
		11			
10	18		**03**		
		14		**02**	
20	32		05		**10**
		19		12	
30	51		17		
		36			
40	87				

Here $x_0 = 0$, $y_0 = 7$, $h = x_1 - x_0 = 10-0 = 10$

$\Delta y_0 = 11$, $\Delta^2 y_0 = 3$,

$\Delta_3 y_0 = 2$, $\Delta^4 y_0 = 10$

$p = (x - x_0)/h = (4 - 0)/10 = 0.4$

$y'(x_0) = (1/h)\ [\ \Delta y_0 - \Delta^2 y_0/2 + \Delta^3 y_0/3 - \Delta^4 y_0/4 + \ldots\ldots\ldots\ldots\ ]$

$= 0.1\ [\ 11 - 3/2 + 2/3 - 10/4\ ] = 0.7666$

$y''(x_0) = (1/h^2)\ [\ \Delta^2 y_0 - \Delta^3 y_0 + (11/12)\,\Delta^4 y_0 \ldots\ldots\ ]$

$= (1/100)\ [\ 3 - 2 + (11/12)\ 10\ ] = 0.10166$

(ii) Method based on Interpolation

Linear Interpolation

$$y'(x_0) = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

Quadratic Interpolation

$y'(x_0) = (-3y_0 + 4y_1 - y_2)/(2h)$

$y'(x_1) = (y_2 - y_0)/(2h)$

$y'(x_2) = (y_0 - 4y_1 + 3y_2)/(2h)$

The second derivative is constant i.e. same at all points because of quadratic interpolation and the interpolating polynomial is of degree two. Hence we must have

$y''(x_0) = (y_0 - 2y_1 + y_2)/(2h)$

$y''(x_1) = (y_0 - 2y_1 + y_2)/(2h)$

$y''(x_2) = (y_0 - 2y_1 + y_2)/(2h)$

Problems

13. Using following data find the value of first and second derivatives of y at x=30

x	10	30	50
y	42	64	88

Solution

Here $x_0 = 10$, $x_1 = 30$, $x_2 = 50$, $h = x_1 - x_0 = 30 - 10 = 20$

$y_0 = 42$, $y_1 = 64$, $y_2 = 88$

Linear Interpolation

$$y'(x_0) = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{64-42}{30-10} = 1.1$$

Quadratic Interpolation

$y'(x_0) = (-3y_0 + 4y_1 - y_2)/(2h) = [-3(42) + 4(64) - 88]/40 = 1.05$

$y'(x_1) = (y_2 - y_0)/(2h) = (88 - 42)/40 = 1.15$

$y'(x_2) = (y_0 - 4y_1 + 3y_2)/(2h) = (42 - 256 + 264)/40 = 1.25$

$y''(x_0) = (y_0 - 2y_1 + y_2)/(2h) = (42 - 128 + 88)/40 = 0.05$

14. Using following data find the value of first and second derivatives of y at x=12

x	0	10	20	30	40
y	7	18	32	48	85

Solution

Here $x_0 = 0, x_1 = 10, x_2 = 20, x_3 = 30, x_4 = 40,$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	**7**				
		11			
10	18		**03**		
		14		**02**	
20	32		05		**10**
		19		12	
30	51		17		
		36			
40	87				

Here $x_0 = 0$, $y_0 = 7$, $h = x_1 - x_0 = 10-0 = 10$

$\Delta y_0 = 11$, $\Delta^2 y_0 = 3$,

$\Delta_3 y_0 = 2$, $\Delta^4 y_0 = 10$

$p = (x - x_0)/h = (x - 0)/10 = 0.1x$

$P_n(x) = y_0 + p\Delta y_0 + [p(p-1)/2!]\Delta^2 y_0 + [p(p-1)(p-2)/3!]\Delta^3 y_0$

$+ [p(p-1)(p-2)(p-3)/4!]\Delta^4 y_0$

$= 7 + 0.1x(11) + [0.1x(0.1x - 1)/2!](3) + [0.1x(0.1x - 1)(0.1x - 2)/3!](2)$

$+ [0.1x(0.1x - 1)(0.1x - 2)(0.1x - 3)/4!](10)$

$= 7 + 1.1x + (0.01x^2 - 0.1x)1.5 + (0.001x^3 - 0.03x^2 + 0.2x)/3$

$+ 0.416(0.0001x^4 - 0.006x^3 + 0.11x^2 - 0.6x)$

$y = P_n(x) = 0.0000416x^4 - 0.0022x^3 + 0.05x^2 + 1.26x + 7$ .... (1)

Differentiating (1) w.r. to x we get

$y' = 0.0001664x^3 - 0.0066x^2 + 0.1x + 1.26$ .... (2)

$y'(12) = 1.7971392$ at $x = 12$

Differentiating (2) w.r. to x we get

$y'' = 0.0004992\, x^2 - 0.0132\, x + 0.1$

$y''(12) = 0.0134848$ at $x = 12$

NUMERICAL INTEGRATION

Consider the integral $I = \int_a^b f(x)\, dx$

Where integrand f(x) is a given function and a, b are known which are end points of the interval [a, b]

Either f(x) is given or a table of values of f(x) are given.

Let us divide the interval [a, b] into n number of equal subintervals so that length of each subinterval is $h = (b - a)/n$

The end points of subintervals are $a = x_0, x_1, x_2, x_3, \ldots\ldots\ldots, x_n = b$

Trapezoidal Rule of integration

Let us approximate integrand f by a line segment in each subinterval. Then coordinate of end points of subintervals are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$. Then from x=a to x=b the area under curve of y = f(x) is approximately equal to sum of the areas of n trapezoids of each n subintervals.

So the integral $I = \int_a^b f(x)\, dx = (h/2)[y_0 + y_1] + (h/2)[y_1 + y_2] + (h/2)[y_2 + y_3]$

$+ \ldots\ldots\ldots + (h/2)[y_{n-1} + y_n]$

$= (h/2)[y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1} + y_n]$

$= (h/2)[y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1})]$

Which is called trapezoidal rule.

The error in trapezoidal rule is $-\frac{b-a}{12} h^2 f''(\theta)$ where $a < \theta < b$

Simpsons rule of Numerical integration (Simpsons 1/3rd rule)

Consider the integral $I = \int_a^b f(x)\, dx$

Where integrand f(x) is a given function and a, b are known which are end points of the interval [a, b]

Either f(x) is given or a table of values of f(x) are given.

Let us approximate integrand f by a line segment in each subinterval. Then coordinate of end points of subintervals are (x_0, y_0), (x_1, y_1), (x_2, y_2), ... ,(x_n, y_n).

We are taking two strips at a time Instead of taking one strip as in trapezoidal rule. For this reason the number of intervals in Simpsons rule of Numerical integration must be even.

The length of each subinterval is h = (b – a)/(2m)

The formula is

$$I = \int_a^b f(x)\,dx = (h/3)\,[\,y_0 + y_{2m} + 4(y_1 + y_3 + \dots\dots + y_{2m-1}) + 2(y_2 + y_4 + \dots\dots + y_{2m-2})\,]$$

The error in Simpson 1/3rd rule is $-\frac{b-a}{180} h^4 f^{iv}(\theta)$ where $a < \theta < b$

Simpsons rule of Numerical integration (Simpsons 3/8th rule)

Consider the integral $I = \int_a^b f(x)\,dx$

Where integrand f(x) is a given function and a, b are known which are end points of the interval [a, b]

Either f(x) is given or a table of values of f(x) are given.

We are taking three strips at a time Instead of taking one strip as in trapezoidal rule. For this reason the number of intervals in Simpsons 3/8th rule of Numerical integration must be multiple of 3.

The length of each subinterval is h = (b – a)/(3m)

The formula is

$$I = \int_a^b f(x)\,dx = (3h/8)\,[\,y_0 + y_{3m} + 3(y_1 + y_2 + y_4 + y_5 + \dots\dots + y_{3m-1}) + 2(y_3 + y_6 + \dots\dots + y_{3m-3})\,]$$

The error in Simpson 1/3rd rule is $-\frac{b-a}{80} h^4 f^{iv}(\theta)$ where $a < \theta < b$

15. Using Trapezoidal and Simpsons rule evaluate the following integral with number of subintervals n =6

$$\int_0^6 e^{(-x^2)}\,dx$$

Solution:

Here integrand $y = f(x) = \exp(-x^2)$

a=0, b=6, h= (b-a)/n= (6-0)/6=1

x	0	1	2	3	4	5	6
Y= $\exp(-x^2)$	1	e^{-1}	e^{-4}	e^{-9}	e^{-16}	e^{-25}	e^{-36}
	y_0	y_1	y_2	y_3	y_4	Y_5	y_6

(i) Using Trapezoidal rule

$I = (h/2)[\ y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots\ldots\ldots + y_{n-1})\ ]$

$= (1/2)[\ y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\ ]$

$= 0.5\ [\ 1 + e^{-36} + 2(e^{-1} + e^{-4} + e^{-9} + e^{-16} + e^{-25})\ ]$

(ii) Using Simpsons rule

$I = (h/3)\ [\ y_0 + y_{2m} + 4(y_1 + y_3 + \ldots\ldots\ldots\ldots + y_{2m-1}) + 2(\ y_2 + y_4 + \ldots\ldots\ldots\ldots + y_{2m-2})\ ]$

$= (h/3)\ [\ y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(\ y_2 + y_4\ )\ ]$

$= (1/3)\ [\ 1 + e^{-36} + 4\ (e^{-1} + e^{-9} + e^{-25}) + 2(e^{-4} + e^{-16}\ )\ ]$

(iii) Using Simpsons 3/8th rule

$I = (3h/8)\ [\ y_0 + y_{3m} + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{3m-1}) + 2(\ y_3 + y_6 + \ldots\ldots + y_{3m-3})\ ]$

$= (3h/8)\ [\ y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(\ y_3\ )\ ]$

$= (3/8)\ [\ 1 + e^{-36} + 3\ (e^{-1} + e^{-4} + e^{-16} + e^{-25}) + 2(e^{-9})\ ]$

16. Using Trapezoidal and Simpsons rule evaluate the following integral with number of subintervals n =8 and compare the result

$$\int_0^{0.8} \frac{dx}{4 + x^2}$$

Solution:

Here integrand $y = f(x) = (4 + x^2)^{-1}$

$a=0$, $b= 0.8$, $h= (b-a)/n= (0.8-0)/8= 0.1$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Y= $(4 + x^2)^{-1}$	1/4	1/4.01	1/4.04	1/4.09	1/4.16	1/4.25	1/4.36	1/4.49	1/4.64
	y_0	y_1	y_2	y_3	y_4	Y_5	y_6	Y_7	y_8

(i) Using Trapezoidal rule

$$I = (h/2)[y_0 + y_n + 2(y_1 + y_2 + y_3 + \dots\dots\dots\dots + y_{n-1})]$$

$$= (0.1/2)[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

= 0.05 [0.25+ 1/4.64+ 2(1/4.01+ 1/4.04+ 1/4.09+1/4.16+ 1/4.25+ 1/4.36+1/4.49)]

(ii) Using Simpsons rule

$$I = (h/3) [y_0 + y_{2m} + 4(y_1 + y_3 + \dots\dots\dots + y_{2m-1}) + 2(y_2 + y_4 + \dots\dots\dots + y_{2m-2})]$$

$$= (h/3) [y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

= (0.1/3) [0.25+ 1/4.64+ 4(1/4.01+ 1/4.09+ 1/4.25+ 1/4.49)

+2(1/4.04+1/4.16+ 1/4.36)]

By direct integration we get

$$\int_0^{0.8} \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^{0.8} = 0.5\left[\tan^{-1}0.4 - \tan^{-1}0\right] = 0.5\tan^{-1}0.4$$

=10.900704743176

Comparing the result we get error in Trapezoidal and Simpsons rule.

17. Using Trapezoidal and Simpsons rule evaluate the following integral with number of subintervals n =6

$$I = \int_0^{0.6} \frac{dx}{\sqrt{1+x}}$$

Solution:

Here integrand $y = f(x) = \frac{1}{\sqrt{1+x}}$

a=0, b= 0.6 , h= (b-a)/n = (0.6-0)/6 = 0.1

x	0	0.1	0.2	0.3	0.4	0.5	0.6
Y= $\frac{1}{\sqrt{1+x}}$	1	$\frac{1}{\sqrt{1.1}}$ =0.953462	$\frac{1}{\sqrt{1.2}}$ =0.912871	$\frac{1}{\sqrt{1.3}}$ =0.877058	$\frac{1}{\sqrt{1.4}}$ =0.845154	$\frac{1}{\sqrt{1.5}}$ =0.816496	$\frac{1}{\sqrt{1.6}}$ =0.790569
	y_0	y_1	y_2	y_3	y_4	Y_5	y_6

(i) Using Trapezoidal rule

$I = (h/2)[\ y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots + y_{n-1})\ ]$

$= (0.1/2)[\ y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\ ]$

$= 0.05\ [\ 1 + 0.790569 + 2(0.953462 + 0.912871 + 0.877058 + 0.845154 + 0.816496)\ ]$

(ii) Using Simpsons rule

$I = (h/3)\ [\ y_0 + y_{2m} + 4(y_1 + y_3 + \ldots\ldots\ldots + y_{2m-1}) + 2(y_2 + y_4 + \ldots\ldots\ldots + y_{2m-2})\ ]$

$= (h/3)\ [\ y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\ ]$

$= (0.1/3)\ [\ 1 + 0.790569 + 4(0.953462 + 0.877058 + 0.816496) + 2(0.912871 + 0.845154)\ ]$