SYNERGY INSTITUTE OF ENGINEERING & TECHNOLOGY
Department of Mechanical Engineering

<u>LECTURE NOTE (Academic Session 2023-2024)</u>

Name of Faculty: Mrs. Chandana Subudhi

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Name of Subject: Heat Transfer

MODULE I

BASICS OF HEAT TRANSFER

While teaching heat transfer, one of the first questions students commonly ask is the difference between *heat* and *temperature*. Another common question concerns the difference between the subjects of *heat transfer* and *thermodynamics*. Let me begin this chapter by trying to address these two questions.

1.1 Difference between heat and temperature

In heat transfer problems, we often interchangeably use the terms *heat* and *temperature*. Actually, there is a distinct difference between the two. *Temperature* is a measure of the amount of energy possessed by the molecules of a substance. It manifests itself as a degree of hotness, and can be used to predict the direction of heat transfer. The usual symbol for temperature is *T*. The scales for measuring temperature in SI units are the Celsius and Kelvin temperature scales. *Heat*, on the other hand, is energy in transit. Spontaneously, heat flows from a hotter body to a colder one. The usual symbol for heat is Q. In the SI system, common units for measuring heat are the Joule and calorie.

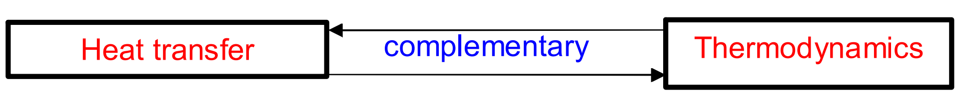
1.2 Difference between thermodynamics and heat transfer

Thermodynamics tells us:

- how much heat is transferred (δQ)
- how much work is done (δW)
- final state of the system

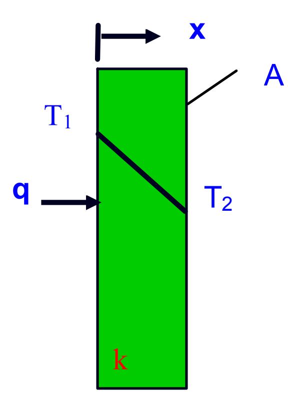
Heat transfer tells us:

- how (with what **modes**) δQ is transferred
- at what **rate** δQ is transferred
- temperature distribution inside the body



1.3 Modes of Heat Transfer

- **Conduction:** An energy transfer across a system boundary due to a temperature difference by the mechanism of inter-molecular interactions. Conduction needs matter and does not require any bulk motion of matter.



Conduction rate equation is described by the Fourier Law:

$$\vec{q} = -kA\nabla T$$

where: q = heat flow vector, (W)
k = thermal conductivity, a thermodynamic property of the material. (W/m K)
A = Cross sectional area in direction of heat flow. (m^2)
∇T = Gradient of temperature (K/m)
$= \partial T/\partial x\ \mathbf{i} + \partial T/\partial y\ \mathbf{j} + \partial T/\partial z\ \mathbf{k}$

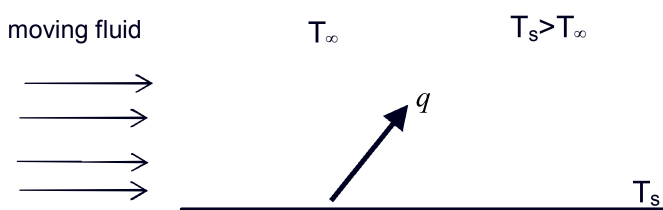
Note: Since this is a vector equation, it is often convenient to work with one component of the vector. For example, in the x direction:

$$q_x = -\ k\ A_x\ dT/dx$$

In circular coordinates it may convenient to work in the radial direction:

$$q_r = -\ k\ A_r\ dT/dr$$

- **Convection:** An energy transfer across a system boundary due to a temperature difference by the combined mechanisms of intermolecular interactions and bulk transport. Convection needs fluid matter.



Newton's Law of Cooling:

$$q = h\ A_s\ \Delta T$$

where: q = heat flow from surface, a scalar, (W)
h = heat transfer coefficient (which is not a thermodynamic property of the material, but may depend on geometry of surface, flow characteristics, thermodynamic properties of the fluid, etc. (W/m^2 K)
A_s = Surface area from which convection is occurring. (m^2)
$\Delta T = T_S - T_\infty$ = Temperature Difference between surface and coolant. (K)

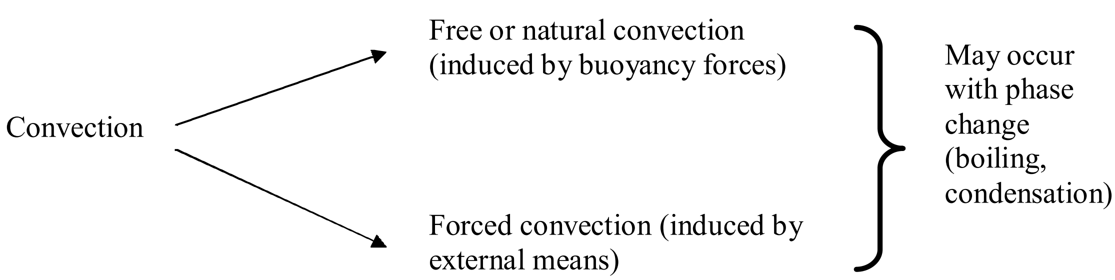


Table 1. Typical values of h (W/m^2K)

Free convection	gases: 2 - 25 liquid: 50 – 100
Forced convection	gases: 25 - 250 liquid: 50 - 20,000
Boiling/Condensation	2500 -100,000

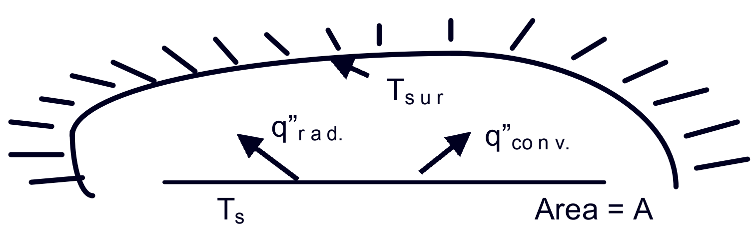
- **Radiation:** Radiation heat transfer involves the transfer of heat by electromagnetic radiation that arises due to the temperature of the body. Radiation does not need matter.

  Emissive power of a surface:

$$E=\sigma\varepsilon T_s^4 \ (W/\ m^2)$$

where: ε = emissivity, which is a surface property (ε = 1 is black body)
σ = Steffan Boltzman constant = 5.67×10^{-8} $W/m^2\ K^4$.
T_s = Absolute temperature of the surface (K)

The above equation is derived from Stefan Boltzman law, which describes a gross heat emission rather than heat transfer. The expression for the actual radiation heat transfer rate between surfaces having arbitrary orientations can be quite complex, and will be dealt with in Module 9. However, the rate of radiation heat exchange between a small surface and a large surrounding is given by the following expression:



$$q = \varepsilon\cdot\sigma\cdot A\cdot(T_s^4 - T_{sur}^4)$$

where: ε = Surface Emissivity
A= Surface Area
T_s = Absolute temperature of surface. (K)
T_{sur} = Absolute temperature of surroundings.(K)

1.4 Thermal Conductivity, k

As noted previously, thermal conductivity is a thermodynamic property of a material. From the State Postulate given in thermodynamics, it may be recalled that thermodynamic properties of pure substances are functions of two independent thermodynamic intensive properties, say temperature and pressure. Thermal conductivity of real gases is largely independent of pressure and may be considered a function of temperature alone. For solids and liquids, properties are largely independent of pressure and depend on temperature alone.

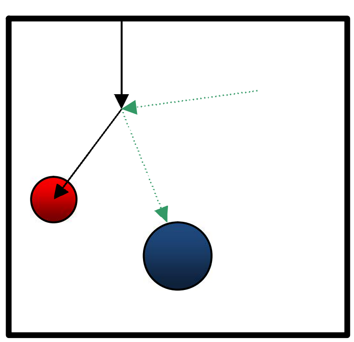
$$k = k\,(T)$$

Table 2 gives the values of thermal conductivity for a variety of materials.

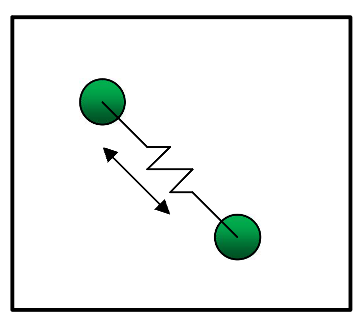
Table 2. Thermal Conductivities of Selected Materials at Room Temperature.

Material	Thermal Conductivity, W/m K
Copper	401
Silver	429
Gold	317
Aluminum	237
Steel	60.5
Limestone	2.15
Bakelite	1.4
Water	0.613
Air	0.0263

Let us try to gain an insight into the basic concept of thermal conductivity for various materials. The fundamental concept comes from the molecular or atomic scale activities. Molecules/atoms of various materials gain energy through different mechanisms. Gases, in which molecules are free to move with a mean free path sufficiently large compared to their diameters, possess energy in the form of kinetic energy of the molecules. Energy is gained or lost through collisions/interactions of gas molecules.



Kinetic energy transfer between gas molecules.



Lattice vibration may be transferred between molecules as nuclei attract/repel each other.

Solids, on the other hand, have atoms/molecules which are more closely packed which cannot move as freely as in gases. Hence, they cannot effectively transfer energy through these same mechanisms. Instead, solids may exhibit energy through vibration or rotation of the nucleus. Hence, the energy transfer is typically through lattice vibrations.

Another important mechanism in which materials maintain energy is by shifting electrons into higher orbital rings. In the case of electrical conductors the electrons are weakly bonded to the molecule and can drift from one molecule to another, transporting their energy in the process. Hence, flow of electrons, which is commonly observed in metals, is an effective transport mechanism, resulting in a correlation that materials which are excellent electrical conductors are usually excellent thermal conductors.

Combined Mechanisms of Heat Transfer

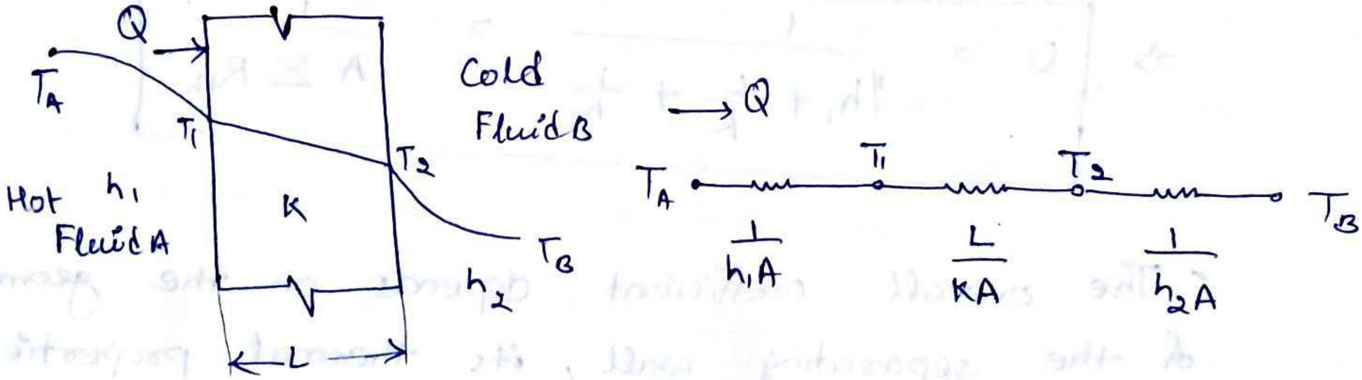
It is observed that the heat-transfer is taking place due to two, or perhaps all three mechanisms.

Example – A solid wall which separates two convecting fluids, e.g. the tubes of a heat exchanger.

- Steam generating tubes of a boiler receive heat from the products of combustion by all 3 modes of heat transfer.

The overall heat transfer by combined modes is usually expressed in terms of overall heat transfer coefficient 'U'.

$$Q = UA(\Delta T) \quad \text{——— (1)}$$



The heat transfer rate is given by

$$Q = h_1A(T_A - T_1) = \frac{KA}{L}(T_1 - T_2) = h_2A(T_2 - T_B)$$

$$\Rightarrow T_A - T_1 = \frac{Q}{h_1A}$$

$$T_1 - T_2 = \frac{Q}{KA/L}$$

$$T_2 - T_B = \frac{Q}{h_2A}$$

Adding all these we have

$$T_A - T_B = \frac{Q}{h_1 A} + \frac{Q}{\frac{KA}{L}} + \frac{Q}{h_2 A}$$

$$\Rightarrow Q = \frac{T_A - T_B}{\frac{1}{h_1 A} + \frac{L}{KA} + \frac{1}{h_2 A}} \quad \text{——— (2)}$$

$\frac{1}{hA}$ – Convection resistance

$\frac{L}{KA}$ = Conduction resistance

Comparing equ^n (1) & (2) we have

$$U A \Delta T = \frac{\Delta T \times A}{\left(\frac{1}{h_1} + \frac{L}{K} + \frac{1}{h_2}\right)}$$

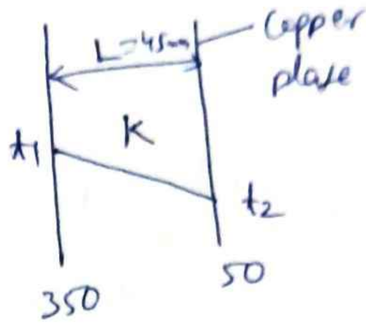
$$\Rightarrow \boxed{U = \frac{1}{1/h_1 + \frac{L}{K} + \frac{1}{h_2}} = \frac{1}{A \Sigma R_{th}}}$$

- The overall coefficient depends on the geometry of the separating wall, its thermal properties, and the convective coefficients at the two surfaces.
- It is useful in the case of composite walls, such as in the design of structural walls for boilers, refrigerators, airconditioned buildings etc.
  - in the design of heat exchangers.

Q. Calculate the rate of heat transfer per unit area through a copper plate 45 mm thick, whose one face is maintained at 350°C & the other face at 50°C. Take thermal conductivity of copper as 370 W/m°C.

Given temp difference $dt = t_2 - t_1 = 50 - 350$

thickness of copper plate $L = 45mm$
$= 0.045m$



Thermal conductivity of copper

$K = 370 W/m°C$

$\frac{Q}{A} = ?$

From Fourier's law $Q = KA\frac{dt}{dx}$

$\Rightarrow \frac{Q}{A} = -k \times \frac{dt}{dx} = -k\frac{(t_2 - t_1)}{L}$

Heat flux $= \frac{-370 \times (-300)}{0.045}$

$= 2.466 \times 10^6 \ W/m^2$

Q. A plane wall is 150mm thick and its wall area is 4.5m². If its conductivity is 9.35 W/m°C and surface temps are steady at 150°C & 45°C, determine

i) Heat flow across the plane wall

ii) Temp gradient in the flow direction

Given

thickness $L = 150mm = 0.15m$

Area $A = 4.5m^2$

$k = 9.35 W/m°C$

Temp difference $dt = t_2 - t_1 = 45 - 150 = -105°C$

i) Heat flow across the plane wall, $Q = ?$

From Fourier's law we have

$$Q = -kA\frac{dt}{dx} = -kA\frac{(t_2 - t_1)}{L}$$

$$= -9.35 \times 4.5 \times \frac{(-105)}{0.15} = 29452.5W$$

ii) Temp gradient, $\frac{dt}{dx} = ?$

Again $Q = -kA\frac{dt}{dx}$

$$\Rightarrow \frac{dt}{dx} = -\frac{Q}{KA} = -\frac{29452.5}{9.35 \times 4.5}$$

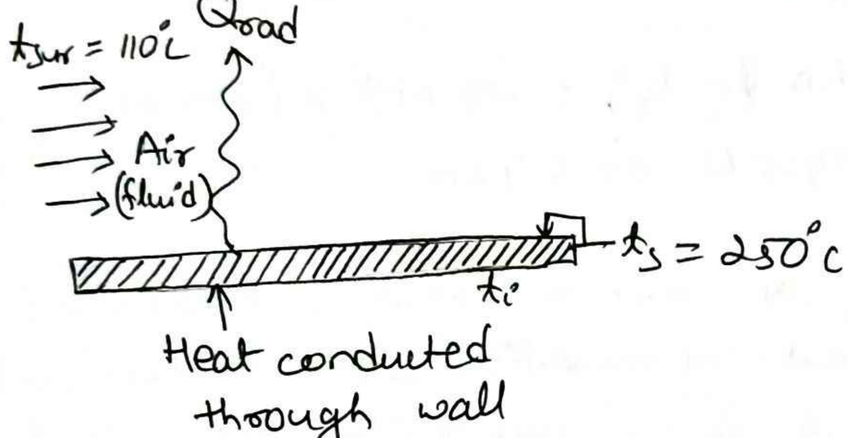
$$= -700°C/m$$

Q - The wall of an industrial furnace is constructed from 0.15m thick fireclay brick having thermal conductn of 1.7 W/mK. Measurements made during steady-state operⁿ temps of 1400 & 1150K at the inner & outer surface what is the rate of heat loss through a

$$Q = -KA\frac{dT}{dx} = -KA\left(\frac{T_2 - T_1}{L}\right) = KA\left(\frac{T_1 - T_2}{L}\right)$$

$A(t_1 - t_2)$ $1.5(300-140)$ $/m^\circ C$

Q. A surface at 250°C exposed to the surroundings at 110°C convects & radiates heat to the surroundings. The convection coefficient & radiation factor are 75 W/m²°C and unity respectively. If the heat is conducted to the surface through a solid of conductivity 10 W/m°C, what is the temp gradient at the surface in the solid?



combination of conduction, convection & radiation

Given $t_s = 250^\circ C$, $t_{sur} = 110^\circ C$, $h = 75\ W/m^2{}^\circ C$

$F = 1$, $\sigma = 5.67 \times 10^{-8}\ W/m^2K^4$, $K = 10\ W/m^\circ C$

$\frac{dt}{dx} = ?$

Heat conducted through the plate = convection heat losses + radiation heat losses

$\Rightarrow Q_{cond} = Q_{conv} + Q_{rad}$

$\Rightarrow -KA\frac{dt}{dx} = hA(t_s - t_{sur}) + F\sigma A(T_s^4 - T_{sur}^4)$

$\Rightarrow -10 \times \frac{dt}{dx} = 75(250-110) + 1 \times 5.67 \times 10^{-8}\left[(250+273)^4 - (110+273)^4\right]$

$\Rightarrow \frac{dt}{dx} = -1352.21\ ^\circ C/m$

General Heat Conduction Equ^n in Cartesian coordinates →

Consider a rectangular parallelopiped of sides dx, dy & dz parallel, to the 3 axes (X, Y, Z) in a medium in which temp is varying with location & time.

Let t = Temp. at the left face ABCD; this temp may be assumed uniform over the entire surface, since the area of this face can be made arbitarily small

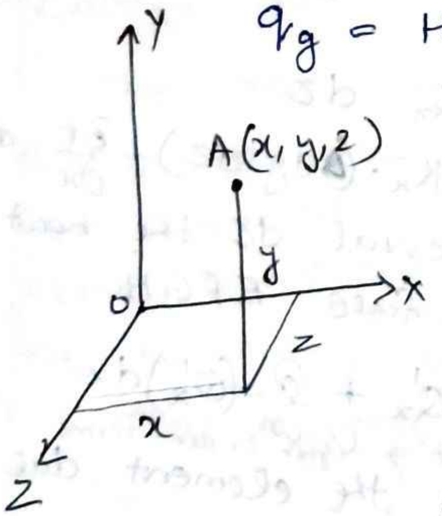
$\frac{dt}{dx}$ = Temp changes & rate of change along x-direction.

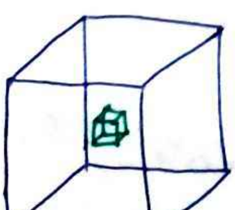
Then $\frac{\partial t}{\partial x} \cdot dx$ = Change of temp through distance dx

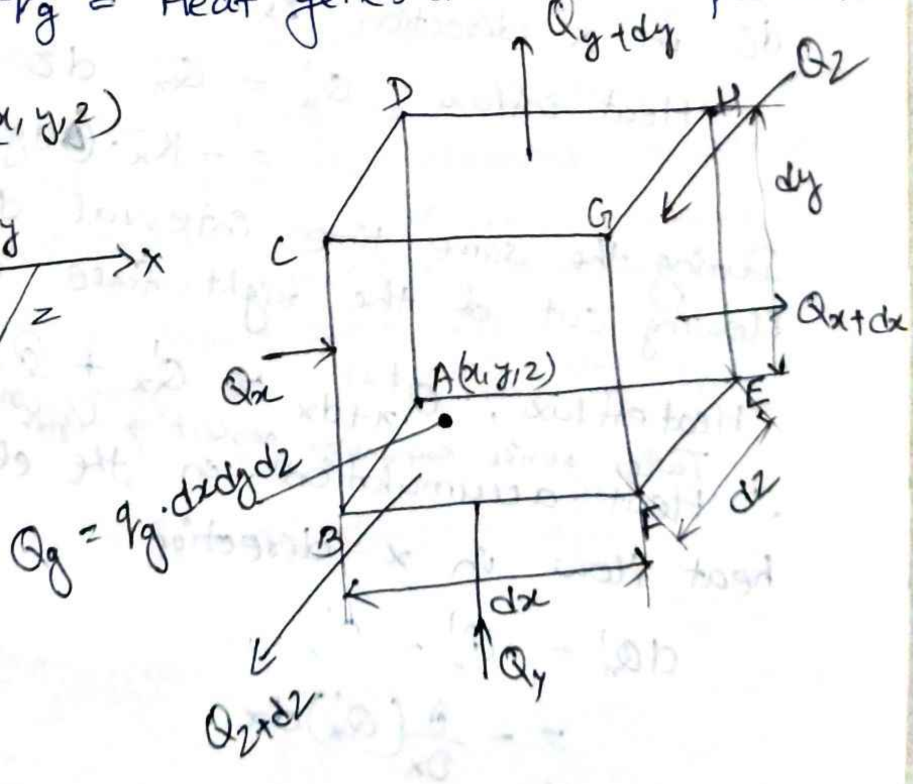
$t + \left(\frac{\partial t}{\partial x}\right) dx$ = Temp on the right face EFGH at a distance dx from left face ABCD.

Let k_x, k_y, k_z = Thermal conductivities along x, Y & Z ~~direction~~ axes.

q_g = Heat generated per unit volume per unit time







If the directional characteristics of a material are equal / same, it is called Isotropic. if unequal / different, it is called Anisotropic material.

(Inside the control volume there may be heat sources due to flow of electric current & electric motors & generators, nuclear fission etc)

Energy balance / equ^n for a volume element

Net heat accumulated in the element due to conduction of heat from all the coordinates directions considered (A) + heat generated within the element (B) = Energy stored in the element (C) —— (1)

Let Q = Rate of heat flow in a direction

$Q' = Q \cdot d\tau$ = Total heat flow in that direction in time $d\tau$.

A. Net heat accumulated in the element due to conduction of heat from all the directions →

Quantity of heat flowing into the element from the left face ABCD during time interval $d\tau$ in x-direction is

Heat influx $Q'_x = Q_x \cdot d\tau$

$= -K_x \cdot (dy \cdot dz) \cdot \frac{\partial t}{\partial x} \cdot d\tau$

During the same time interval $d\tau$ the heat flowing out of the right face EFGH

Heat efflux, $Q'_{x+dx} = Q'_x + \frac{\partial}{\partial x}(Q'_x)dx$ neglecting higher order terms

Tailor series expansion

∴ Heat accumulation in the element due to heat flow in x-direction

$dQ'_x = Q'_x - Q'_{x+dx}$

$= -\frac{\partial}{\partial x}(Q'_x)dx$

$$= -\frac{\partial}{\partial x}\left[-K_x (dy\,dz)\frac{\partial t}{\partial x}\cdot d\tau\right] dx$$

$$= \frac{\partial}{\partial x}\left(K_x \frac{\partial t}{\partial x}\right) dx\,dy\,dz\,d\tau$$

Similarly along y & z directions in time $d\tau$ will be

$$dQ'_y = \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial y}\right) dx\,dy\,dz\,d\tau$$

$$dQ'_z = \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right) dx\,dy\,dz\,d\tau$$

∴ Net heat accumulated due to condⁿ of heat from all coordinate directions

$$= \frac{\partial}{\partial x}\left(k_x \frac{\partial t}{\partial x}\right) dx\,dy\,dz\,d\tau + \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial y}\right) dx\,dy\,dz\,d\tau + \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right) dx\,dy\,dz\,d\tau$$

$$= \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right)\right] dx\,dy\,dz\,d\tau$$

B. Total heat generated within the element

$$Q'_g = q_g \cdot (dx\,dy\,dz) \cdot d\tau$$

C Energy stored in the element

The total heat accumulated in the element due to heat flow along co-ordinate axes & the heat generated within the element together serve to increase the thermal energy of the element. (If the material is not experiencing a change in phase → latent energy = 0)

This increase in thermal energy

$$= m \cdot c \cdot dT$$

$$= \rho\,(dx \cdot dy \cdot dz) \cdot c \cdot \frac{\partial t}{\partial \tau} \cdot d\tau$$

∴ Equⁿ ① becomes

$$\left[\frac{\partial}{\partial x}\left(k_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right)\right] dx\,dy\,dz\,d\tau$$

$$+ q_g (dx \cdot dy \cdot dz) \cdot d\tau = \rho\,(dx\,dy\,dz) \cdot c\,\frac{\partial t}{\partial \tau}\,d\tau$$

$$\Rightarrow \frac{\partial}{\partial x}\left(k_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right) + q_g \quad \text{--- (2)}$$

$$= \rho \cdot c \cdot \frac{\partial t}{\partial \tau}$$

$$\Rightarrow \boxed{\nabla \cdot (K \nabla t) + q_g = \rho \cdot c \cdot \frac{\partial t}{\partial \tau}}$$

This is known as the general heat condn equn for non-homogeneous material, self heat generating & unsteady 3-D heat flow.

General heat condn equn for constant thermal conductivity →

$$K_x = K_y = K_z$$

∴ Equn (2) becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial t}{\partial z}\right) + \frac{q_g}{K} = \frac{\rho c}{K} \cdot \frac{\partial t}{\partial \tau}$$

$$\Rightarrow \boxed{\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{K} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}} \quad \text{--- (3)}$$

where $\alpha = \frac{K}{\rho c} = \frac{\text{Thermal conductivity}}{\text{Thermal capacity}}$

= Thermal diffusivity

The larger the value of α, the faster will the heat diffuse through the material & its temp. will change with time. This will be either due to high value of thermal conductivity (K) or a low value of thermal capacity ($\rho \cdot c$). Low value of thermal capacity means the less amount of heat entering the element, would be absorbed & used to raise its temp

- Metals & gases have high value of α so their response to temp changes is quite rapid.

- The non-metallic solids & liquids have small value of α so they respond slowly to temp. changes.

$\therefore$ We can write

$$\nabla^2 t + q_g / K = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \text{— (4)}$$

This is the equ^n for homogeneous & isotropic matl. (for homogeneous & isotropic matl $K_x = K_y = K_z$)

Other simplified forms

i) When no internal source of heat generation is present, then equ^n (3) becomes

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}$$

$$\Rightarrow \nabla^2 t = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \qquad \text{Fourier's equ}^n$$

ii) When temp. does not depend on time, steady state $\left(\frac{\partial t}{\partial \tau} = 0\right)$

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{K} = 0$$

$$\Rightarrow \nabla^2 t + \frac{q_g}{K} = 0 \qquad \text{Poisson's equ}^n$$

In the absence of internal energy

$$\nabla^2 t = 0 \qquad \text{Laplace equ}^n$$

iii) Steady state & 1-D heat transfer

$$\frac{\partial^2 t}{\partial x^2} + \frac{q_g}{K} = 0$$

iv) Steady state, 1-D, without heat genern

$$\frac{\partial^2 t}{\partial x^2} = 0$$

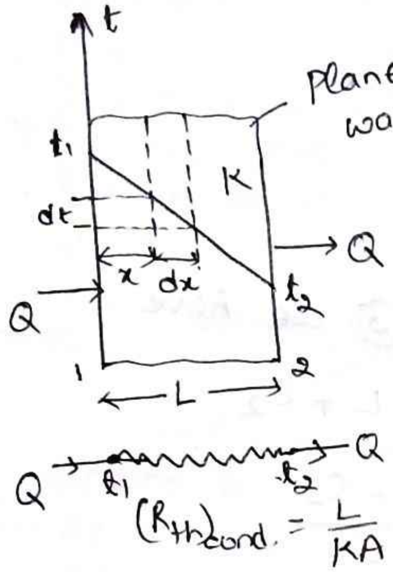
v) Steady state, 2-D, without internal heat generation.

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0$$

vi) Unsteady state, 1-D, without heat genern

$$\frac{\partial^2 t}{\partial x^2} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}$$

Heat Conduction Through a Plane Wall



Let's consider a plane wall of homogeneous material through which heat is flowing only in x-direction

Let L = Thickness of the plane wall

A = Cross-sectional area of the wall

K = Thermal conductivity of the wall material

t_1, t_2 = Temp. maintained at two surfaces 1 & 2 of the wall

Case-I → Uniform Thermal conductivity

The general heat conduction equ^n in cartesian co-ordinates is

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{K} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}$$

If the heat cond^n takes place under the cond^s → steady state $\left(\frac{\partial t}{\partial \tau} = 0\right)$, 1-D $\left(\frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial z^2} = 0\right)$ & with no internal heat gener^n $\left(\frac{q_g}{K} = 0\right)$ then the equ^n becomes

$$\frac{\partial^2 t}{\partial x^2} = 0$$

$$\Rightarrow \frac{d^2 t}{dx^2} = 0 \quad \text{——— (1)}$$

By integrating equ^n (1) we have

$$\frac{dt}{dx} = C_1$$

Again integrating

$$\Rightarrow t = C_1 x + C_2 \quad \text{——— (2)}$$

C_1 & C_2 are calculated by applying boundary cond^ns

The boundary cond^ns are

at $x = 0$, $t = t_1$

at $x = L$, $t = t_2$

Substituting this in equ^n (2) we have

$C_2 = t_1$ & $t_2 = C_1 L + C_2$

$$\Rightarrow C_1 = \frac{t_2 - C_2}{L}$$

$$\Rightarrow C_1 = \frac{t_2 - t_1}{L}$$

∴ Equ^n (2) becomes

$$t = \left(\frac{t_2 - t_1}{L}\right)x + t_1$$

This equ^n indicates that temp. distribution across a wall is linear & is independent of thermal conductivity.

Now heat through plane wall

$$Q = -KA\frac{dt}{dx}$$

$$= -KA \times C_1$$

$$= -KA\left(\frac{t_2 - t_1}{L}\right)$$

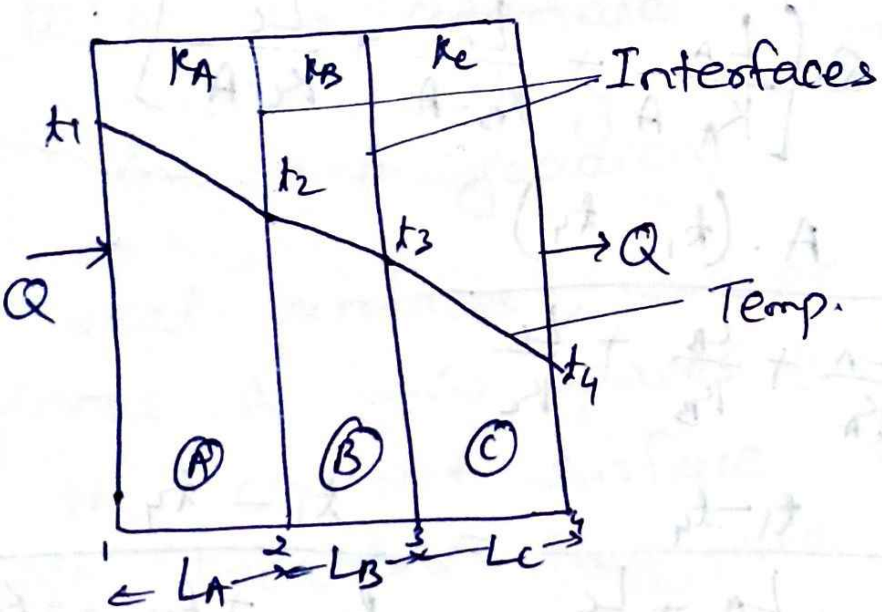
$$= \frac{t_1 - t_2}{(L/KA)} = \frac{t_1 - t_2}{(R_{th})_{cond^n}}$$

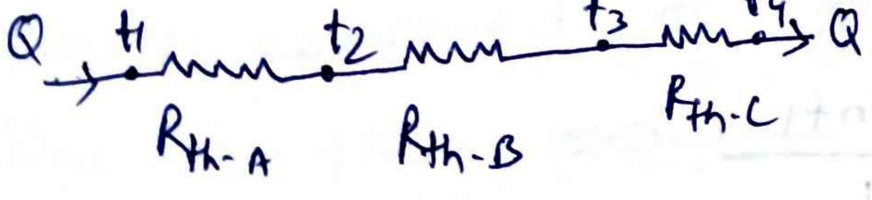
$(R_{th})_{cond}$ = Thermal resistance to heat cond^n

$$= \frac{L}{KA}$$

Heat Conduction Through a Composite Wall

Let's consider the transmission of heat through a composite wall consisting of no. of slabs.





$$R_{th \cdot A} = \frac{L_A}{K_A A}$$

$$R_{th \cdot B} = \frac{L_B}{K_B \cdot A}$$

$$R_{th \cdot C} = \frac{L_C}{K_C \cdot A}$$

Let L_A, L_B, L_C = Thickness of slabs, A, B & C

K_A, K_B, K_C = Thermal conductivities of slabs

t_1, t_4 $(t_1 > t_4)$ = Temp at wall surfaces 1 & 4

t_2, t_3 - Temp at interface 2 & 3

Since the quantity of heat transmitted per unit time through each layer is same

$$Q = \frac{K_A A (t_1 - t_2)}{L_A} = \frac{K_B A (t_2 - t_3)}{L_B} = \frac{K_C A (t_3 - t_4)}{L_C}$$

(Assuming there is a perfect contact betn the layers & no temp. drop across the interface betn the materials)

$$\Rightarrow t_1 - t_2 = \frac{Q \cdot L_A}{K_A \cdot A}$$

$$t_2 - t_3 = \frac{Q L_B}{K_B \cdot A}$$

$$t_3 - t_4 = \frac{Q L_C}{K_C \cdot A}$$

Adding all we have

$$t_1 - t_4 = Q \left[\frac{L_A}{K_A \cdot A} + \frac{L_B}{K_B \cdot A} + \frac{L_C}{K_C \cdot A} \right]$$

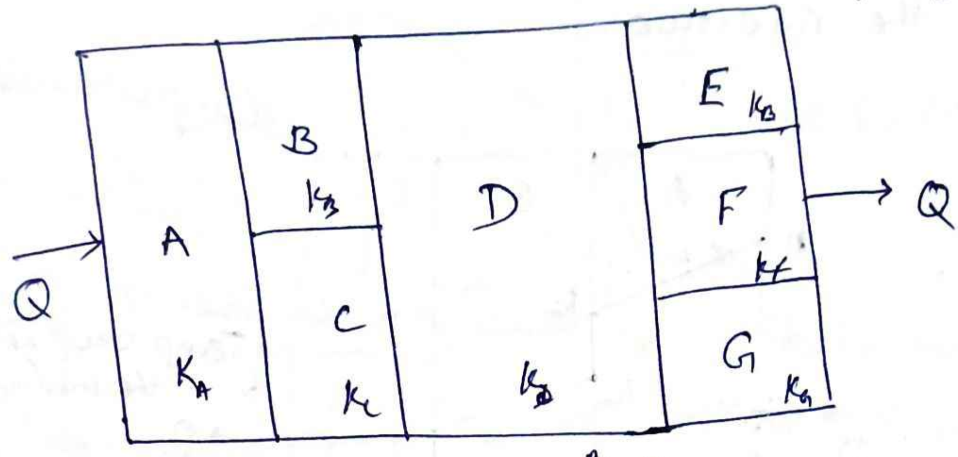
$$\Rightarrow Q = \frac{A \cdot (t_1 - t_4)}{\frac{L_A}{K_A} + \frac{L_B}{K_B} + \frac{L_C}{K_C}}$$

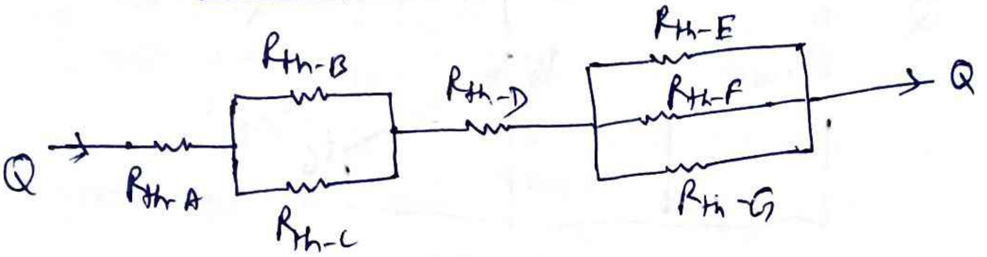
$$= \frac{t_1 - t_4}{\frac{L_A}{K_A \cdot A} + \frac{L_B}{K_B \cdot A} + \frac{L_C}{K_C \cdot A}} = \frac{t_1 - t_4}{R_{th-A} + R_{th-B} + R_{th-C}}$$

$$\Rightarrow Q = \frac{t_1 - t_{n+1}}{\sum_{1}^{n} \frac{L}{KA}}$$

$Q = \frac{\Delta t_{overall}}{\Sigma R_{th}}$ Resistance Series → $R_1 + R_2 + \dots = R_{eq}$

" Parallel → $\frac{1}{R_1} + \frac{1}{R_2} + \dots = \frac{1}{R_{eq}}$





Thermal Contact Resistance

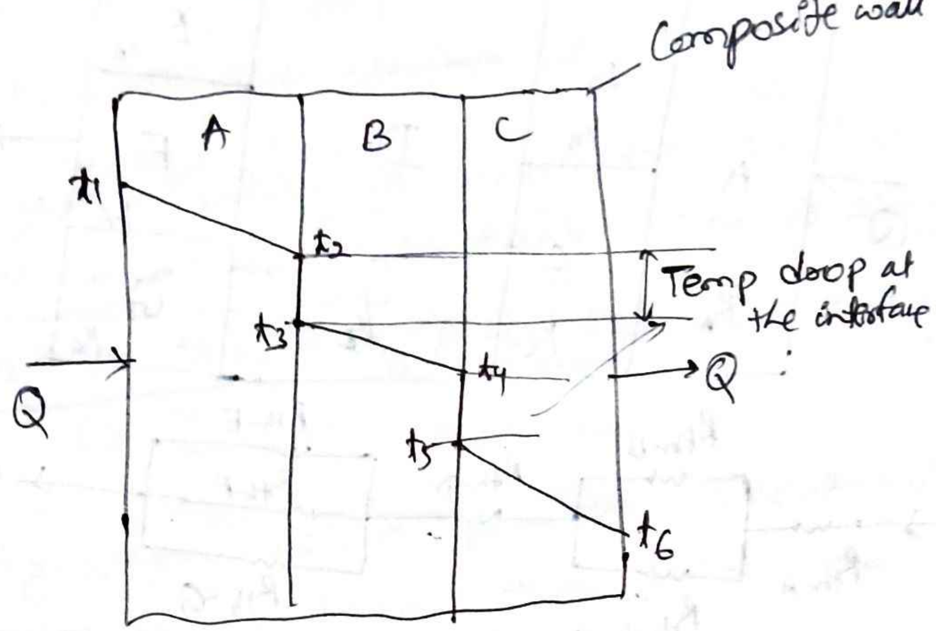
In the calculation of heat flow in composite wall it is assumed that →

i) The contact betn the adjacent layers is perfect

ii) At the interface there is no fall of temp

iii) At the interface the temp is continuous, although there is discontinuity in temp. gradient.

In real surfaces, however due to surface roughness & void spaces (usually filled with air) the contact surface touches only at discrete locations. Thus the area available for the flow of heat at interface will be small compared to geometric face area. Due to this reduced area & presence of air voids, a large resistance to heat flow at interface occurs. This resistance is called thermal contact resistance & it causes temp. drop bet^n two materials at the interface.



$$(R_{th-AB})_{cont} = \frac{t_2 - t_3}{Q/A}$$

$$(R_{th-BC})_{cont} = \frac{t_4 - t_5}{Q/A}$$

Overall Heat-Transfer coefficient (U)

Let t = Thickness of the metal wall

K = Thermal conductivity of wall matl

t_1 = Temp of the surface-1

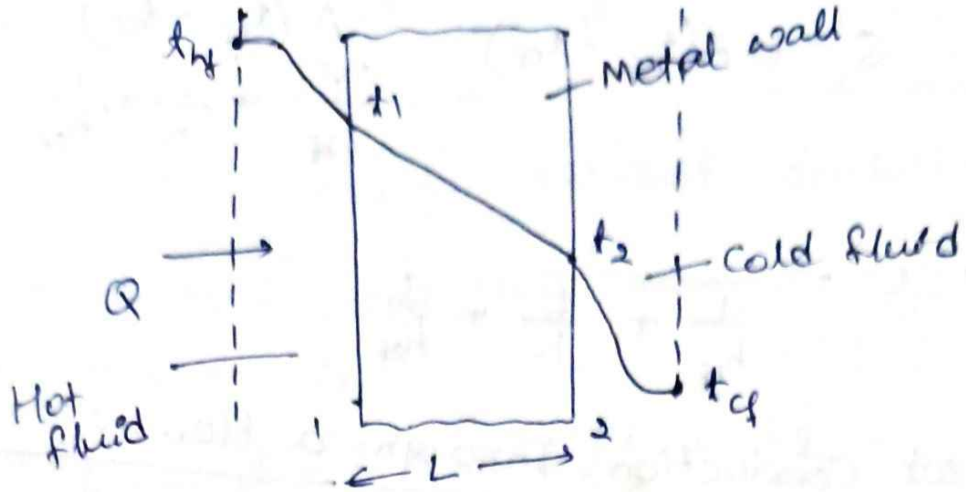
t_2 = " " " " -2

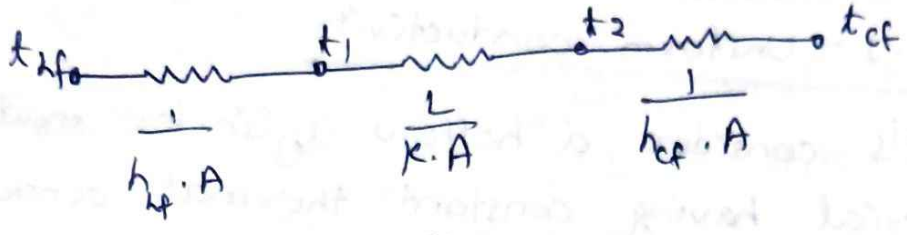
t_{hf} = Temp of the hot fluid

t_{cf} = " " " cold "

h_{hf} = Heat transfer coefficient from hot fluid to metal surface

h_{cf} = Heat transfer coefficient from metal surface to cold fluid.





The equn of heat flow

$$Q = h_{hf} A (t_{hf} - t_1)$$

$$Q = \frac{K A (t_1 - t_2)}{L}$$

$$Q = h_{cf} A (t_2 - t_{cf})$$

$$\Rightarrow t_{hf} - t_1 = \frac{Q}{h_{hf} A}$$

$$t_1 - t_2 = \frac{QL}{KA}$$

$$t_2 - t_{cf} = \frac{Q}{h_{cf} A}$$

Adding all we have

$$t_{hf} - t_{cf} = Q\left[\frac{1}{h_{hf} A} + \frac{L}{KA} + \frac{1}{h_{cf} A}\right]$$

$$\Rightarrow Q = \frac{A(t_{hf} - t_{cf})}{\frac{1}{h_{hf}} + \frac{L}{K} + \frac{1}{h_{cf}}}$$

If U is the overall coefficient of heat transfer then

$$Q = U \cdot A(t_{hf} - t_{cf}) = \frac{A(t_{hf} - t_{cf})}{\frac{1}{h_{hf}} + \frac{L}{K} + \frac{1}{h_{cf}}}$$

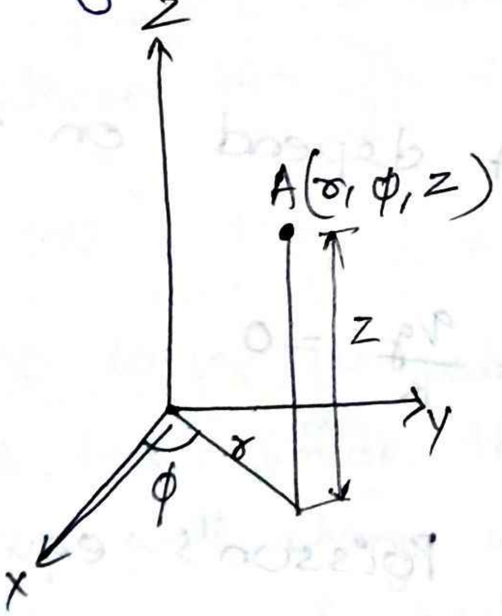
$$\Rightarrow U = \frac{1}{\frac{1}{h_{hf}} + \frac{L}{K} + \frac{1}{h_{cf}}}$$

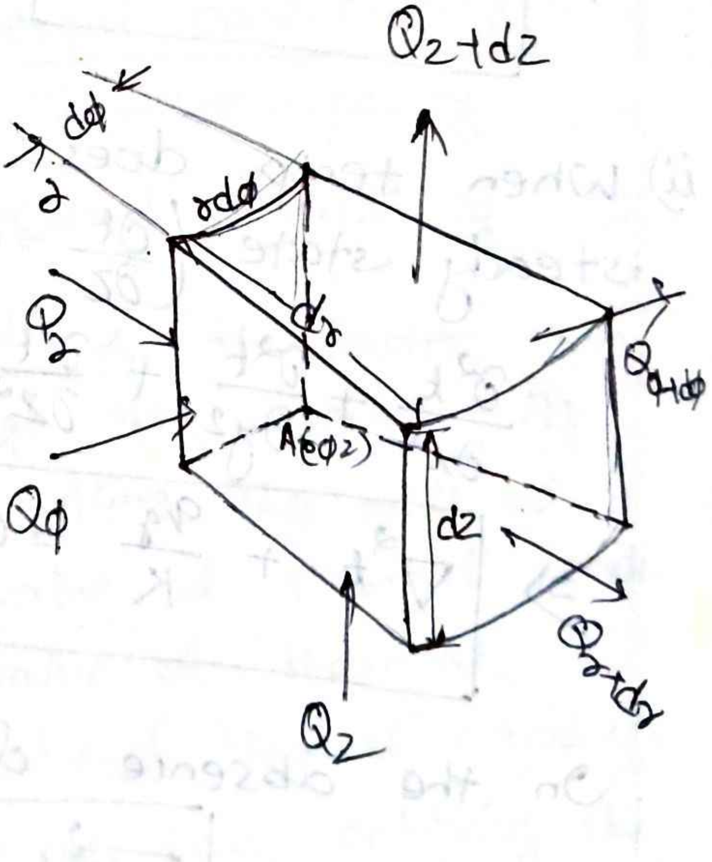
$\partial x^2 \quad \partial t$

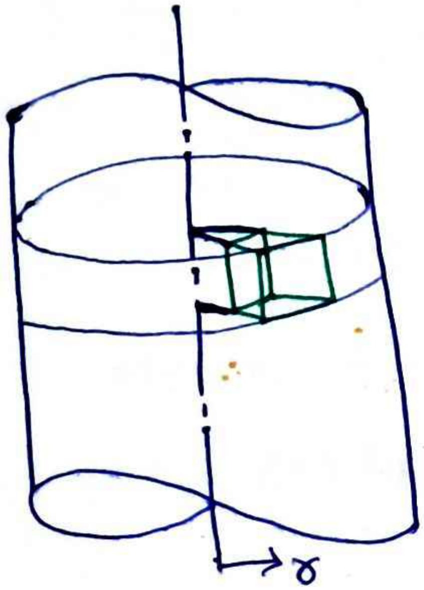
General Heat Conduction equn in cylindrical co-ordinates →

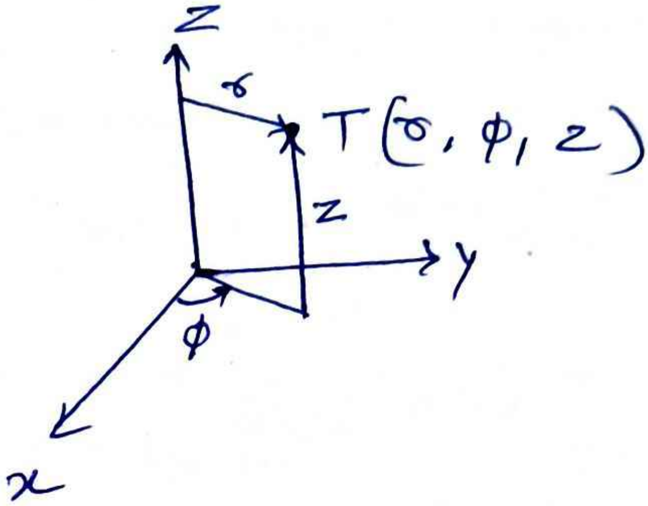
For cylindrical geometries (e.g. rods & pipes) it is convenient to use cylindrical coordinates.

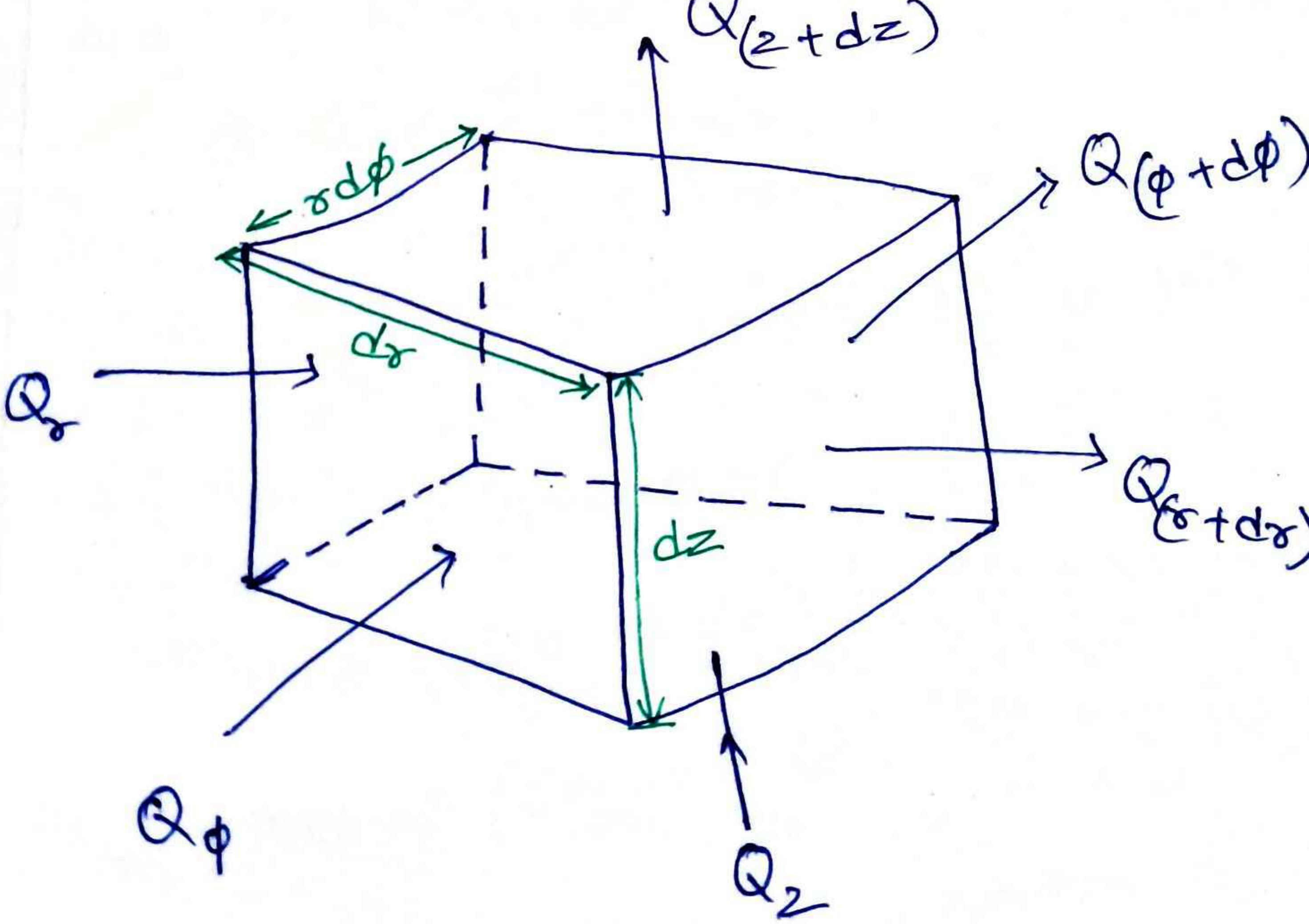
Consider an elemental volume having the co-ordinates (r, ϕ, z) for 3-D heat conduction analysis.






r


Z
r
T (r, φ, z)
z
y
φ
x


Q(z+dz)
rdφ
Q(φ+dφ)
dr
Qr
Q(r+dr)
dz
Qφ
Qz

The volume of the element $= r\,d\phi \cdot dr \cdot dz$

Let q_g = Heat generation per unit volume per unit time.

Let's assume that K, ρ, & C do not vary with position.

- A. Net heat accumulated in the element due to conduction of heat from all co-ordinate directions.

Heat flow in radial direction ($z-\phi$) plane

Heat influx $Q'_r = Q_r \cdot dz$

$$= -K\,(r\,d\phi \cdot dz)\frac{\partial t}{\partial r} \cdot dz$$

Heat efflux $Q'_{r+dr} = Q'_r + \frac{\partial}{\partial r}(Q'_r)\,dr$

$\therefore$ Heat accumulation in the element due to heat flow in radial direction.

$$dQ'_r = Q'_r - Q'_{r+dr}$$

$$= -\frac{\partial}{\partial r}(Q'_r)\,dr$$

$$= -\frac{\partial}{\partial r}\left(-K\,(r\,d\phi \cdot dz)\frac{\partial t}{\partial r} \cdot dz\right)dr$$

$$= K(dr \cdot d\phi \cdot dz)\frac{\partial}{\partial r}\left(r \cdot \frac{\partial t}{\partial r}\right)dz$$

$$= K\,(dr \cdot d\phi \cdot dz)\left(r\frac{\partial^2 t}{\partial r^2} + \frac{\partial t}{\partial r}\right)dz$$

$$= K \cdot (dr \cdot r\,d\phi \cdot dz)\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right)dz$$

Heat flow in tangential direction ($r-z$) plane

Heat influx $Q'_\phi = -K\,(dr \cdot dz)\,\frac{\partial t}{r\,d\phi} \cdot dz$

Heat efflux $Q'_{\phi+d\phi} = Q'_\phi + \frac{\partial}{r\,d\phi}(Q'_\phi)\,r\,d\phi$

$\therefore$ Heat accumulation $= dQ'_\phi = Q'_\phi - Q'_{\phi+d\phi}$

$$= -\frac{\partial}{r\,d\phi}(Q'_\phi)\,r\,d\phi$$

$$= -\frac{\partial}{r\partial\phi}\left[K(dr\,dz)\cdot\frac{\partial t}{r d\phi}\cdot d\tau\right] r d\phi$$

$$= K(dr \cdot r d\phi \cdot dz)\cdot\frac{1}{r^2}\frac{\partial}{\partial\phi}\left(\frac{\partial t}{\partial\phi}\right) d\tau$$

$$= K(dr \cdot r d\phi \cdot dz)\cdot\frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2}\cdot d\tau$$

Heat flow in axial dir^n ($r-\phi$ plane)

Heat influx $Q_z' = -K(dr \cdot r d\phi)\frac{\partial t}{\partial z} d\tau$

Heat efflux $Q'_{z+dz} = Q_z + \frac{\partial}{\partial z}(Q'_z) dz$

∴ Heat accumulation

$$dQ_z' = Q_z' - Q'_{z+dz}$$

$$= -\frac{\partial}{\partial z}(Q'_z)\, dz$$

$$= -\frac{\partial}{\partial z}\left[-K(r d\phi \cdot dr)\frac{\partial t}{\partial z}\cdot d\tau\right] dz$$

$$= K(dr \cdot r d\phi \cdot dz)\frac{\partial^2 t}{\partial z^2} d\tau$$

∴ Net heat accumulated in the element

$$= K\cdot(dr \cdot r d\phi \cdot dz)\left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}\right] d\tau$$

B. Heat generated within the element

$$Q_g' = q_g(dr \cdot r d\phi \cdot dz)\, d\tau$$

C Energy stored in the element

The increase in thermal energy in the element

$$= \rho\,(dr \cdot r d\phi \cdot dz)\cdot c\,\frac{\partial t}{\partial \tau}\cdot d\tau$$

Now A + B = C

$$\therefore K(dr \cdot r\,d\phi \cdot dz)\left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}\right]d\tau$$

$$+ q_g(dr \cdot r\,d\phi \cdot dz) \cdot d\tau = \rho(r\,d\phi \cdot dr \cdot dz) \cdot c\frac{\partial t}{\partial \tau} \cdot d\tau$$

$$\Rightarrow \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{K} = \frac{\rho c}{K}\frac{\partial t}{\partial \tau}$$

$$= \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}$$

This is the general heat conduction equ^n in cylindrical co-ordinates

When there is no heat generation & the flow is steady & 1-D then

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} = 0$$

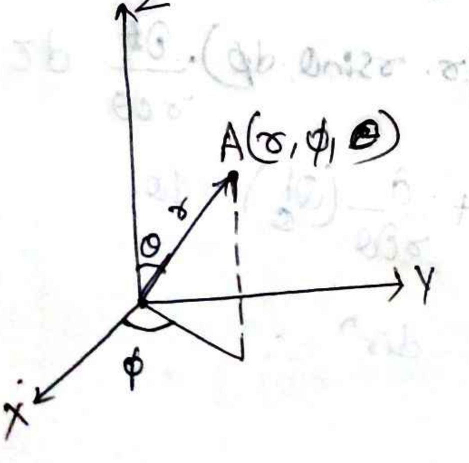
$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r \cdot \frac{dt}{dr}\right) = 0$$

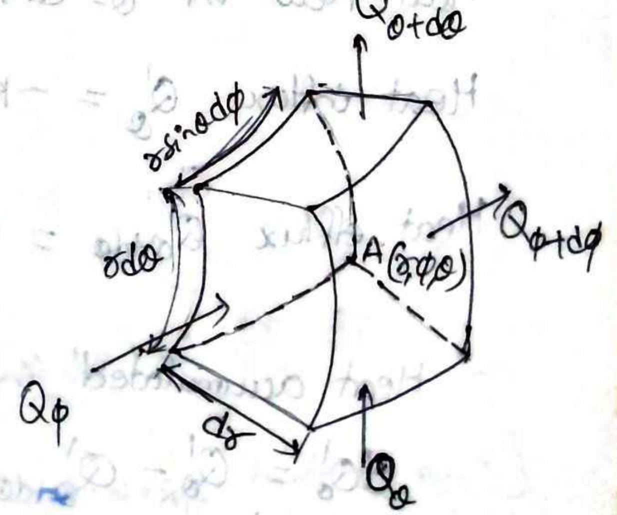
$$\Rightarrow \frac{d}{dr}\left(r \cdot \frac{dt}{dr}\right) = 0$$

$$\Rightarrow r \cdot \frac{dt}{dr} = \text{constant}$$

General Heat Conduction equ^n in Spherical co-ordinates →

Consider an elemental volume having the co-ordinates (r, ϕ, θ)





The volume of the element $= r d\theta . r\sin\theta d\phi . dr$

Let q_g = Heat generation per unit volume per unit time.

Let's assume that K, ρ, c do not alter with position. (Isotropic matl)

A. Net Heat accumulated in the element due to conduction of heat from all the co-ordinates

Heat flow through ϕ- direction ($r-\theta$ plane)

Heat influx $Q'_\phi = -K(dr . r d\theta) . \frac{\partial t}{r\sin\theta d\phi} d\tau$

Heat efflux $Q'_{\phi+d\phi} = Q'_\phi + \frac{\partial}{r\sin\theta \partial\phi}(Q'_\phi) r\sin\theta d\phi$

∴ Heat accumulated in the element in ϕ-dirn

$$dQ'_\phi = Q'_\phi - Q'_{\phi+d\phi}$$

$$= -\frac{\partial}{r\sin\theta \partial\phi}(Q'_\phi) r\sin\theta d\phi$$

$$= \frac{-1}{r\sin\theta} \frac{\partial}{\partial\phi}\left[-K(dr . r d\theta)\frac{\partial t}{r\sin\theta \partial\phi} . d\tau\right] r\sin\theta d\phi$$

$$= K(dr . r d\theta . r\sin\theta d\phi) . \frac{1}{r^2\sin^2\theta} . \frac{\partial^2 t}{\partial\phi^2} . d\tau$$

Heat flow in θ- dirn ($r-\phi$ plane)

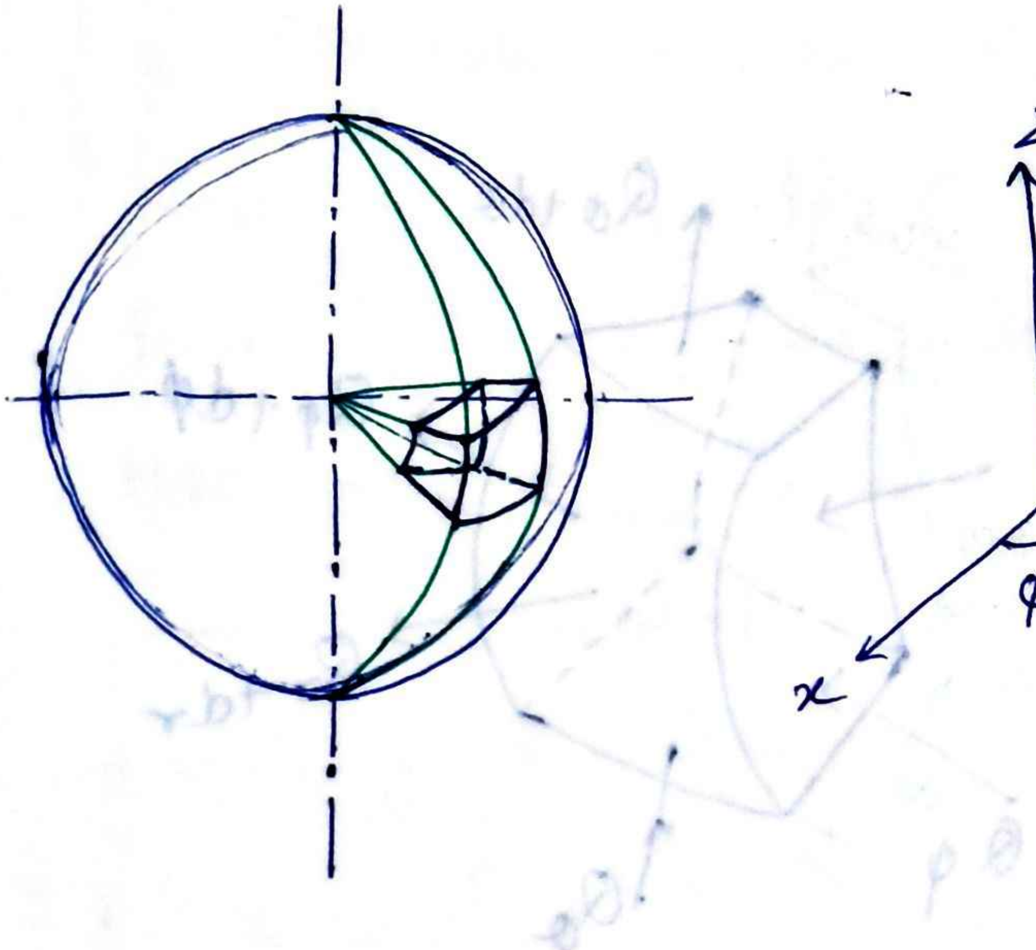
Heat influx $Q'_\theta = -K(dr . r\sin\theta . d\phi) . \frac{\partial t}{r\partial\theta} . d\tau$

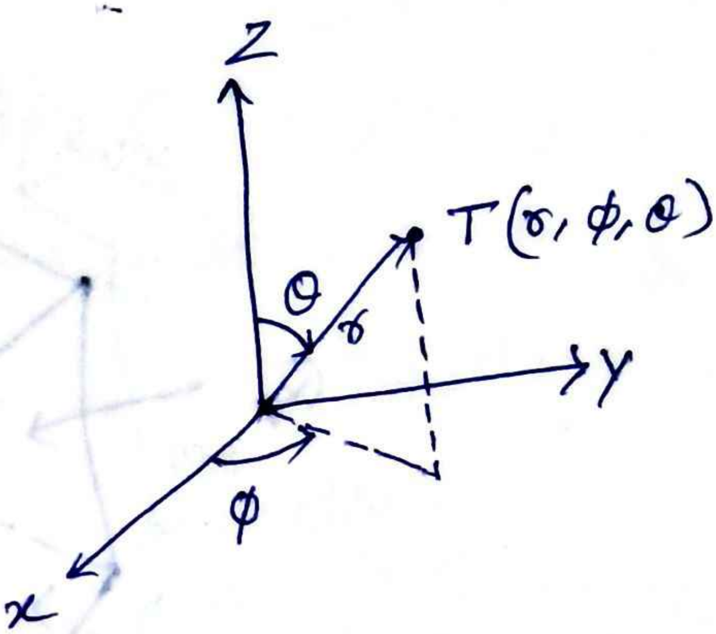
Heat efflux $Q'_{\theta+d\theta} = Q'_\theta + \frac{\partial}{r\partial\theta}(Q'_\theta) r d\theta$

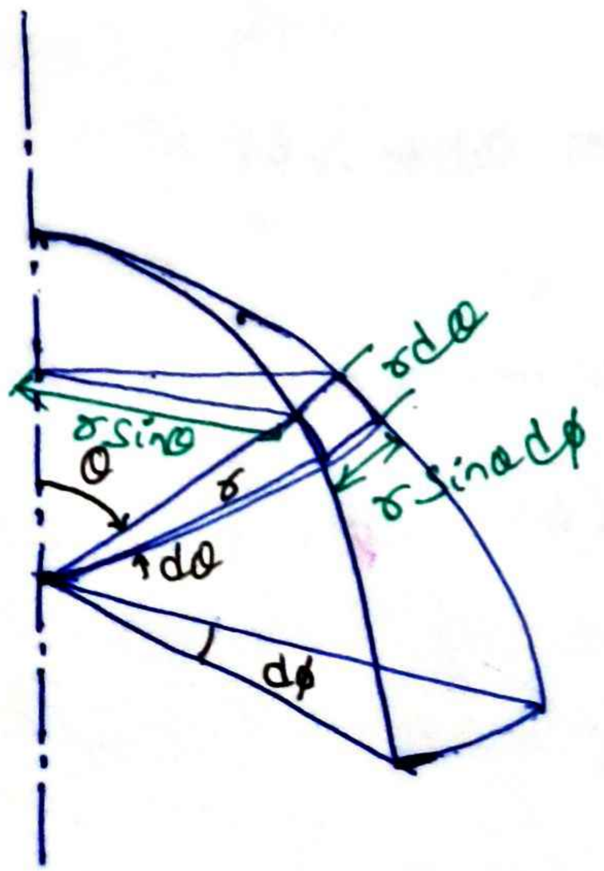
∴ Heat accumulated in θ - dirn

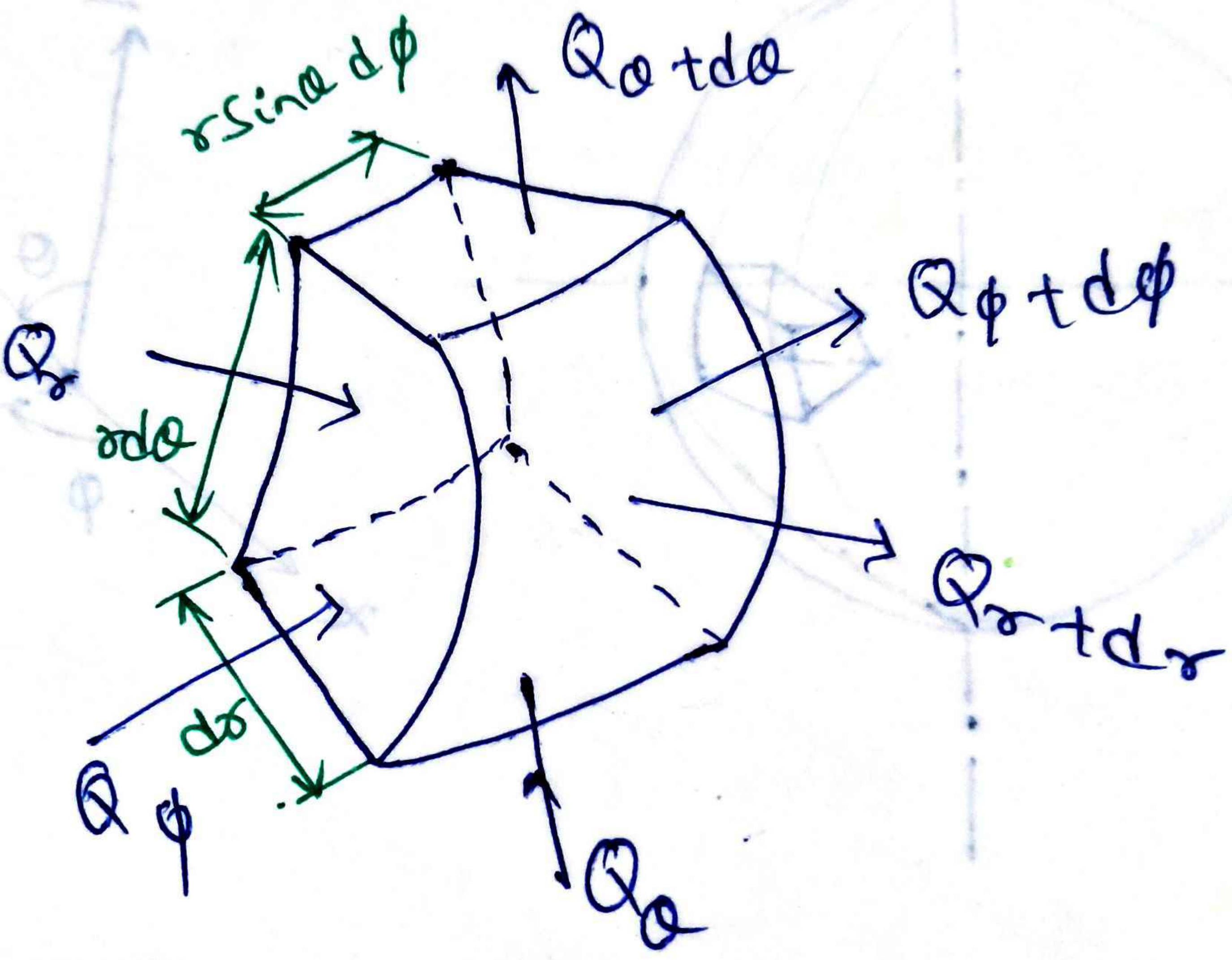
$$dQ'_\theta = Q'_\theta - Q'_{\theta+d\theta}$$

$$= -\frac{\partial}{r\partial\theta}(Q'_\theta) r d\theta$$




Z
T(r, φ, θ)
θ
r
Y
φ
x


r dθ
r sinθ
θ
r
r sinθ dφ
dθ
dφ



rSinθ dφ
Qθ+dθ
Qφ+dφ
Qr
rdθ
Qr+dr
dr
Qφ
Qθ

$$= -\frac{1}{r}\frac{\partial}{\partial\theta}\left[-K\,(dr \cdot r\sin\theta \cdot d\phi)\frac{\partial t}{r\,\partial\theta}\,d\tau\right] r\,d\theta$$

$$= K(dr \cdot r\,d\theta \cdot r\sin\theta \cdot d\phi) \cdot \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial t}{\partial\theta}\right) d\tau$$

Heat flow in r-dir^n (θ-ϕ plane)

Heat influx $Q'_r = -K\,(r\,d\theta \cdot r\sin\theta\,d\phi)\frac{\partial t}{\partial r} \cdot d\tau$

Heat efflux $Q'_{r+dr} = Q'_r + \frac{\partial}{\partial r}(Q'_r)\,dr$

∴ Heat accumulated in r-dir^n

$$dQ'_r = Q'_r - Q'_{r+dr}$$

$$= -\frac{\partial}{\partial r}(Q'_r)\,dr$$

$$= -\frac{\partial}{\partial r}\left[-K\,(r\,d\theta \cdot r\sin\theta\,d\phi)\frac{\partial t}{\partial r} \cdot d\tau\right] dr$$

$$= K\,(dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi) \cdot \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) d\tau$$

∴ Net heat accumulated in the element

$$= K\,(dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi)\left[\frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2} + \frac{1}{r^2\sin\theta} \times \frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)\right] d\tau$$

B Heat generated within the element

$$Q'_g = q_g\,(dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi)\,d\tau$$

C Energy stored in the element

$$= m \cdot c \cdot dT$$

$$= \rho\,(dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi) \cdot c \cdot \frac{\partial t}{\partial\tau} \cdot d\tau$$

Now A+B = C (Energy balance equ^n)

$$\therefore K(dr.\, rd\theta.\, r\sin\theta\, d\phi)\left[\frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)\right] d\tau$$

$$+ q_g(dr.\, rd\theta.\, r\sin\theta\, d\phi)\, d\tau = \rho\,(dr.\, rd\theta.\, r\sin\theta\, d\phi) \times C.\frac{\partial t}{\partial \tau}\, d\tau$$

$$\Rightarrow \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)$$

$$+ q_g/K = \frac{\rho c}{K}\frac{\partial t}{\partial \tau} = \frac{1}{\alpha}\cdot\frac{\partial t}{\partial \tau}$$

This is the general heat conduction equ^n in spherical coordinates.

When there is no heat source & the heat flow is steady & 1-D, then we have

$$\cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2} = 0}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2.\frac{\partial t}{\partial r}\right) = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2.\frac{dt}{dr}\right) = 0$$

For general $\frac{1}{r^n}\frac{d}{dr}\left(r^n\frac{dt}{dr}\right) = 0$

$n = 0 \rightarrow$ Plane wall

$n = 1 \rightarrow$ cylinder

$n = 2 \rightarrow$ sphere

Heat Conduction With Internal Heat Generation →

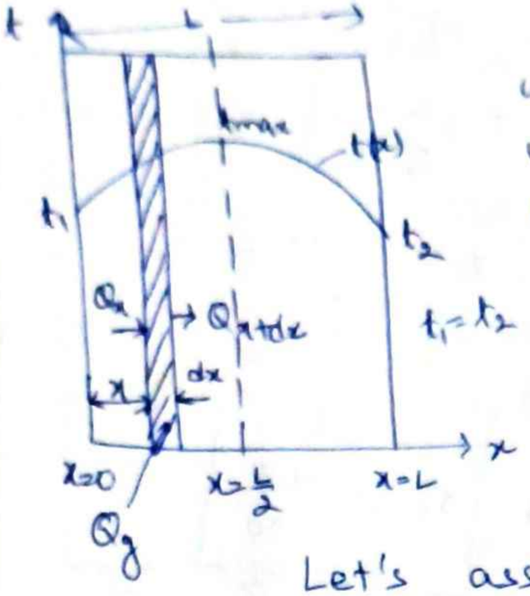
Some cases where heat generation & heat conduction are encountered →

i) Fuel rods - nuclear reactor
ii) Electrical conductors
iii) Chemical & combustion processes
iv) Drying & setting of concrete

In these cases the heat generation rate should be controlled otherwise the equipment may fail.

- Thus in the design of the thermal systems temp distribution within the medium & the rate of heat dissipation to the surroundin are very important.

Plane Wall with Uniform Heat Generation



Let's consider a plane wall of thickness L & uniform thermal conductivity K & with heat source.

Let the wall surfaces are maintained at temp. t_1 & t_2

Let's assume that heat flow is 1-D, steady & there is a uniform volumetric heat generation within the wall.

Now consider an element of thickness dx at a distance x from the left hand face of the wall.

Heat conducted in

$$Q_x = -KA\frac{dt}{dx}$$

Heat generated in the element

$$Q_g = A.dx.q_g$$

where q_g = heat generated per unit volume per unit time in the element

Heat conducted out

$$Q_{x+dx} = Q_x + \frac{d}{dx}(Q_x)dx$$

The energy balance on the element

$$Q_x + Q_g = Q_{x+dx} = Q_x + \frac{d}{dx}(Q_x)dx$$

$$\Rightarrow Q_g = \frac{d}{dx}(Q_x)dx$$

$$\Rightarrow q_g . A.dx = \frac{d}{dx}\left(-KA\frac{dt}{dx}\right) = -KA\frac{d^2t}{dx^2}.dx$$

$$\Rightarrow \frac{d^2t}{dx^2} + \frac{q_g}{K} = 0$$

Integrating this we have

$$\frac{dt}{dx} = -\frac{q_g}{K}x + C_1$$

Again integrating we have

$$t = -\frac{q_g}{2K}x^2 + C_1 x + C_2 \quad \text{—— (1)}$$

Case-1

Both surfaces have same temp.

The boundary cond's are

i) At $x = 0$, $t = t_1 = t_w$ $\quad$ t_w = Temp of wall surface

ii) At $x = L$, $t = t_2 = t_w$

Applying these boundary cond's in equ^n (1) we have

$$t_w = C_2$$

$$t_w = -\frac{q_g}{2K} \cdot L^2 + C_1 L + t_w$$

$$\Rightarrow C_1 = \frac{q_g}{2K} \cdot L$$

∴ Equ^n (1) becomes

$$t = -\frac{q_g}{2K}x^2 + \frac{q_g}{2K} \cdot L \cdot x + t_w$$

$$t = \frac{q_g}{2K}(L - x)x + t_w \quad \text{—— (2)}$$

Location of max^m temp.

Differentiating equ^n (2) & equating to zero we have

$$\frac{dt}{dx} = \frac{d}{dx}\left(\frac{q_g}{2K}(L-x)x + t_w\right) = 0$$

$$\Rightarrow -\frac{q_g \cdot 2x}{2K} + \frac{q_g}{2K} \cdot L = 0$$

$$\Rightarrow \frac{q_g}{2K}(L - 2x) = 0$$

$$\frac{q_g}{2k} \neq 0 \Rightarrow L - 2x = 0$$

$$\Rightarrow \boxed{x = \frac{L}{2}}$$

From equⁿ ② we have the temp. distribut is parabolic & symmetrical about the midpoint.

$$\therefore t_{max} = \left[\frac{q_g}{2K}(L - x) \cdot x\right]_{x = \frac{L}{2}} + t_w$$

$$= \frac{q_g}{2k}\left(L - \frac{L}{2}\right) \cdot \frac{L}{2} + t_w$$

$$\boxed{t_{max} = \frac{q_g}{8k} \cdot L^2 + t_w}$$

Heat transfer takes place towards both the surfaces.

For each surface heat transfer

$$Q = -kA\left(\frac{dt}{dx}\right)_{x=0 \text{ or } x=L}$$

$$= -KA\left[\frac{q_g}{2K}(L - 2x)\right]_{x=0 \text{ or } x=L}$$

$$\Rightarrow Q = \frac{AL}{2} q_g$$

When both surfaces are considered

$$Q = 2 \times \frac{AL}{2} q_g = AL \cdot q_g$$

Also heat conducted to each wall surface is further dissipated to the surrounding atmosphere t_a

Thus $\frac{AL}{2} \cdot q_g = hA(t_w - t_a)$

$$\Rightarrow \boxed{t_w = t_a + \frac{q_g}{2h} \cdot L}$$

$\therefore$ Equ^n ② becomes

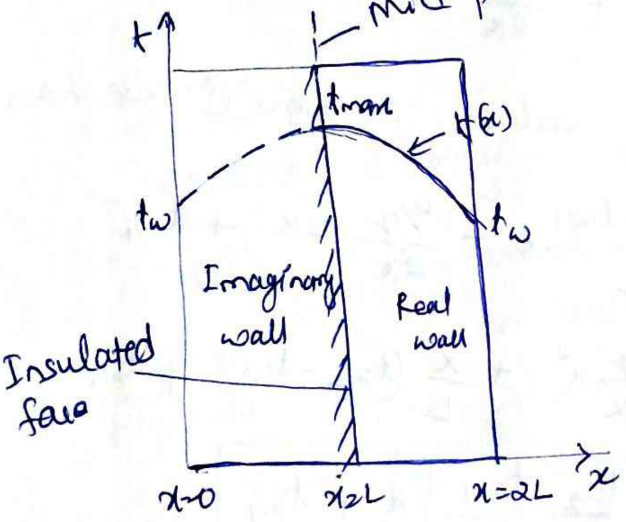
$$\boxed{t = t_a + \frac{q_g}{2h} \cdot L + \frac{q_g}{2K}(L-x)x} \quad -③$$

At $x = \frac{L}{2}$

$$t = t_{max} = t_a + \frac{q_g}{2h} \cdot L + \frac{q_g \cdot L^2}{8K}$$

$$\Rightarrow \boxed{t_{max} = t_a + q_g\left(\frac{L}{2h} + \frac{L^2}{8K}\right)}$$

Equ^n ③ is also works well in case of conduction in an insulated wall.



The boundary cond^{n}s for full hypothetical wall of thickness 2L

i) At $x = L$, $\frac{dt}{dx} = 0$

ii) At $x = 2L$, $t = t_w$

$x = L$ refers to the mid-plane of the hypothetical wall or insulated face of given wall.

The temp distribution & max^m temp at the mid-plane (insulated end of the given wall)

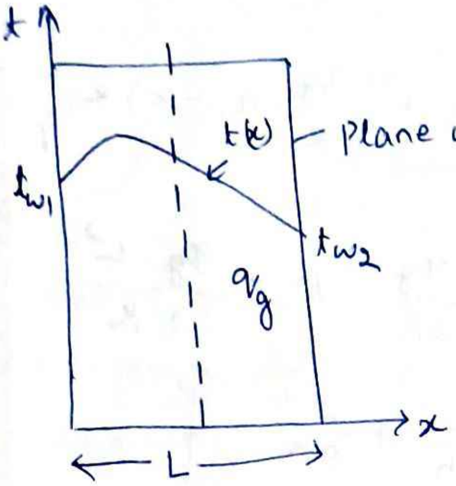
i.e. L = 2L

$$\therefore t = \frac{q_g}{2K}(2L-x)x + t_w$$

$$t_{max} = \frac{q_g}{2K}L^2 + t_w$$

Case-II

Both the surfaces of the wall have different temp. →



The boundary cond^{n}s are
i) At $x = 0$, $t = t_{w1}$
ii) At $x = L$, $t = t_{w2}$

Applying these boundary cond^{n}s in equ^n ① we have

$$C_2 = t_{w1}$$

$$t_{w2} = -\frac{q_g}{2K} L^2 + C_1 L + t_{w1}$$

$$\Rightarrow C_1 = \frac{t_{w2} - t_{w1}}{L} + \frac{q_g}{2K} \cdot L$$

∴ Substituting these values in equ^n ① we have

$$t = -\frac{q_g}{2K} x^2 + \frac{t_{w2} - t_{w1}}{L} x + \frac{q_g}{2K} \cdot Lx + t_{w1}$$

$$= \frac{q_g}{2K} \cdot L \cdot x - \frac{q_g}{2K} x^2 + \frac{x}{L}(t_{w2} - t_{w1}) + t_{w1}$$

$$\Rightarrow \boxed{t = \left[\frac{q_g}{2K}(L - x) + \frac{t_{w2} - t_{w1}}{L}\right] x + t_{w1}}$$

$$\Rightarrow t - t_{w2} = \left[\frac{q_g}{2K}(L - x) + \frac{t_{w2} - t_{w1}}{L}\right] x + t_{w1} - t_{w2}$$

$$\Rightarrow \frac{t - t_{w2}}{t_{w1} - t_{w2}} = \frac{q_g}{2K} \cdot \frac{L^2}{t_{w1} - t_{w2}} \left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] - \frac{x}{L} + 1$$

$$\Rightarrow \boxed{\frac{t - t_{w2}}{t_{w1} - t_{w2}} = \frac{q_g}{2K} \cdot \frac{L^2}{t_{w1} - t_{w2}} \cdot \frac{x}{L}\left(1 - \frac{x}{L}\right) + \left(1 - \frac{x}{L}\right)}$$

$$\left(1 - \frac{x}{L}\right)\left(Z\frac{x}{L} + 1\right)$$

Let $Z = \frac{q_g}{2K} \cdot \frac{L^2}{(t_{w1} - t_{w2})}$

Location of Max^n Temp.

$$\frac{dt}{d(x/L)} = \left(1-\frac{x}{L}\right)Z + \left(\frac{Zx}{L}+1\right)(-1) = 0$$

$$\Rightarrow Z - \frac{Zx}{L} - \frac{Zx}{L} - 1 = 0$$

$$\Rightarrow \boxed{\frac{x}{L} = \frac{Z-1}{2Z}}$$

$\therefore$ Max^n temp $\rightarrow$

$$\frac{t_{max} - t_{w_2}}{t_{w_1} - t_{w_2}} = \left[1 - \frac{Z-1}{2Z}\right]\left[Z \times \frac{Z-1}{2Z} + 1\right]$$

$$= \frac{Z+1}{2Z} \times \frac{Z+1}{2}$$

$$= \frac{(Z+1)^2}{4Z}$$

Case-III

Current carrying electrical conductor

When electrical current passes through a conductor, heat is generated (Q_g) in it is given by

$$Q_g = I^2 R \quad \text{where } R = \frac{\rho L}{A} \quad \rho = \text{Sp. resistance}$$

Also $Q_g = q_g \times A \times L$

$$\Rightarrow I^2 \times \frac{\rho L}{A} = q_g \times A \times L$$

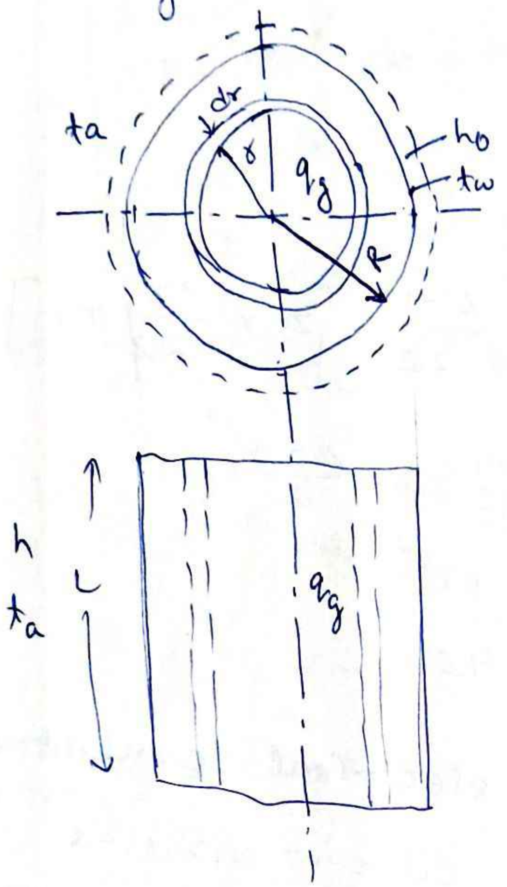
$$\Rightarrow q_g = \left(\frac{I}{A}\right)^2 \rho = J^2 \rho = \frac{J^2}{K_e}$$

J = Current density

K_e = Electrical conductivity (reciprocal of ρ)

Cylinder with uniform Heat Generation

Let's consider a cylindrical rod in which there is 1-D radial conduction under steady state cond's.



Let R = Radius of the rod

L = Length of rod

K = Thermal conductivity

q_g = Uniform volumetric heat genr" per unit volume per unit time

h = Heat transfer coefficient

t_a = Ambient temp

Now let's consider an element of radius r & thickness dr

Heat conducted in

$$Q_r = -K \cdot 2\pi r \cdot L \frac{dt}{dr}$$

Heat generated in the element

$$Q_g = q_g \cdot 2\pi r \cdot dr \cdot L$$

Heat conducted out

$$Q_{r+dr} = Q_r + \frac{d}{dr}(Q_r)\,dr$$

Under steady state conditions

$$Q_r + Q_g = Q_{r+dr} = Q_r + \frac{d}{dr}(Q_r)\,dr$$

$$\Rightarrow Q_g = \frac{d}{dr}\left(-K \cdot 2\pi r \cdot L \cdot \frac{dt}{dr}\right)dr$$

$$\Rightarrow q_g \cdot 2\pi r \cdot dr \cdot L = -K \cdot 2\pi L \frac{d}{dr}\left(r\frac{dt}{dr}\right) dr$$

$$\Rightarrow \frac{d}{dr}\left(r \cdot \frac{dt}{dr}\right) = -\frac{q_g}{K} \cdot r$$

Integrating this we have

$$r \cdot \frac{dt}{dr} = -\frac{q_g}{K} \cdot \frac{r^2}{2} + C_1$$

Again integrating we have

$$t = -\frac{q_g}{K} \cdot \frac{r^2}{4} + C_1 \log_e r + C_2$$

The boundary cond's are

i) At $r = R$, $t = t_w$

ii) Heat generated = Heat lost by cond^n

iii) At $r = 0$, $\frac{dt}{dr} = 0$ (∵ In case of cylinder, centre line is line of symmetry for temp distribution & temp gradient must be zero)

From B.Cs (ii) we have

~~$q_g \cdot 2\pi r \cdot dr \cdot L = -K \cdot 2\pi r$~~

$$q_g \cdot \pi R^2 \cdot L = -K \cdot 2\pi R L \cdot \frac{dt}{dr}\Big]_{r=R}$$

$$\Rightarrow \frac{dt}{dr}\Big|_{r=R} = -\frac{q_g R}{2K}$$

Again $$\frac{dt}{dr}\Big|_{r=R} = -\frac{q_g}{K} \cdot \frac{R}{2} + \frac{C_1}{R}$$

$$\Rightarrow -\frac{q_g R}{2K} = -\frac{q_g \cdot R}{2K} + \frac{C_1}{R}$$

$$\Rightarrow \boxed{C_1 = 0}$$

Applying B.C. (i)

$$\therefore t_w = -\frac{q_g R^2}{4K} + C_2$$

$$\Rightarrow \boxed{C_2 = t_w + \frac{q_g}{K} \cdot \frac{R^2}{4}}$$

$$\therefore t = -\frac{q_g}{K} \cdot \frac{r^2}{4} + t_w + \frac{q_g}{K} \cdot \frac{R^2}{4}$$

$$\Rightarrow \boxed{t = t_w + \frac{q_g}{4K}\left(R^2 - r^2\right)}$$

⇒ The temp. distribution is parabolic & max^n temp. occurs at centre $(r=0)$

$$\boxed{t_{max} = t_w + \frac{q_g}{4K} \cdot R^2}$$

$$t_{max} - t_w = \frac{q_g}{4K} \cdot R^2$$

$$t - t_w = \frac{q_g}{4K}\left(R^2 - r^2\right)$$

$$\Rightarrow \frac{t - t_w}{t_{max} - t_w} = \frac{\frac{q_g}{4K}\left(R^2 - r^2\right)}{\frac{q_g}{4K} \cdot R^2} = \frac{R^2 - r^2}{R^2} = 1 - \left(\frac{r}{R}\right)^2$$

$$\Rightarrow \boxed{\frac{t - t_w}{t_{max} - t_w} = 1 - \left(\frac{r}{R}\right)^2}$$

Also energy generated = energy dissipated by convection at the boundary

$$q_g \cdot \pi R^2 \cdot L = h \times 2\pi R L \left(t_w - t_a\right)$$

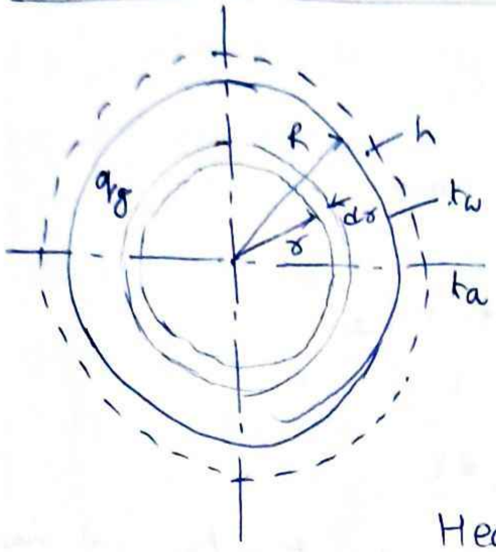
$$\Rightarrow \boxed{t_w = t_a + \frac{q_g}{2h} \cdot R}$$

$$\therefore \boxed{t = t_a + \frac{q_g}{2h} \cdot R + \frac{q_g}{4K}\left(R^2 - r^2\right)}$$

$\therefore t_{max}$ at $r = 0$

$$t_{max} = t_a + \frac{q_g}{2h} \cdot R + \frac{q_g}{4K} \cdot R^2$$

Sphere with uniform Heat Genera?



Heat conducted in

$$Q_r = -KA\frac{dt}{dr}$$

$$= -K\cdot 4\pi r^2\cdot\frac{dt}{dr}$$

Heat generated

$$Q_g = q_g\cdot 4\pi r^2\cdot dr$$

Heat conducted Out

$$Q_{r+dr} = Q_r + \frac{d}{dr}(Q_r)\,dr$$

Under steady state

$$Q_r + Q_g = Q_{r+dr}$$

$$\Rightarrow Q_g = \frac{d}{dr}(Q_r)\,dr$$

$$\Rightarrow q_g\cdot 4\pi r^2\,dr = \frac{d}{dr}\left(-K\,4\pi r^2\cdot\frac{dt}{dr}\right)dr$$

$$\Rightarrow q_g\cdot 4\pi r^2\cdot dr = -4\pi K\frac{d}{dr}\left(r^2\frac{dt}{dr}\right)dr$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dt}{dr}\right) + \frac{q_g}{K} = 0$$

$$\Rightarrow \frac{1}{r^2}\left[r^2\frac{d^2t}{dr^2} + 2r\cdot\frac{dt}{dr}\right] + \frac{q_g}{K} = 0$$

$$\Rightarrow \frac{d^2t}{dr^2} + \frac{2}{r}\cdot\frac{dt}{dr} + \frac{q_g}{K} = 0$$

$$\Rightarrow r\frac{d^2t}{dr^2} + 2\frac{dt}{dr} + \frac{q_g r}{K} = 0$$

$$\Rightarrow r\frac{d^2t}{dr^2} + \frac{dt}{dr} + \frac{dt}{dr} + \frac{q_g r}{K} = 0$$

$$\Rightarrow \frac{dt}{dr}\left(r\frac{dt}{dr}\right) + \frac{dt}{dr} + \frac{q_g r}{K} = 0$$

Integrating this we have

$$r\frac{dt}{dr} + t + \frac{q_g}{K}\frac{r^2}{2} = C_1$$

$$\Rightarrow \frac{d}{dr}(rt) + \frac{q_g}{K}\frac{r^2}{2} = C_1$$

Again integrating we have

$$rt + \frac{q_g}{K}\cdot\frac{r^3}{6} = C_1 r + C_2$$

At centre $r=0$ $C_2 = 0$

Boundary condⁿ at $r = R$ $t = t_w$ we have

$$Rt_w + \frac{q_g}{K}\cdot\frac{R^3}{6} = C_1 R$$

$$\Rightarrow C_1 = t_w + \frac{q_g}{6K}R^2$$

$$\therefore \boxed{t = t_w + \frac{q_g}{6K}(R^2 - r^2)}$$

$\Rightarrow$ The temp. distribution is parabolic & maxⁿ temp occurs at centre

$$t_{max} = t_w + \frac{q_g}{6K}R^2$$

$$\therefore \frac{t - t_w}{t_{max} - t_w} = \frac{R^2 - r^2}{R^2} = 1 - \left(\frac{r}{R}\right)^2$$

$$\Rightarrow \boxed{\frac{t - t_w}{t_{max} - t_w} = 1 - \left(\frac{r}{R}\right)^2}$$ → Temp. distribution in dimensionless form.

Under steady state heat generated = heat convected from the sphere

$$\Rightarrow q_g \cdot \frac{4}{3}\pi R^3 = h \times 4\pi R^2 (t_w - t_a)$$

$$\Rightarrow t_w = t_a + \frac{q_g R}{3h}$$

$$\therefore \boxed{t = t_a + \frac{q_g R}{3h} + \frac{q_g}{6K}(R^2 - r^2)}$$

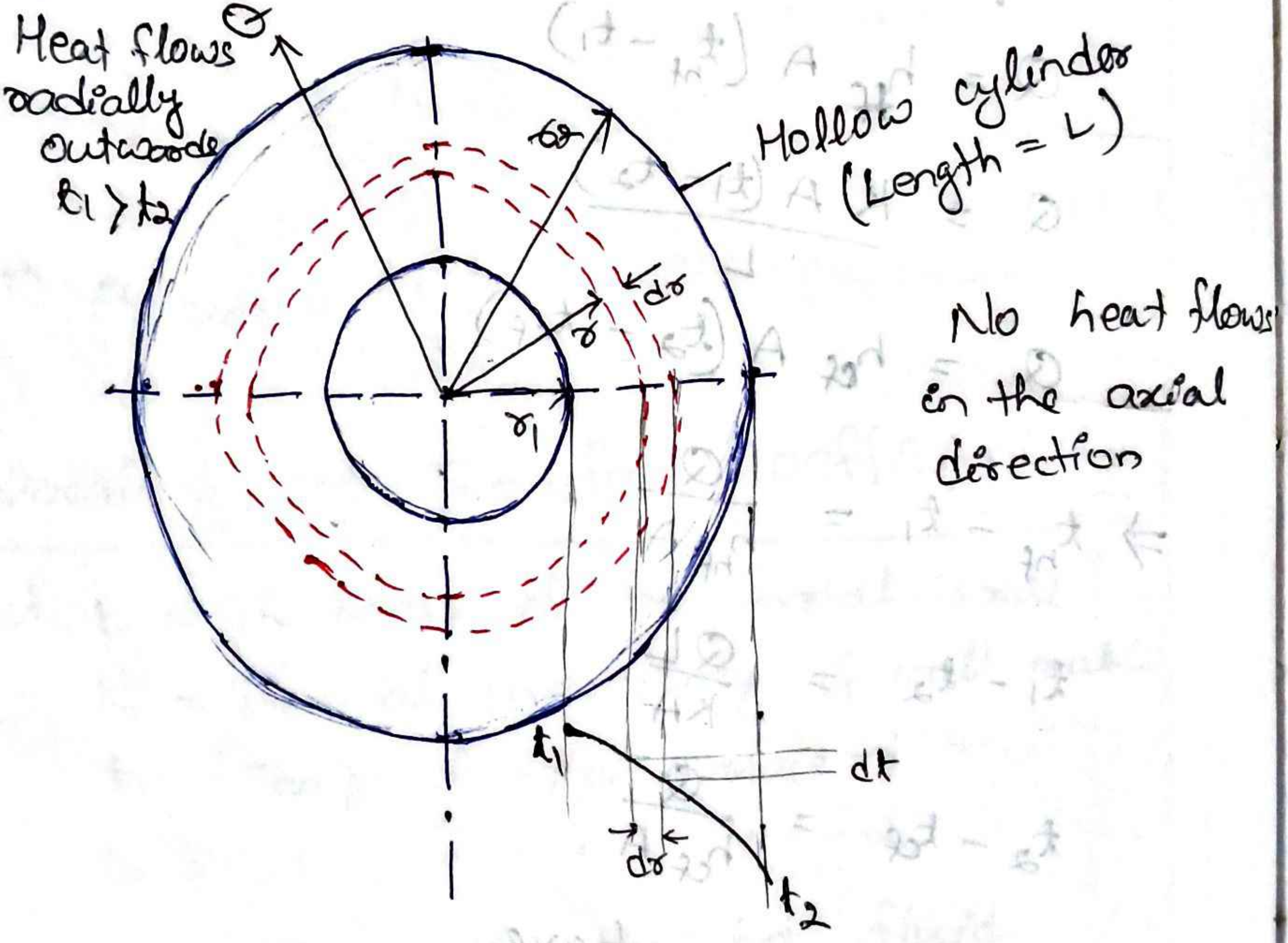
max^n temp

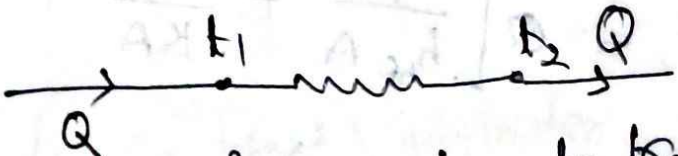
$$\boxed{t_{max} = t_a + \frac{q_g}{3h} \cdot R + \frac{q_g}{6K} R^2} \quad (r=0)$$

Heat Conduction Through a Hollow Cylinder

Case-1 → Uniform conductivity

Let's consider a hollow cylinder made of material having constant thermal conductivity & insulated at both ends.





$$R_{th} = \frac{1}{2\pi KL} \ln (r_2/r_1)$$

Let r_1, r_2 = Inner & outer radii

t_1, t_2 = Temp. of inner & outer surfa

K = Constant thermal conductivity

The general heat condⁿ equⁿ in cylindrical co-ordinates is

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{K} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

For steady state, unidirectional heat flow in axial direction & with no internal heat generation, the equⁿ becomes

$$\frac{d^2 t}{dr^2} + \frac{1}{r}\frac{dt}{dr} = 0$$

$$\Rightarrow \frac{d}{dr}\left(r \cdot \frac{dt}{dr}\right) = 0$$

$$\Rightarrow r \cdot \frac{dt}{dr} = c$$

Integrating this we have

$$t = c \ln(r) + C_1$$

The boundary condⁿs are

i) At $r = r_1$, $t = t_1$

ii) At $r = r_2$, $t = t_2$

Applying this we have

$$t_1 = c \ln r_1 + C_1$$

$$t_2 = c \ln r_2 + C_1$$

$$\Rightarrow c = -\frac{(t_1 - t_2)}{\ln(r_2/r_1)} \quad \& \quad C_1 = t_1 + \frac{t_1 - t_2}{\ln(r_2/r_1)} \ln r_1$$

$$\therefore t = t_1 + \frac{t_1 - t_2}{\ln(r_2/r_1)} \ln r_1 - \frac{t_1 - t_2}{\ln(r_2/r_1)} \ln r$$

$\Rightarrow (t - t_1) \ln(r_2/r_1) = (t_1 - t_2) \ln r_1 - (t_1 - t_2) \ln r$

$= (t_2 - t_1) \ln r - (t_2 - t_1) \ln r_1$

$= (t_2 - t_1) \ln (r/r_1)$

$$\Rightarrow \boxed{\frac{t - t_1}{t_2 - t_1} = \frac{\ln (r/r_1)}{\ln (r_2/r_1)}}$$

From this equ^n we have

i) The temp. distribution is logarithmic

ii) Temp. at any point in the cylinder can be expressed as a fun^n of radius only. Isotherms (or lines of constant temp) are concentric circles lying bet^n the inner & outer ~~surface~~ boundaries of the hollow cylinder

iii) The temp. profile is nearly linear for values of $\frac{r_2}{r_1}$ of the order of 1 but non-linear for large values of r_2/r_1.

Conduction Heat transfer Rate (Q)

Let's consider an element at radius r & thickness 'dr' for a length of hollow cylinder. Let dt be the temp. drop over the element.

Area through which heat is transmitted

$A = 2\pi r . L$

Path length = dr

$\therefore Q = - KA \frac{dt}{dr}$

$= - K \times 2\pi r L . \frac{dt}{dr}$ per unit time

$\Rightarrow Q . \frac{dr}{r} = - K 2\pi L . dt$

Integrating both sides we have

$Q \int_{r_1}^{r_2} \frac{dr}{r} = - K . 2\pi L \int_{t_1}^{t_2} dt$

$$\Rightarrow Q \ln(r)\Big]_{r_1}^{r_2} = -k\, 2\pi L \cdot t\Big]_{t_1}^{t_2}$$

$$\Rightarrow Q \cdot \ln(r_2/r_1) = -k \cdot 2\pi L \cdot (t_2 - t_1)$$

$$\Rightarrow Q = \frac{k \cdot 2\pi L \cdot (t_1 - t_2)}{\ln(r_2/r_1)} = \frac{t_1 - t_2}{\frac{\ln(r_2/r_1)}{2\pi k L}}$$

Case-II Variable Thermal Conductivity

A) Temp. variation in terms of t_1 & t_2

We know $Q = -KA\frac{dt}{dr}$ $\quad k = k_0(1+\beta t)$

$$= -k_0(1+\beta t)\, 2\pi r L \frac{dt}{dr}$$

$$\Rightarrow Q\frac{dr}{r} = -k_0 2\pi L (1+\beta t)\, dt$$

$$\Rightarrow Q\int_{r_1}^{r_2}\frac{dr}{r} = -k_0 2\pi L\int_{t_1}^{t_2}(1+\beta t)\, dt$$

$$\Rightarrow Q\ln(r_2/r_1) = -k_0 2\pi L\left[(t_2 - t_1) + \frac{\beta}{2}(t_2^2 - t_1^2)\right]$$

$$= k_0 2\pi L\left[1 + \frac{\beta}{2}(t_1 + t_2)\right](t_1 - t_2)$$

$$\Rightarrow Q = \frac{k_0 2\pi L\left[1+\frac{\beta}{2}(t_1+t_2)\right](t_1 - t_2)}{\ln(r_2/r_1)}$$

Integrating betn r_1 & r we have

$$Q = \frac{k_0 2\pi L\left[1+\frac{\beta}{2}(t_1+t)\right](t_1 - t)}{\ln(r/r_1)}$$

$$\Rightarrow \frac{k_0 2\pi L\left[1+\frac{\beta}{2}(t_1+t_2)\right](t_1 - t_2)}{\ln(r_2/r_1)} = \frac{k_0 2\pi L\left[1+\frac{\beta}{2}(t_1+t)\right](t_1 - t)}{\ln(r/r_1)}$$

$$\Rightarrow t = -\frac{1}{\beta} \pm \frac{1}{\beta}\left[(1+\beta t_1)^2 - \frac{\ln(r/r_1)}{\ln(r_2/r_1)}\left\{(1+\beta t_1)^2 - (1+\beta t_2)^2\right\}\right]^{1/2}$$

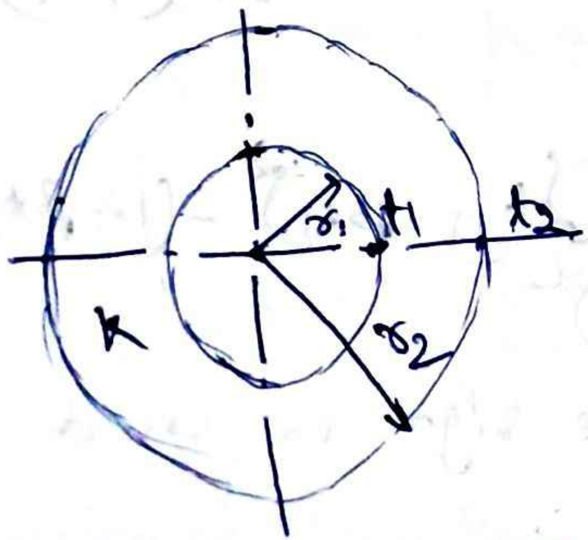
Since $t = t_2$ where $r = r_2$ only +ve sign is used.

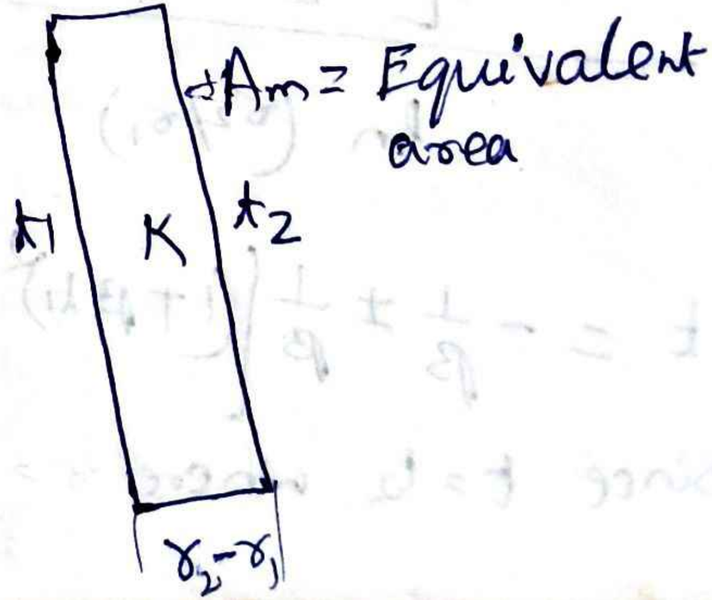
Logarithmic Mean Area for the Hollow cylinder →

It is convienient to have an expression for the heat flow through a hollow cylinder of the same form as that for a plane wall.

Then thickness $= r_2 - r_1$

Area A will be an equivalent area A_m





We know heat flow through cylinder,

$$Q = \frac{t_1 - t_2}{\frac{\ln(r_2/r_1)}{2\pi K L}}$$

Heat flow through plane wall

$$Q = \frac{t_1 - t_2}{\frac{r_2 - r_1}{K A_m}}$$

Am is so choosen that heat flow through cylinder & plane wall will be equal

$$\therefore \frac{t_1 - t_2}{\frac{\ln(r_2/r_1)}{2\pi K L}} = \frac{t_1 - t_2}{\frac{r_2 - r_1}{K A_m}}$$

$$\Rightarrow A_m = \frac{2\pi L (r_2 - r_1)}{\ln(r_2/r_1)} = \frac{2\pi L r_2 - 2\pi L r_1}{\ln\left(\frac{2\pi L r_2}{2\pi L r_1}\right)}$$

$$\Rightarrow \boxed{A_m = \frac{A_o - A_i}{\ln(A_o/A_i)}}$$

o - outside
i - inside

This expression is known as logarithmic mean area of the plane wall & hollow cylinder.

- By using this expression a cylinder can be transformed into a plane wall & the problem can be solved easily.

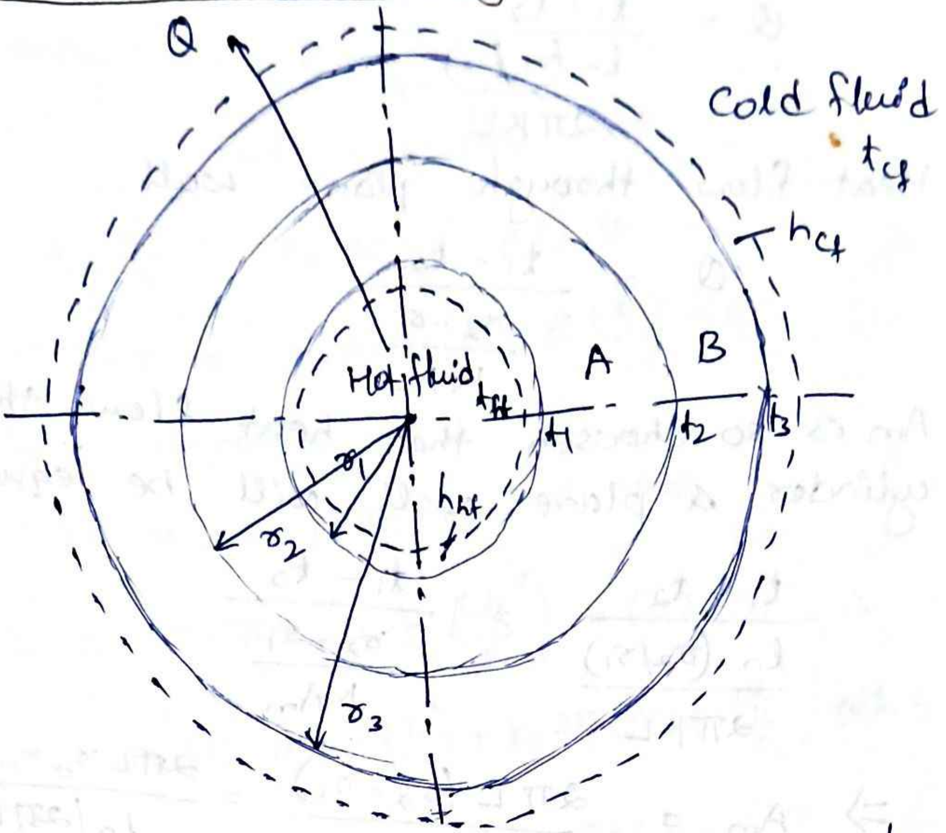
If $\frac{A_o}{A_i} < 2$ then we can take $A_{av} = \frac{A_i + A_o}{2}$

Further $A_m = 2\pi r_m L = \frac{2\pi L (r_2 - r_1)}{\ln(r_2/r_1)}$

$\therefore$ Logarithmic mean radius

$$\boxed{r_m = \frac{r_2 - r_1}{\ln(r_2/r_1)}}$$

Heat Conduction through a Composite Cylinder



Let's consider flow of heat through a composite cylinder.

Let t_{hf} = Temp of hot fluid

t_{cf} = " " cold "

K_A = Thermal conductivity of layer A

K_B = " " " " B

t_1, t_2, t_3 = Temp. at 1, 2, 3

L = Length of the composite cylinder

h_{cf}, h_{hf} = outside & inside heat transfer coefficients

The rate of heat transfer is

$$Q = h_{hf} \cdot 2\pi r_1 L (t_{hf} - t_1) = \frac{K_A \, 2\pi L (t_1 - t_2)}{\ln(r_2/r_1)}$$

$$= \frac{K_B \, 2\pi L (t_2 - t_3)}{\ln(r_3/r_2)} = h_{cf} \cdot 2\pi r_3 L (t_3 - t_{cf})$$

$$\Rightarrow t_{hf} - t_1 = \frac{Q}{h_{hf}\, r_1\, 2\pi L}$$

$$t_1 - t_2 = \frac{Q}{\frac{K_A \cdot 2\pi L}{\ln(r_2/r_1)}}$$

$$t_2 - t_3 = \frac{Q}{\frac{K_B \cdot 2\pi L}{\ln(r_3/r_2)}}$$

$$t_3 - t_{cf} = \frac{Q}{h_{cf} \cdot r_3 \cdot 2\pi L}$$

Adding all we have

$$t_{hf} - t_{cf} = \frac{Q}{2\pi L}\left[\frac{1}{h_{hf} \cdot r_1} + \frac{1}{\frac{K_A}{\ln(r_2/r_1)}} + \frac{1}{\frac{K_B}{\ln(r_3/r_2)}} + \frac{1}{h_{cf} \cdot r_3}\right]$$

$$\Rightarrow Q = \frac{2\pi L \cdot (t_{hf} - t_{cf})}{\frac{1}{h_{hf} \cdot r_1} + \frac{\ln(r_2/r_1)}{K_A} + \frac{\ln(r_3/r_2)}{K_B} + \frac{1}{h_{cf} \cdot r_3}}$$

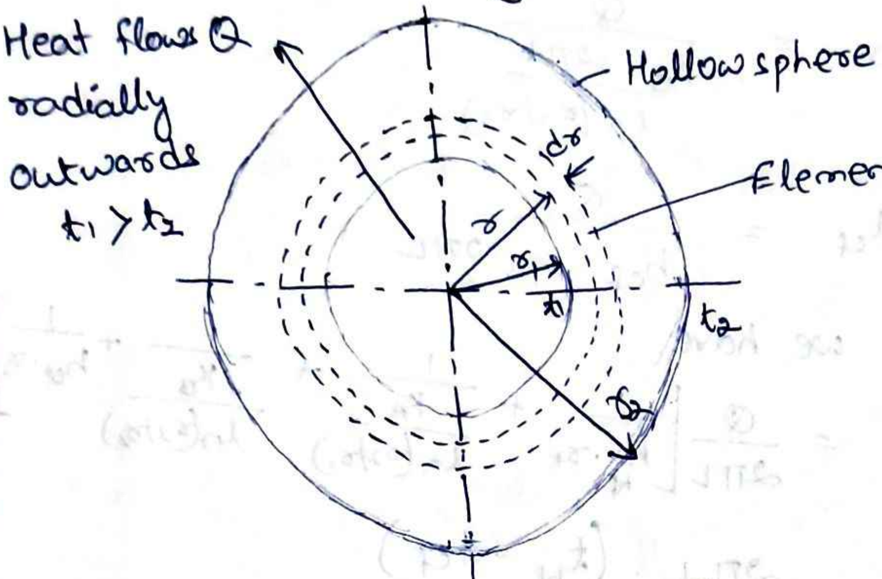
If there are n concentric cylinders, then

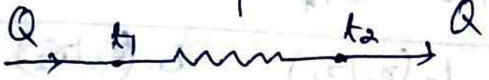
$$Q = \frac{2\pi L\, (t_{hf} - t_{cf})}{\frac{1}{h_{hf} \cdot r_1} + \sum_{n=1}^{n} \frac{1}{K_n} \ln(r_{n+1}/r_n) + \frac{1}{h_{cf} \cdot r_{n+1}}}$$

Heat Conduction Through Hollow Sphere

Case-I Uniform conductivity

Let's consider a hollow sphere made of material having constant thermal conductivity





$$R_{th} = \frac{r_2 - r_1}{4\pi K r_1 r_2}$$

Let r_1, r_2 = Inner & outer radii

t_1, t_2 = Temp. of inner & outer surfaces

K = Constant thermal conductivity

The general heat condⁿ equⁿ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{q_g}{K} = \frac{1}{\alpha}\cdot\frac{\partial t}{\partial \tau}$$

For steady state, unidirectional flow in the radial dirⁿ & with no heat generation the equⁿ becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dt}{dr}\right) = 0$$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dt}{dr}\right) = 0$$

$$\Rightarrow r^2\frac{dt}{dr} = C$$

Integrating this we have

$$t = -\frac{c}{r} + C_1$$

Applying boundary cond's we have

At $r = r_1$, $t = t_1$

At $r = r_2$, $t = t_2$

$$\Rightarrow t_1 = -\frac{c}{r_1} + C_1$$

$$t_2 = -\frac{c}{r_2} + C_1$$

$$\therefore c = \frac{(t_1 - t_2) r_1 r_2}{r_1 - r_2}$$

$$\& \ C_2 = t_1 + \frac{(t_1 - t_2) r_1 r_2}{r_1 (r_1 - r_2)}$$

$$\therefore t = -\frac{(t_1 - t_2) r_1 r_2}{(r_1 - r_2) r} + t_1 + \frac{(t_1 - t_2) r_1 r_2}{r_1 (r_1 - r_2)}$$

$$= t_1 + \left[\frac{t_1 - t_2}{r_1 - r_2}\right]\left(\frac{1}{r_1} - \frac{1}{r}\right)$$

$$\Rightarrow \boxed{\frac{t - t_1}{t_2 - t_1} = \frac{r_2}{r}\left(\frac{r - r_1}{r_2 - r_1}\right)}$$

$\therefore$ The temp. distribution through a sphere is represented by a hyperbola.

<u>The cond^n heat transfer rate</u>

$$Q = -KA\frac{dt}{dr}$$

$$= -K \cdot 4\pi r^2 \cdot \frac{dt}{dr}$$

$$\Rightarrow Q\int_{r_1}^{r_2} \frac{dr}{r^2} = -\int_{t_1}^{t_2} 4\pi K \, dt$$

$$\Rightarrow -Q\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = -4\pi K(t_2 - t_1)$$

$$\Rightarrow Q = \frac{4\pi K r_1 r_2 (t_1 - t_2)}{r_2 - r_1} = \frac{t_1 - t_2}{\frac{r_2 - r_1}{4\pi K r_1 r_2}}$$

Case-II Variable conductivity

$$K = K_0(1+\beta t)$$

$$\therefore Q = \frac{4\pi K_0 r_1 r_2}{r_2 - r_1}\left[1 + \frac{\beta}{2}(t_1 + t_2)\right](t_1 - t_2)$$

$$\& \; t = \frac{1}{\beta}\left[(1+\beta t_1)^2 - \left(\frac{r - r_1}{r_2 - r_1}\right)\frac{r_2}{r}\left\{(1+\beta t_1)^2 - (1+\beta t_2)^2\right\}\right] - \frac{1}{\beta}$$

Temp. variation in terms of Q

$$t = -\frac{1}{\beta} + \left[\left(t_1 - \frac{1}{\beta}\right)^2 - \frac{Q}{\beta K_0}\cdot\frac{1}{2\pi}\left(\frac{1}{r_1} - \frac{1}{r}\right)\right]^{1/2}$$

Logarithmic mean area for the hollow sphere →

We know $Q_{sphere} = \dfrac{t_1 - t_2}{\left(\frac{r_2 - r_1}{4\pi K r_1 r_2}\right)}$

$$Q_{plane\ wall} = \frac{t_1 - t_2}{\frac{r_2 - r_1}{K A_m}}$$

A_m is so choosen that the heat flow through cylinder & plane wall will be equal ~~for the~~

$$Q_{sphere} = Q_{plane\ wall}$$

$$\Rightarrow \frac{t_1 - t_2}{\frac{r_2 - r_1}{4\pi K r_1 r_2}} = \frac{t_1 - t_2}{\frac{r_2 - r_1}{K A_m}}$$

$$\Rightarrow \boxed{A_m = 4\pi r_1 r_2}$$

$$\Rightarrow A_m^2 = (4\pi r_1 r_2)^2 = 4\pi r_1^2 \times 4\pi r_2^2$$

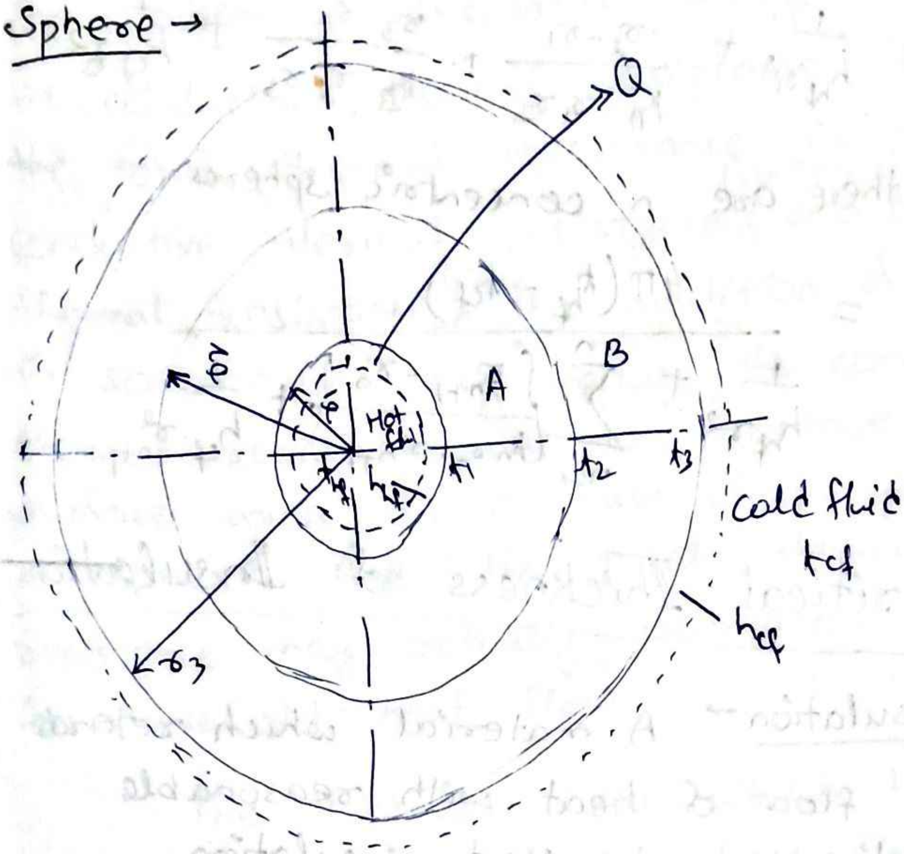
$$= A_1 \times A_0$$

$$\Rightarrow \boxed{A_m = \sqrt{A_1 A_0}}$$

Again $A_m = 4\pi r_m^2 = 4\pi r_1 r_2$

$$\Rightarrow \boxed{r_m = \sqrt{r_1 r_2}}$$

Heat Conduction Through a Composite Sphere →



The heat flow equⁿ can be written as

$$Q = h_{hf} \cdot 4\pi r_1^2 (t_{hf} - t_1) = \frac{4\pi K_A r_1 r_2 (t_1 - t_2)}{r_2 - r_1}$$

$$= \frac{4\pi K_B r_2 r_3 (t_2 - t_3)}{r_3 - r_2} = h_{cf} \cdot 4\pi r_3^2 (t_3 - t_{cf})$$

$$\Rightarrow t_{hf} - t_1 = \frac{Q}{h_{hf} \cdot 4\pi r_1^2}$$

$$t_1 - t_2 = \frac{Q(r_2 - r_1)}{4\pi K_A r_1 r_2}$$

$$t_2 - t_3 = \frac{Q(r_3 - r_2)}{4\pi K_B r_2 r_3}$$

$$t_3 - t_{cf} = \frac{Q}{h_{cf} \cdot 4\pi r_3^2}$$

Adding all we have

$$\frac{Q}{4\pi}\left[\frac{1}{h_{hf}\cdot r_1^2}+\frac{r_2-r_1}{k_A\cdot r_1 r_2}+\frac{r_3-r_2}{k_B\cdot r_2 r_3}+\frac{1}{h_{cf}\cdot r_3^2}\right]=t_{hf}-t_{cf}$$

$$\Rightarrow Q=\frac{4\pi(t_{hf}-t_{cf})}{\frac{1}{h_{hf}r_1^2}+\frac{r_2-r_1}{k_A\cdot r_1 r_2}+\frac{r_3-r_2}{k_B\cdot r_2 r_3}+\frac{1}{h_{cf}r_3^2}}$$

If there are n concentric spheres

$$Q=\frac{4\pi(t_{hf}-t_{cf})}{\frac{1}{h_{hf}r_1^2}+\sum_{n=1}^{n}\left\{\frac{r_{n+1}-r_n}{k_n\cdot r_n\cdot r_{n+1}}\right\}+\frac{1}{h_{cf}\cdot r_{n+1}^2}}$$

Critical Thickness of Insulation

Insulation → A material which retards the flow of heat with reasonable effectiveness is called insulation.

Purpose

- It prevents the heat flow from the system to the surrounding
- It prevents the heat from the surrounding to the system.

Application

- Boilers & Steam pipes
- Air conditioning system
- Food preserving stores & refrigerators
- Insulating bricks
- Preservation of liquid gases

- The value of K increases with increase in temp
- K decreases with decrease in pressure.

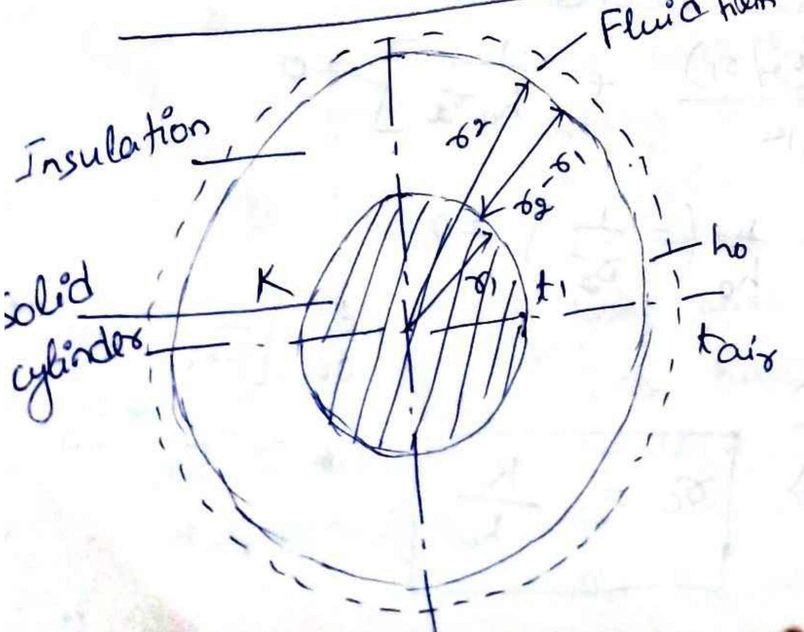
Critical Thickness of Insulation

- The addition of insulation always increases the conductive thermal resistance, but when the total thermal resistance is made of conductive thermal resistance $[\ln(r_2/r_1)/k]$ and convective thermal resistance $(\frac{1}{h r_2})$ the addition of insulation in some cases may reduce the convective thermal resistance due to increase in surface area, as in case of a cylinder & sphere. And the total thermal resistance may actually decrease resulting in increased heat flow.

The thickness upto which heat flow increases & after which heat flow decreases is called <u>Critical Thickness</u>.

- In case of cylinders & spheres it is called Critical radius.

Critical thickness of insulation for cylinder



Let's consider a solid cylinder of radius r_1 insulated with an insulation of thickness $(r_2 - r_1)$

Let L = Length of the cylinder

t_1 = surface temp. of the cylinder

t_{air} = Temp. of air

h_o = Heat transfer coefficient at the outer surface of the insulation

K = Thermal conductivity of insulating material.

The rate of heat transfer

$$Q = \frac{2\pi L (t_1 - t_{air})}{\frac{\ln(r_2/r_1)}{K} + \frac{1}{h_o r_2}} \quad \text{--- (1)}$$

From this equ^n we have as r_2 increases, the factor $\frac{\ln(r_2/r_1)}{K}$ increases but the factor $\frac{1}{h_o r_2}$ decreases

Thus Q becomes max^m when

$\left[\frac{\ln(r_2/r_1)}{K} + \frac{1}{h_o r_2}\right]$ becomes min^m

$\therefore$ The required cond^n is

$$\frac{d}{dr_2}\left[\frac{\ln(r_2/r_1)}{K} + \frac{1}{h_o r_2}\right] = 0$$

$$\Rightarrow \frac{r_1}{r_2} \times \frac{1}{r_1} \times \frac{1}{K} + \frac{1}{h_o}\left(-\frac{1}{r_2^2}\right) = 0$$

$$\Rightarrow \frac{1}{r_2 K} = \frac{1}{h_o r_2^2}$$

$$\Rightarrow r_2 = \frac{K}{h_o} \quad \Rightarrow \boxed{r_c = \frac{K}{h_o}}$$

$\frac{d}{dr_2}\left[\frac{1}{K r_2} - \frac{1}{h_o r_2^2}\right] = -\frac{1}{K r_2^2} + \frac{2}{h_o r^3}$

$r = \frac{k}{h}$ $\quad \frac{d^2L}{dr^2} = +ve$

This is the condⁿ for minᵐ resistance & consiquently maxⁿ heat flow rate.

- The insulation radius at which resistance to heat flow is minᵐ is called the <u>critical radius</u> (r_c)

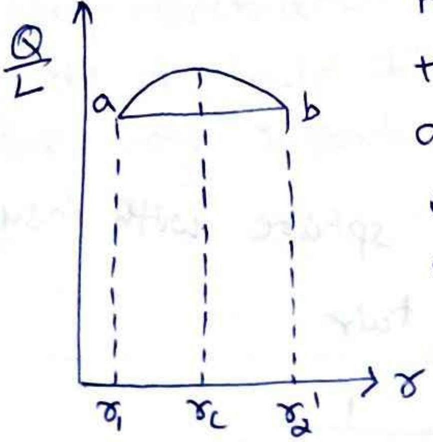
- The critical radius (r_c) depends on K, h_o
- r_c is independent of r_1 (cylinder radius)
- The 2nd derivative of the denominator $\left[\frac{\ln(r_2/r_1)}{K} + \frac{1}{h_o r_2}\right]$ is +ve so heat flow rate will be maxⁿ when $r_2 = r_c$

In equⁿ-① the conductive thermal resistance $\frac{\ln(r_2/r_1)}{K}$ will increase with increase in r_2 & convective thermal resistance $\frac{1}{h_o r_2}$ will decrease with increase in r_2.

At $r_2 = r_c$, the rate of increase of conductive resistance of insulation is equal to the rate of decrease of convective resistance thus, the sum of thermal resistances is minᵐ.

Conclusions →

①



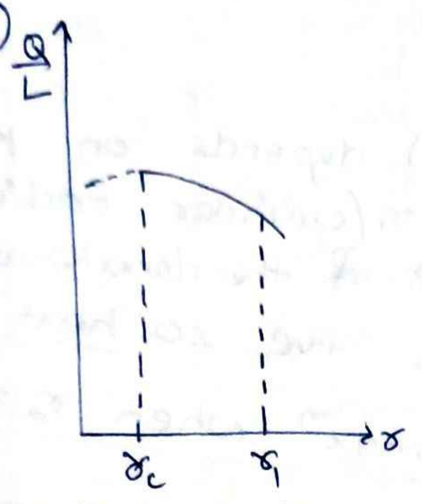
cylinder radius
$r_1 \leq r_c = \frac{K}{h_o}$

For cylindrical bodies with $r_1 < r_c$ the heat transfer increases by adding insulation till $r_2 = r_c$, If insulation thickness is further increased, the rate of heat loss will decrease from peak

This happens when r_1 is small & r_c is large i.e the thermal conductivity of the insulation (K) is high (poor insulating material) & h_o is low.

Practical application :- The insulation of electric cables which should be a good insulator for current but poor for heat.

(2)



For cylindrical bodies with $r_1 > r_c$, the heat transfer decreases by adding insulation. This happens when r_1 is large & r_c is small i.e. a good insulating material is used with low K & h_o is high.

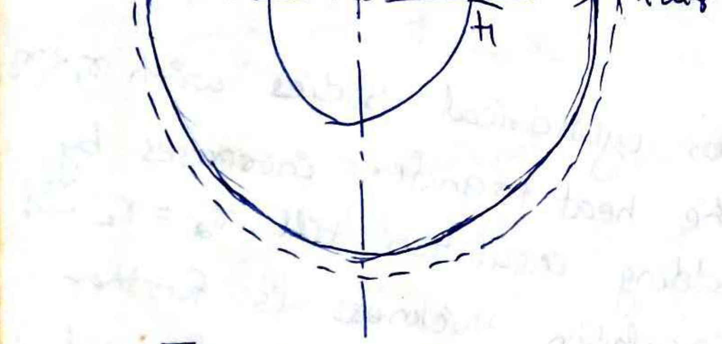
cylinder radius

$r_1 > r_c = \frac{K}{h_o}$

Application → steam & refrigeration pipes

∴ For effective insulation the outer radius must be greater than or equal to the critical radius.

Critical thickness of insulation for sphere



The heat flow through sphere with insulation

$$Q = \frac{t_1 - t_{air}}{\frac{r_2 - r_1}{4\pi K r_1 r_2} + \frac{1}{h_o \cdot 4\pi r_2^2}}$$

Our requirement condⁿ is

$$\frac{d}{dr_2}\left[\frac{r_2-r_1}{4\pi K r_1 r_2} + \frac{1}{4\pi r_2^2 \cdot h_0}\right] = 0$$

$$\Rightarrow \frac{d}{dr_2}\left[\frac{1}{Kr_1} - \frac{1}{Kr_2} + \frac{1}{r_2^2 h_0}\right] = 0$$

$$\Rightarrow \frac{1}{Kr_2^2} - \frac{2}{r_2^3 h_0} = 0$$

$$\Rightarrow r_2^3 h_0 = 2Kr_2^2$$

$$\Rightarrow r_2 = \frac{2K}{h_0} \Rightarrow \boxed{r_c = \frac{2K}{h_0}}$$

Effectiveness - NTU Method

LMTD method is useful when the inlet & outlet temps are known or easily determined. Then

$$\dot{Q} = UA_s \Delta T_{lm}$$

When the inlet or exit temps are to be evaluated for a given HX then NTU method is used.

This method is based on a dimensionless parameter called the heat transfer effectiveness,

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{max}} = \frac{\text{Actual heat transfer rate}}{\text{Max}^m \text{ possible heat transfer rate}}$$

The actual heat transfer rate

$$\dot{Q} = m_h C_{ph}(T_{h,i} - T_{h,o}) = m_c C_{pc}(T_{c,o} - T_{c,i})$$

$$= C_h(T_{h,i} - T_{h,o}) = C_c(T_{c,o} - T_{c,i})$$

<u>Max^m possible heat transfer rate</u> →

We know max^m temp. difference in a HX is

$$\Delta T_{max} = T_{h,i} - T_{c,i}$$

Usually $C_c \neq C_h$. In this case the fluid with the smaller heat capacity rate will experience a larger temp. change.

∴ Max^m possible heat transfer rate

$$\dot{Q}_{max} = C_{min}(T_{h,i} - T_{c,i})$$

Where C_{min} = smaller of C_h & C_c

∴ The actual heat transfer rate

$$\dot{Q} = \varepsilon \dot{Q}_{max}$$

$$= \varepsilon \cdot C_{min}(T_{hi} - T_{ci})$$

Therefore, the effectiveness of a HX enables us to determine the heat transfer rate without knowing the outlet temps of the fluids.

The effectiveness of a HX depends on the geometry of the HX & the flow arrangement

Let's consider a double-pipe parallel-flow HX.

We know ——— ①

$$d\dot{Q} = -m_h c_{ph} dT_h = m_c c_{pc} dT_c$$

$$\Rightarrow dT_h = \frac{-d\dot{Q}}{m_h c_{ph}} \quad \& \quad dT_c = \frac{d\dot{Q}}{m_c c_{pc}}$$

Now

$$dT_h - dT_c = d(T_h - T_c)$$

$$= -d\dot{Q}\left(\frac{1}{m_h c_{ph}} + \frac{1}{m_c c_{pc}}\right)$$

Again

$$d\dot{Q} = U(T_h - T_c)\,dA_s$$

$$\Rightarrow d(T_h - T_c) = -U(T_h - T_c)\,dA_s\left(\frac{1}{m_h c_{ph}} + \frac{1}{m_c c_{pc}}\right)$$

$$\Rightarrow \frac{d(T_h - T_c)}{T_h - T_c} = -U\,dA_s\left(\frac{1}{m_h c_{ph}} + \frac{1}{m_c c_{pc}}\right)$$

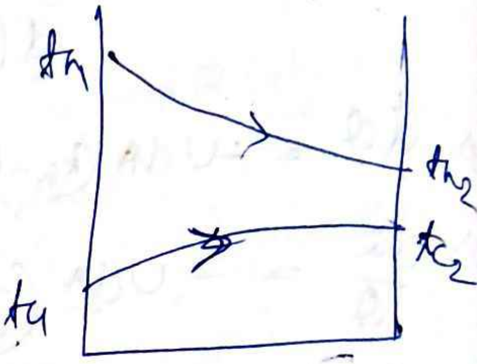
Integrating this, from inlet to outlet, we have

NTU

$$\varepsilon = \frac{\text{Actual heat flow}}{\text{Max}^n \text{ heat flow}}$$

$$= \frac{C_c(t_{c_2} - t_{c_1})}{C_{min}(t_{h_1} - t_{c_1})} = \frac{C_h(t_{h_1} - t_{h_2})}{C_{min}(t_{h_1} - t_{c_1})}$$

Parallel flow



$$dQ = -m_h c_{p_h} dT_h = -C_h dT_h$$

$$= C_c dT_c$$

$$\Rightarrow dT_h = -\frac{dQ}{C_h} \quad dT_c = \frac{dQ}{C_c}$$

$$dT_h - dT_c = -\frac{dQ}{C_h} - \frac{dQ}{C_c} = -dQ\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

$$\Rightarrow d\theta = -U\,dA\,(t_h - t_c)\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

$$\Rightarrow \frac{d\theta}{\theta} = -U\,dA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

$$\Rightarrow \ln\left(\frac{\theta_2}{\theta_1}\right) = -UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

$$\Rightarrow \frac{\theta_2}{\theta_1} = \exp\left[-UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\right]$$

$$\Rightarrow \frac{t_{h_2} - t_{c_2}}{t_{h_1} - t_{c_1}} = \exp\left[-UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\right]$$

$$Q = m_c c_{p_c}(t_{c_2} - t_{c_1}) = m_h c_{p_h}(t_{h_1} - t_{h_2})$$

$$= C_c(t_{c_2} - t_{c_1}) = C_h(t_{h_1} - t_{h_2})$$

$$\varepsilon = \frac{C_c(t_{c_2} - t_{c_1})}{C_{min}(t_{h_1} - t_{c_1})} = \frac{C_h(t_{h_1} - t_{h_2})}{C_{min}(t_{h_1} - t_{c_1})}$$

$$\Rightarrow t_{c_2} = \frac{\varepsilon C_{min}(t_{h_1} - t_{c_1})}{C_c} + t_{c_1}$$

$$t_{h_2} = -\frac{\varepsilon C_{min}(t_{h_1} - t_{c_1})}{C_h} + t_{h_1}$$

$$\therefore \frac{t_{h_1} - \frac{\varepsilon C_{min}(t_{h_1} - t_{c_1})}{C_h} - \left(t_{c_1} + \frac{\varepsilon C_{min}(t_{h_1} - t_{c_1})}{C_c}\right)}{t_{h_1} - t_{c_1}}$$

$$= \frac{t_{h_1} - t_{c_1} - \varepsilon C_{min}(t_{h_1} - t_{c_1})\left(\frac{1}{C_h} + \frac{1}{C_c}\right)}{t_{h_1} - t_{c_1}}$$

$$\Rightarrow 1 - \varepsilon C_{min}\left(\frac{1}{C_h} + \frac{1}{C_c}\right) = \exp\left[UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\right]$$

Let $C_h = C_{min}$ & $C_c = C_{max}$, $C = \frac{C_{min}}{C_{max}}$

$$\Rightarrow 1 - \varepsilon \left(1 + \frac{C_{min}}{C_{max}}\right) = \exp\left[-\frac{UA}{C_{min}}\left(1 + \frac{C_{min}}{C_{max}}\right)\right]$$

$$\Rightarrow 1 - \varepsilon(1 + C) = \exp[-NTU(1 + C)]$$

$$\Rightarrow \varepsilon = \frac{1 - \exp[-NTU(1 + C)]}{1 + C}$$

Transient (Unsteady-state) Heat Conduction

- If the temp. of a body does not vary with time, it is said to be in a steady state.
- If the temp of a body varies with time, it is said to be in an unsteady or transient state.
- Transient conditions occur in →
  - i) Cooling of I.C. engines
  - ii) Automobile engines
  - iii) Cooling & freezing of food
  - iv) brick burning
  - v) heat treatment of metals by quenching
- In general the temp. field in any transient problem is given by

$$t = f(x, y, z, \tau)$$

- The transient heat conduction problems may be solved by following methods
  - i) Analytical
  - ii) Graphical
  - iii) Analogical
  - iv) Numerical

Heat Condn in solids having infinite thermal conductivity → LUMPED PARAMETER ANALYSIS

- All solids have a finite thermal conductivity & there will be a temp. gradient inside the solid whenever heat is added or removed.
- For solids of large thermal conductivity with surface areas that are large in proportion to their volume like plates & thin metallic wires, the internal resistance $\left(\frac{L}{KA}\right)$ can be assumed to be small in comparison with the convective resistance $\left(\frac{1}{hA}\right)$ at the surface.

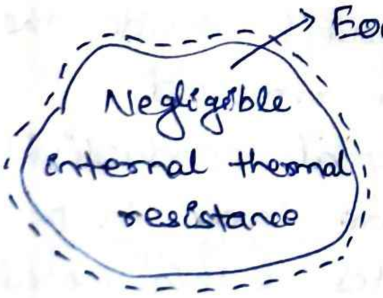
- Examples of this type of heat flow are →
  i) Heat treatment of metals
  ii) Time response of thermocouples & thermometers

- The process in which the internal resistance is assumed negligible in comparison with its surface resistance is called the <u>Newtonian heating or cooling process</u>.
In this process the temp is considered uniform at a given time. Such an analysis is called <u>Lumped parameter analysis</u>. because the whole solid, whose energy at any time is a fun^n of its temp. & total heat capacity is called as <u>one lump</u>.

Let's consider a body whose initial temp. is t_i & which is placed suddenly in ambient air or any liquid at a constant temp. t_a.

The transient response of the body can be determined by relating its rate of change of internal energy with convective exchange at the surface.

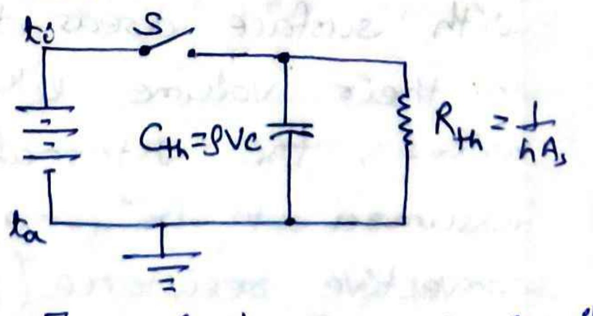
i.e. $Q = -\rho V c \frac{dt}{d\tau} = hA_s(t - t_a)$



$= hA(t - t_a)$
$= -\rho V c \frac{dt}{d\tau}$

$\tau = 0,\ t = t_i$
$\tau > 0,\ t = f(\tau)$

General system for unsteady heat cond^n



Equivalent thermal circuit for lumped capacitance solid

Where ρ = density of solid, kg/m³
V = Volume of the body, m³
c = specific heat of body J/kg°c
h = unit surface conductance, W/m²°c
t = temp. at any time, °c
A_s = Surface area of the body, m²
t_a = ambient temp, °c
τ = time, s

$$\Rightarrow \frac{dt}{t-t_a} = -\frac{hA_s}{\rho Vc}\, d\tau$$

Integrating the equⁿ we have

$$\int \frac{dt}{t-t_a} = -\frac{hA_s}{\rho Vc}\int d\tau$$

$$\Rightarrow \ln(t-t_a) = -\frac{hA_s}{\rho Vc}\tau + C_1$$

The boundary condⁿ →
At $\tau=0$, $t = t_i$ (initial surface temp.)

Applying this we have

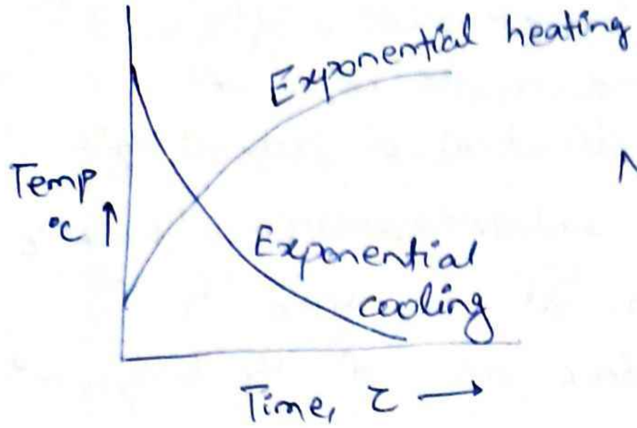
$$C_1 = \ln(t_i - t_a)$$

$$\therefore \ln(t-t_a) = -\frac{hA_s}{\rho Vc}\tau + \ln(t_i - t_a)$$

$$\Rightarrow \ln\left(\frac{t-t_a}{t_i-t_a}\right) = -\frac{hA_s}{\rho Vc}\tau$$

$$\Rightarrow \boxed{\frac{t-t_a}{t_i-t_a} = \frac{\theta}{\theta_i} = \exp\left[-\frac{hA_s}{\rho Vc}\cdot\tau\right]} \quad \text{--- (1)}$$

- Equⁿ-(1) gives the temp. distribution in the body for Newtonian heating & cooling.
- It indicates that the temp. rises exponentially with time.



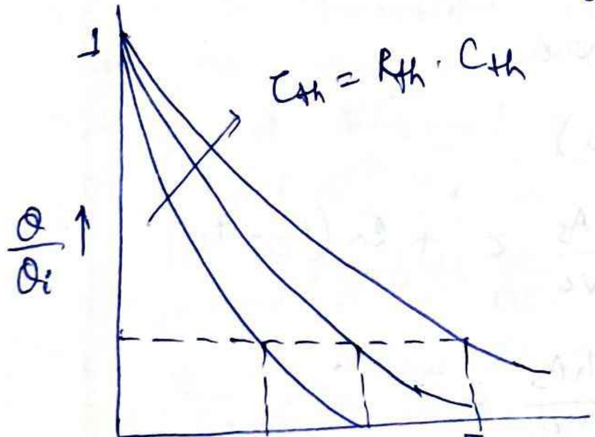
Newtonian heating or cooling.

- The dimension of the quantity $\frac{\rho V c}{h A_s}$ is Time. It is called <u>thermal time constant</u>. (τ_{th}) It indicates the rate of response of a system to a sudden change in its environmental temp. ie. how fast a body will response to a change in the environmental temp.

$$\tau_{th} = \frac{\rho V c}{h A_s} = \left(\frac{1}{hA_s}\right)(\rho V c) = R_{th} C_{th}$$

where $R_{th} = \frac{1}{hA_s}$ = resistance to convection heat transfer

$C_{th} = \rho V c$ = lumped thermal capacitance of solid



- This fig indicates that any increase in R_{th} or C_{th} will cause a solid to responde more slowly to changes in its thermal environment & will increase the time required to attain the thermal equilibrium ($\theta = 0$)

- We can have an analogus electric network for a lumped heat capacity system in which $C_{th} = \rho V c$ represents the thermal capacity of the system.

$Q = (\rho V c)\theta = C_{th} \cdot \theta$

When switch is closed the solid is charged to the temp θ. On opening the switch, the thermal energy stored as C_{th} is dissipated through the thermal resistance R_{th} & the temp. of the body decreases with time.

Hence RC circuits may be used to determine the transient behaviour of thermal systems.

The term $\frac{hA_s}{\rho V c}\tau$ can be arranged as

$$\frac{hA_s}{\rho V c}\tau = \left(\frac{hV}{KA_s}\right) \times \left(\frac{A_s^2 K}{\rho V^2 c}\tau\right) = \left(\frac{hL_c}{K}\right)\left(\frac{\alpha\tau}{L_c^2}\right)$$

Where $\alpha = \frac{K}{\rho c}$ = thermal diffusivity

L_c = characteristic length

$$= \frac{\text{Volume } (V)}{\text{Surface area} (A_s)}$$

For Flat plate $L_c = \frac{V}{A_s} = \frac{LBH}{2BH} = \frac{L}{2}$ = Semi-thickness

Cylinder, $L_c = \frac{\pi R^2 L}{2\pi R L} = \frac{R}{2}$

Sphere, $L_c = \frac{\frac{4}{3}\pi R^3}{4\pi R^2} = \frac{R}{3}$

Cube, $L_c = \frac{L^3}{6L^2} = \frac{L}{6}$

$Bi = \frac{hL}{K}$
$K \to$ for solid
$Nu = \frac{hd}{K}$
$K \to$ for fluid

- The non-dimensional factor $\left(\frac{hL_c}{K}\right)$ is called the Biot number (Bi)

ie. $Bi = \frac{hL_c}{K}$ $\frac{L}{KA} \Big/ \frac{1}{hA} = \frac{L}{KA} \times hA = \frac{hL}{K}$

- It is the ratio of internal (conduction) resistance to surface (convection) resistance. $\frac{L}{KA}$

$1/hA$

- When value of Bi is small, it indicates that the system has a small internal (conduction) resistance i.e. small temp. gradient or temp is uniform within the system.
- If $Bi < 0.1$, the lumped heat capacity approach can be used.
- The non-dimensional factor $\frac{\alpha \tau}{L_c^2}$ is called the Fourier number. (F_o)

$$\therefore \boxed{F_o = \frac{\alpha \tau}{L_c^2}}$$

It signifies the degree of penetration of heating or cooling effect through a solid.

$\therefore$ Using non-dimensional terms, we have

$$\boxed{\frac{\theta}{\theta_0} = \frac{t - t_a}{t_i - t_a} = e^{-Bi \cdot F_o}}$$

Instantaneous rate of heat flow (Q_i)

$$Q_i = \rho V c \frac{dt}{d\tau}$$

$$= \rho V c \frac{d}{d\tau}\left[t_a + (t_i - t_a) \exp\left(-\frac{h A_s}{\rho V c}\tau\right)\right]$$

$$= \rho V c \left[(t_i - t_a)\left(-\frac{h A_s}{\rho V c}\right) \exp\left(-\frac{h A_s \tau}{\rho V c}\right)\right]$$

$$= -h A_s (t_i - t_a) \exp\left(-\frac{h A_s \tau}{\rho V c}\right)$$

$$\Rightarrow \boxed{Q_i = -h A_s (t_i - t_a) e^{-Bi \cdot F_o}}$$

Total or cumulative heat transfer (Q')

$$Q' = \int_0^\tau Q_i \, d\tau$$

$$= \int_0^\tau -h A_s (t_i - t_a) \exp\left(-\frac{h A_s}{\rho V c}\tau\right) d\tau$$

$$= -hA_s(t_i - t_a)\left.\frac{\exp(-hA_s/\rho Vc)\tau}{-hA_s/\rho Vc}\right]_0^{\tau}$$

$$= \rho Vc\,(t_i - t_a)\left.\exp\left\{-\frac{hA_s}{\rho Vc}\tau\right\}\right]_0^{\tau}$$

$$\Rightarrow Q' = \rho Vc\,(t_i - t_a)\left[\exp\left(-\frac{hA_s}{\rho Vc}\right) - 1\right]$$

$$\Rightarrow \boxed{Q' = \rho Vc\,(t_i - t_a)\left[e^{-B_i F_o} - 1\right]}$$

Time constant & Response of Temp. Measuring instrument →

Measurement of temp. by a thermocouple is an application of the lumped parameter analysis.

The response of a thermocouple is defined as the time required for the thermocouple to attain the source temp.

From equ^n $\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA_s}{\rho Vc}\tau\right)$

we have → larger the quantity $\frac{hA_s}{\rho Vc}$, the faster the exponential term will approach zero or the response of the temp. measuring device is rapid.

- The value of the term $\left(\frac{hA_s}{\rho Vc}\right)$ can be increased either by increasing h or by decreasing the wire diameter $\left(\frac{V}{A_s} = \frac{R}{2}\right)$, density and specific heat.

Hence a very thin wire is used in thermocouple to have a rapid response.

- The quantity $\frac{\rho V c}{h A_s}$ is called time constant

- symbol τ^*

$$\therefore \boxed{\tau^* = \text{Time constant} = \frac{\rho V c}{h A_s}}$$

$$= \frac{V}{A_s} \cdot \frac{\rho c}{K} \cdot \frac{K}{h}$$

$$= \frac{V}{A_s} \cdot \frac{K}{\alpha h}$$

$$\Rightarrow \frac{t - t_a}{t_i - t_a} = \frac{\theta}{\theta_i} = e^{-\tau/\tau^*}$$

At $\tau = \tau^*$ we have

$$\frac{\theta}{\theta_i} = \frac{t - t_a}{t_i - t_a} = e^{-1} = 0.368$$

$\Rightarrow \tau^*$ is the time required for the temp change to reach 36.8% of its final value in response to a step change in temp.

Or τ^* is the time required for the temp. difference to be reduced by 63.2%.

- The time required by a thermocouple to reach its 63.2% of the value of initial temp. difference is called its sensitivity.

- The response times for different sizes of thermocouple wires usually vary betn 0.04 to 2.5 secs.

Thermal Resistance (R_{th})

When two physical systems are described by similar equations and have similar boundary conditions, these are said to be analogous. The heat transfer processes may be compared by analogy with the flow of electricity in an electrical resistance

As the flow of electric current in the electrical resistance is directly proportional to potential difference (dV), similarly heat flow rate Q is directly proportional to temp difference (dt).

As per Ohm's law (in electric-circuit theory) we have

$$\text{Current } (I) = \frac{\text{Potential difference } (dV)}{\text{electrical resistance } (R)} \quad \text{—} (1)$$

$$\Rightarrow \text{Resistance}(R) = \frac{\text{driving Potential}}{\text{corresponding transfer rate}}$$

By analogy, the heat flow equ^n (Fourier's equ^n) may be written as

$$\text{Heat flow rate } (Q) = \frac{\text{Temp. difference } (dt)}{\left(\frac{dx}{KA}\right)} \quad \text{—} (2)$$

By comparing equ^n (1) & (2) we find that

I is analogus to Q

dV " " " dt

R " " " $\frac{dx}{KA}$

$R_{th} = \frac{dx}{KA}$

Equivalent circuit

The quantity $\frac{dx}{KA}$ is called thermal conduction resistance $(R_{th})_{cond}$

$$(R_{th})_{cond} = \frac{dx}{KA}$$

$$(R_{th})_{cond} = \frac{dt}{Q} = \frac{dx}{KA}$$

Thermal resistance is helpful in calculations for flow of heat.

The reciprocal of the thermal resistance is called thermal conductance.

Convection

Convection is possible only in the presence of a fluid medium. The transport of heat here is inseparably linked with the movement of the fluid itself. If the fluid motion is set up by buoyancy effects resulting from the density variation caused by the temp difference in the fluid, the heat transfer is said to be free or natural convⁿ.
If the fluid motion is artificially created by means of the an externarl agency like a blower or fan, the heat transfer is termed as forced convection.

Convection heat transfer is comprised of two mechanisms → energy transfer due to random molecular motion (diffusion) & energy transfer by the bulk or macroscopic motion of the fluid.
Convection → used when both cumulative transport
advection → " " transport due to bulk fluid motion.

The rate equ^n for the convective heat transfer bet^n a surface & an adjacent fluid is prescribed by Newton's law of cooling

$$Q = hA(t_s - t_f)$$

Q – rate of heat transfer

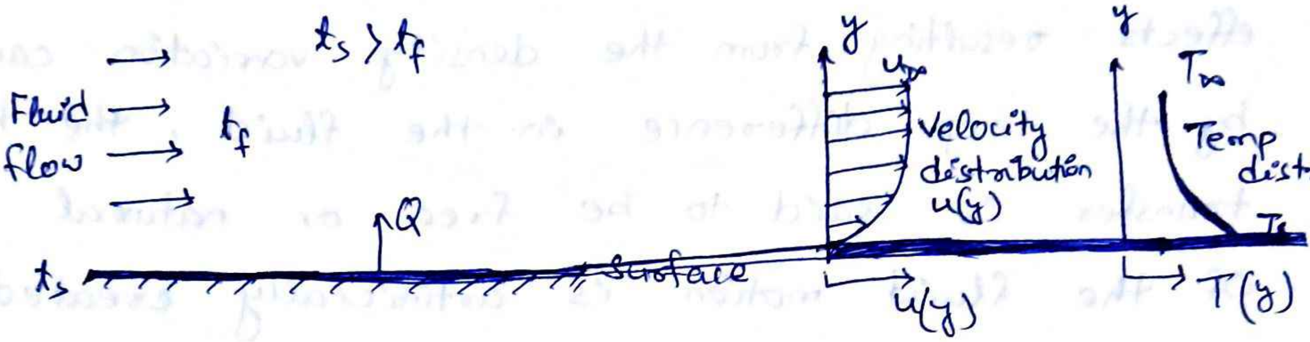
A - Area exposed to heat transfer

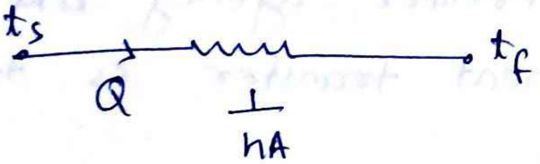
t_s - Surface temp

t_f - fluid temp

h - coefficient of convective heat transfer

$$\therefore h = \frac{Q}{A(t_s - t_f)} = \frac{W}{m^2\,{}^\circ C} \text{ or } W/m^2\,{}^\circ C \text{ or } W/m^2 K$$





(Equivalent circuit)

The <u>coefficient of convective heat transfer</u> h may be defined as the amount of heat transmitted for a unit temp difference bet^n the fluid and unit area of surface in unit time.

<u>Thermal conductivity</u> of a material may be defined as the amount of energy conducted through a body of unit area, and unit thickness in unit time when the difference in temp bet^n the faces causing heat flow is unit temp difference.

Electrical analogy $Q = \frac{(t_s - t_f)}{\frac{1}{hA}}$

$\therefore$ <u>Convection thermal resistance</u> $= \frac{1}{hA}$

Radiation

Radiation is the transfer of heat through space or matter by means other than conduction or convection. Radiation heat is thought of as electromagnetic waves or quanta. All bodies radiate heat ; so a transfer of heat by radiation occurs because hot body emits more heat than it receives and a cold body receives more heat than it emits. Radiant energy requires no medium for propagation and will pass through vacuum. Properties of radiant heat ~~is~~ are similar to light.

Laws of Radiation →

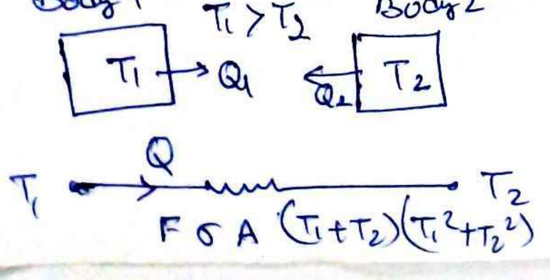
1. Wien's law – It states that the wavelength λ_m corresponding to the max^m energy is inversly proportional to the absolute temp T of the hot body.

$$\lambda_m \propto \frac{1}{T} \Rightarrow \lambda_m T = \text{constant}$$

2. Kirchhoff's law – It states that the emissivity of the body at a particular temp is numerically equal to its absorptivity for radiant energy from body at the same temp.

3. Stefan – Boltzmann law – It states that the emissive power of a black body is directly proportional to fourth power of its absolute temp.

$$Q \propto T^4$$



$$Q = F \sigma A (T_1^4 - T_2^4)$$

where F = a factor depending on geometry & surface properties

σ = Stefan-Boltzmann constant
$= 5.67 \times 10^{-8}$ $W/m^2 K^4$

A = Area, m^2

T_1, T_2 = Temp °K

Or $Q = F \sigma A (T_1 - T_2)(T_1 + T_2)(T_1^2 + T_2^2)$

When two physical systems are described by similar equ^ns and have similar boundary cond's, these are said to be analogous. The heat transfer processes may be compared by analogy with the flow of electricity in an electrical resistance.

- The rules for combining electrical resistances in series and parallel apply equantly well to thermal resistances.
- The concept of thermal resistance is helpful while making calculations for flow of heat.

Convection Heat Transfer

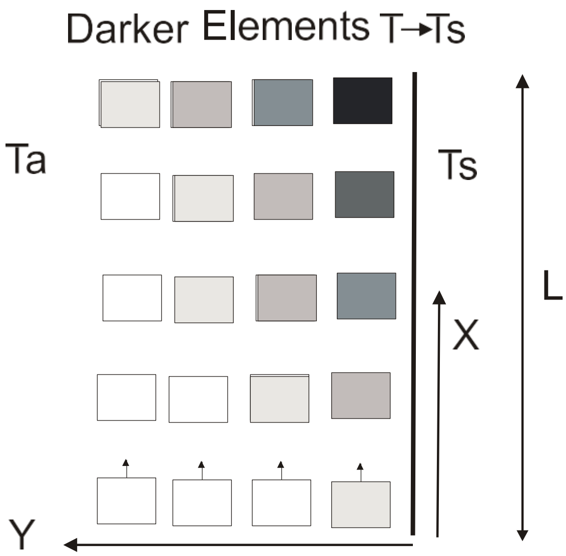
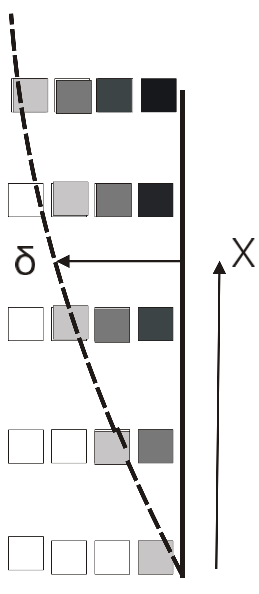


Figure 3.1 Flow over a heated plate

Convective heat transfer is conduction though a gas or liquid augmented by fluid motion. In these notes an introduction to the physics governing convection will be given along with some results for several different conditions.

Consider the case of air at a uniform temperature T_a blown by a fan along a flat surface which is heated to a uniform temperature T_s, Figure 3.1. Say, this is cool air in an air-conditioned room flowing over the surface of a window heated by solar radiation. The element of air closest to the heated surface has a temperature increase as it starts to move up the plate. Elements further away, at a larger y coordinate still are at T_a. As the element becomes hotter it moves up and is replaced with another element at T_a and the process is repeated. Viewed from the point of view of the room as a whole, room air at T_a approaches the plate and a given flow of air leaves the plate at an elevated temperature somewhere between T_a and T_s. Thus there is a net energy transfer from the surface of the window to the room air.

The convective heat transfer coefficient is defined as,



1.2

$$h = \frac{q}{A\ T_s - T_a} \qquad (3.1)$$

To get an estimate of how the rate of heat transfer is influenced by the parameters of the problems we have to look more closely at the layer of air moving along the plate surface. To make the explanation clear we will assume all of the elements of air are moving along the plate surface at uniform velocity and in straight lines. This is an instance of laminar flow. At any location (x,y) fixed relative to the stationary plate the temperature remains constant with time and there is a steady conduction heat transfer from the surface to the air in the y direction, normal to the plate surface. As the air continues up the plate the elements close to the plate increase their temperature. At the same time the elements further away start to rise in temperature due to conduction from the hotter elements at the plate surface..The further along the air moves in the x direction the more elements further from the plate surface feel the conduction heat transfer and rise in temperature as shown in figure 3.2.The maximum y distance at which the thermal effects are felt, at any location x, will be specified as ∗, the thermal boundary layer thickness.

At any position x the variation of air temperature normal to the plate surface, the y axis is shown on figure 3.3. The rate of heat transfer from the plate surface is determined by the conduction into the air at y=0. This can be calculated from,

$$q = -k_a \left. \frac{\partial T}{\partial y} \right|_{y=0} \qquad (3.2)$$

As a good first estimate we can use,

$$q \approx C k_a \frac{T_s - T_a}{\delta} \qquad (3.3)$$

where the constant C should be of order of magnitude unity. Now substituting this into the definition of the convective heat transfer coefficient,

$$h = \frac{q}{(T_s - T_a)} \approx \frac{k_a (T_s - T_a)}{\delta (T_s - T_a)} \approx \frac{k_a}{\delta} \qquad (3.4)$$

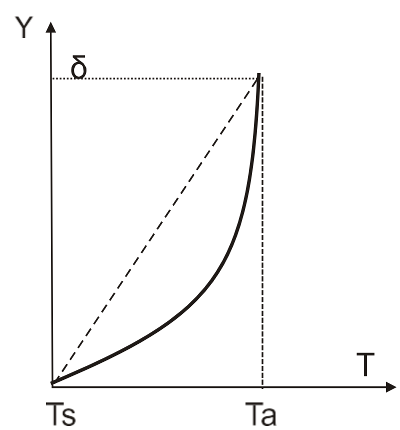


Fig. 3.3 Air temperature in thermal boundary layer

Thus the heat transfer coefficient is proportional to the thermal conductivity of air and inversely proportional to the size of the thermal boundary layer thickness.

We can use the estimate for h to qualitatively predict how the heat transfer coefficient will vary with the main parameters

First, if the velocity of the air is increased over the plate surface then each fluid element spends a shorter time in contact with, or close to the plate. The fluid element temperatures don't increase as much. From figure 3.1 and 3.2, in such an instance the thermal boundary layer, ∗, will be smaller at a given x value. Thus an increase in air velocity should result in an increase in the heat transfer coefficient and the rate of convective heat transfer. This result is summarized in table 3.1

Consider now an increase in the air density, say, by an increase in the air pressure, assuming all other air properties and the air velocity remains the same. For the same magnitude of heat transfer the temperature increase of the element will be less since the element has a larger mass.

A smaller temperature increase will reduce the thermal boundary layer thickness. Therefore, an increase in the density will increase the heat transfer coefficient, one reason that liquid water has a higher convective heat transfer rate than air. An increase in the specific heat of the fluid flowing over the plate has the same behavior as an increase in density.

As the plate length is increased the air will flow over the plate for a longer time. Heat transfer will penetrate a further distance from the plate surface. The thermal boundary layer thickness will grow larger. When averaged over the entire plate length , the average thermal boundary layer thickness will become larger and the average heat transfer coefficient will be smaller. Note, the total heat transfer will be higher for the longer plate but the heat transfer per unit area will be smaller. To augment the heat transfer when possible, designers will break a long surface up into a series of smaller surfaces. This can be done by physically separating sections of the plate or by placing an array of ribs at right angles to the flow to break up the boundary layer and restart it.

Table 3.1 Key Parameters Influence on Convective Heat Transfer

Parameter	∗, thermal boundary layer thickness	h, convective heat transfer coefficient
Velocity increase	∗ decreases	h increases
Density increase	∗ decreases	h increases
Specific heat increase	∗ decreases	h increases
Plate length L increase	∗ averaged over L increases	h averaged over L decreases
Thermal conductivity k_a increase	∗ increase	h increases
Transition to turbulent flow from laminar flow	∗ decreases	h increases

When the thermal conductivity of the fluid passing over the plate is increased, y using a higher conductivity gas or liquid there are two elements in play. The thermal boundary layer thickness will increase because of augmented means of heat transfer through the fluid. Remember that h is proportional to the ratio of conductivity to boundary layer thickness. In this case k increases faster than ∗ and the heat transfer coefficient increases. It should be expected that when we increase the mechanism for heat transfer, in this case the molecular conductivity, that the rate of convection will increase.

Turbulent Flow

At low velocity, the fluid flows in very smooth paths about parallel to the plate surface. As the velocity is increased a point is reached where the fluid motion is much more chaotic characterized by eddies in the flow near the plate surface. This is termed turbulent flow. Turbulent flow over a flat plate is found to occur when the Reynolds number Δ Vx/: exceeds 300,000. x is used to indicate the distance from the leading edge; the front of the flat plate can have laminar flow while the rear experiences turbulent flow. The distinction between laminar and turbulent flow is important because the eddies in the turbulent flow tend to bring fluid at the ambient temperature T_a much closer to the heated plate surface. In effect the eddies reduce the distance $*$ for conduction heat transfer and can markedly increase the heat transfer coefficient, sometimes by an order of magnitude or more.

At can be seen that the convective heat transfer is a function of the fluid properties such as density and conductivity, the flow conditions and the surface geometry. Given below are a few dimensional expressions that can be used for specific cases. The constants in the equations already include the fluid properties.

Laminar flow expressions for h

Using air properties at room temperature h for laminar flow over flat plates can be found as

$$h = 0.71\left[\frac{V}{L}\right]^{0.5} \qquad (3.5)$$

while for water,

$$h = 12.7\left[\frac{V}{L}\right]^{0.5} \qquad (3.6)$$

In this form h is in BTU/hrft^{2}F, V is in ft/sec and L is in ft. Note, water gives a much higher heat transfer coefficient than air because it has a much higher thermal conductivity as well as a higher density and specific heat.

Turbulent Flow

At low velocity, the fluid flows in very smooth paths about parallel to the plate surface. As the velocity is increased a point is reached where the fluid motion is much more chaotic characterized by eddies in the flow near the plate surface. This is termed turbulent flow. Turbulent flow over a flat plate is found to occur at higher velocities and longer plate lengths. Also flowing liquids will reach turbulent flow at lower velocities than gases.

For air near room temperature turbulent flow is reached when the product of plate length and air velocity exceeds,

$$VL \geq 50\ \frac{ft}{s}ft \qquad (3.7)$$

$$h = 0.55\,\frac{V^{0.8}}{L^{0.2}} \qquad (3.8)$$

For water

$$VL \geq 3.9\;\frac{ft}{s}\;ft$$
$$h = 21.2\,\frac{V^{0.8}}{L^{0.2}} \qquad (3.9)$$

Again, all of the parameters are in Imperial units.

Flow Inside Tubes

The other important flow geometry is gas or liquid flow inside tubes. A similar development exists for the convective heat transfer with the exception that h is defined based on the mean temperature T_M of the fluid within the tube at the location in question. For a section within the tube of axial length between x and x+)x,

$$h = \frac{q}{\pi D \Delta x \;\; T_S - T_M} \qquad (3.10)$$

where T_M is the mean fluid temperature at x.

Almost all practical cases of tube flow, the flow is turbulent. Exceptions are flows through very small tube diameters or the flow of viscous fluids such as oil. For turbulent tube at room temperature, the relationship becomes,

$$h = 0.34\,\frac{V^{0.8}}{D^{0.2}} \qquad (3.11)$$

while for water we get,

$$h = 13\,\frac{V^{0.8}}{D^{0.2}} \qquad (3.12)$$

where V is given in ft/sec and D is in feet.

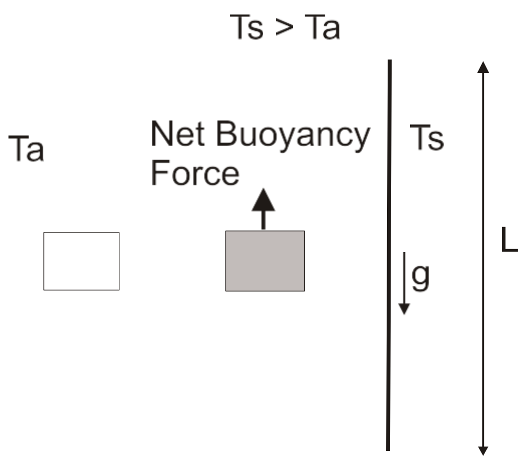


Figure 3.4 natural convection on a vertical heated plate

Natural or Free Convection

In natural convection the fluid motion is solely due to buoyancy effects caused by the heating or cooling process of the fluid. Natural convection flow over vertical flat plates has similar physical considerations and analogous expressions to those used for forced convection, see Figure 3.4. However, in this case the fluid velocity V is not set by external fans or by the motion of the heated body. The air density varies with 1/T. The air close to the plate is at a higher temperature and has a lower mass. This results in a net upward buoyancy force on these elements that accelerates them. We would expect the velocity and the heat transfer coefficient is a function on the temperature difference T_s-T_a.

Just as forced flow, laminar flow exists at low velocities. The exact expression for natural convection of air over a vertical plate at room temperature is

$$\Delta T L^3 < 1000\ ^oF\ \ ft^3$$
$$h = 0.29\left(\frac{\Delta T}{L}\right)^{1/4} \qquad (3.13)$$

For turbulent flow,

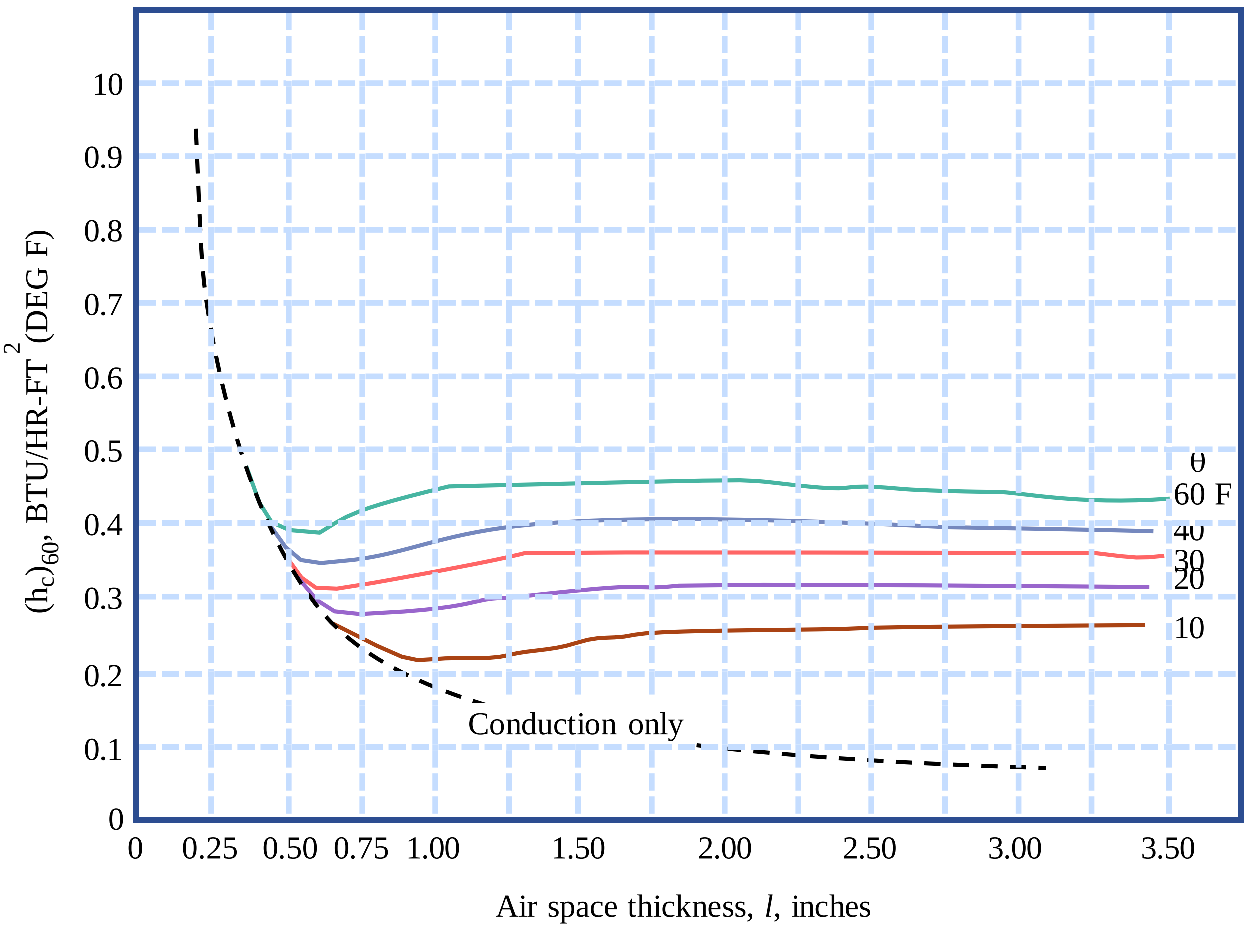
$$\Delta T L^3 \geq 1000$$
$$h = 0.21\ \ \Delta T^{\ 1/3} \qquad (3.14)$$

Natural Convection in Enclosed Spaces

In an open wall cavity between interior and exterior walls air circulation will take place from the hot to the cold wall. The air will rise along the hot wall move horizontally to the cold wall at the upper end of the wall cavity (and at other vertical locations as well). The warm air will then flow down the cold wall. This process will result in energy transfer from the hot to the cold wall. The overall heat transfer can be represented by a heat transfer coefficient defined in terms of the two wall temperatures,

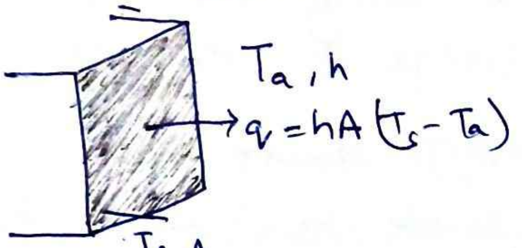
$$q = h_C A \; T_{hot\ wall} - T_{cold\ wall} \qquad (3.15)$$

Figure 3.5 shows measured results for eight foot high walls. h_c is a function of the spacing between the walls and l, the temperature difference from the hot to the cold wall. At a small spacing and/or a small temperature difference the buoyancy effects are minimal and h_c is simply the ratio of air conductivity to wall spacing. As the spacing is increased, h_c reaches a constant, typically a spacing between 1/2 and 3/4 inch is optimum. Radiation heat transfer across the cavity must be added to the convection. If there is infiltration of outside air into the cavity the energy transfer may be increased considerably.

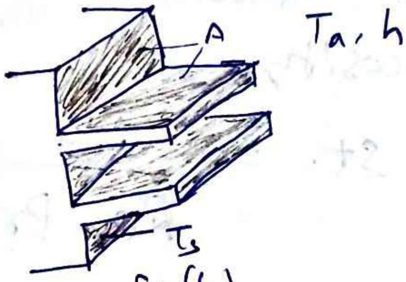


Data taken from Housing Research Paper 32, "The Thermal Insulating Value of Airspaces", Housing and Home Finance Agency Director of Housing Research, Washington D.C., April 1954.

The extended surface is commonly used in special case involving heat transfer by conduction within a solid & heat transfer by convection from the boundaries of the solid. Until now, we have considered heat transfer from the boundaries of a solid to be in the same direction as heat transfer by conduction in the solid. For an extended surface, the direction of heat transfer from the boundaries is perpendicular to the principal direction of heat transfer in the solid.



fig(a)



fig(b)

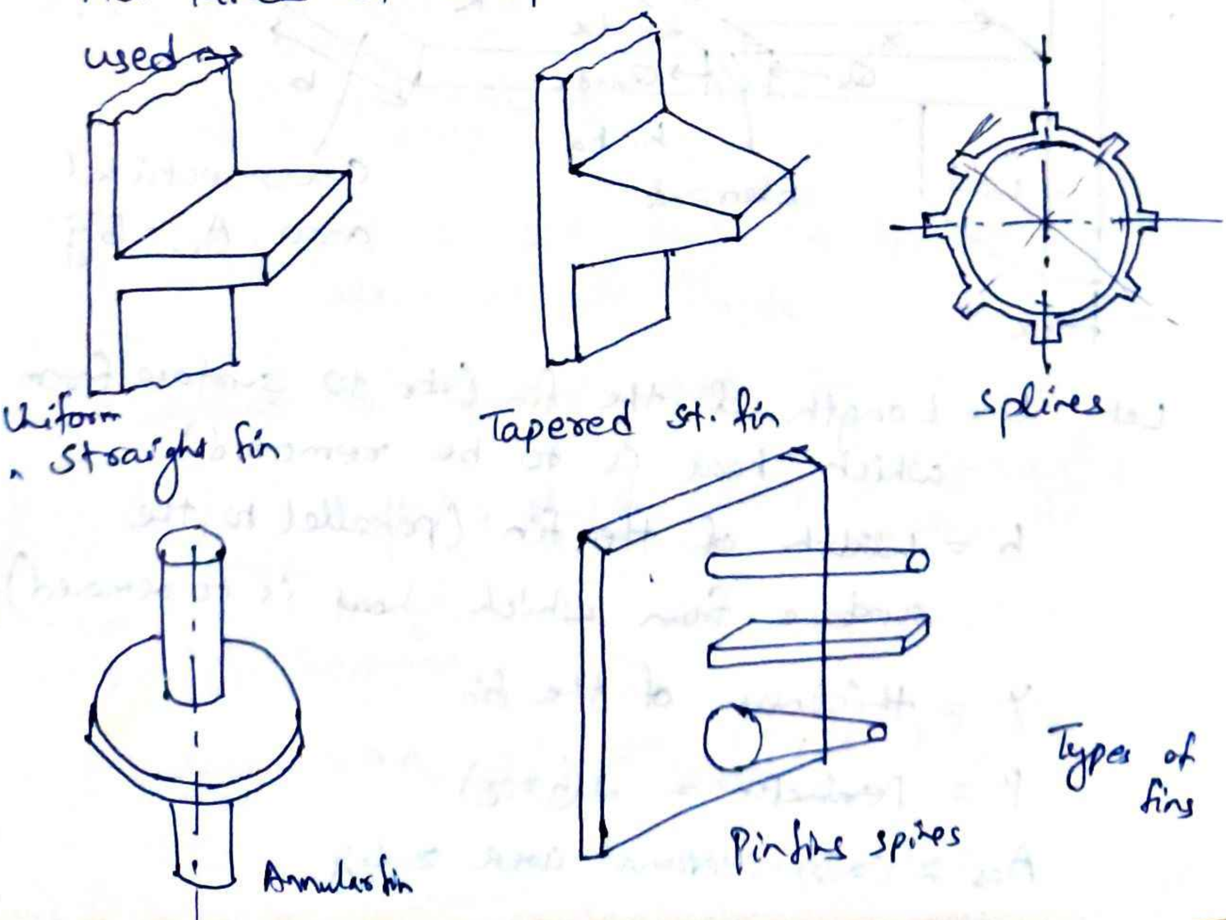
Consider a plane wall of fig(a). If Ts is fixed there are two ways in which the heat transfer rate may be increased. The convection coefficient h could be increased by increasing the fluid velocity, and/or the fluid temp Ta could be reduced. There are many situations for which increasing h to the maxm possible value is either insufficient to obtain the desired heat transfer rate or the associated costs are prohibitive. Such costs are related to the blower or pump power requirements needed to increase h through increased fluid motion. The 2nd option of reducing Ta is often impractical.

From fig(b) we have, the heat transfer rate may be increased by increasing the surface area across which the convection occurs.

Heat Transfer From Extended Surfaces (Fins)

Whenever the available surface is found inadequate to transfer the required quantity of heat with the available temp. drop & convective heat transfer coefficient, extended surfaces or fins are used.

- It is extensively used in heat transfer betⁿ a surface & gas as the convective heat transfer coefficient is low.
- The finned surfaces are widely used in
  - Economisers for steam power plants
  - Radiators of automobiles
  - cooling coils & condenser coils in refrigerators & air conditioners
  - Transformers
- All kinds of shapes & sizes of fins are used →



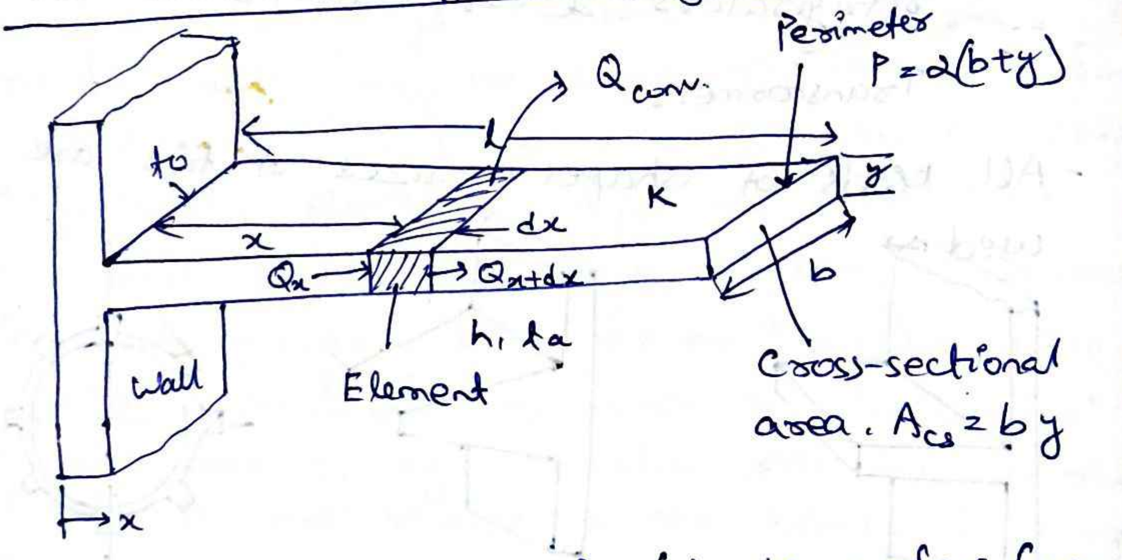
Types of fins

For proper design of fins we have to know the temp. distribution along the fin. Here we know the mathematical analysis for finding out the temp. distribution & heat flow from different types of fins.

Assumptions →

1. Steady state heat conduction
2. No heat generation within the fin
3. Uniform heat transfer coefficient 'h' over the entire surface of the fin
4. Homogeneous & isotropic fin matl (K is const.)
5. One-dimensional heat conduction
6. Negligible radiation.

Heat flow Through Rectangular Fin



Let l = Length of the fin (⊥r to surface from which heat is to be removed)

b = width of the fin (parallel to the surface from which heat is to removed)

Y = thickness of the fin

P = Perimeter = $2(b+y)$

A_{cs} = cross-sectional area = by

t_0 = temp. at the base of the fin

t_a = Temp. of the ambient fluid

k = Thermal conductivity

h = convective heat transfer coefficient.

Let's consider the heat flow to & from an element dx thick at a distance x from the base.

Heat conducted into the element at plane x

$$Q_x = -k A_{cs} \left[\frac{dt}{dx}\right]_x$$

Heat conducted out of the element at plane x+dx

$$Q_{x+dx} = -k A_{cs} \left[\frac{dt}{dx}\right]_{x+dx}$$

Heat convected out of the element betn the planes x & x+dx

$$Q_{conv} = h\,(P \times dx)(t - t_a)$$

∴ Energy balance on the element

$$Q_x = Q_{x+dx} + Q_{conv.}$$

$$\Rightarrow -kA_{cs}\left.\frac{dt}{dx}\right|_x = -kA_{cs}\left.\frac{dt}{dx}\right|_{x+dx} + h(P\cdot dx)(t - t_a) \quad \text{--- (1)}$$

According to Taylor's expansion

$$\left.\frac{dt}{dx}\right|_{x+dx} = \left.\frac{dt}{dx}\right|_x + \frac{d}{dx}\left(\frac{dt}{dx}\right)_x dx + \frac{d^2}{dx^2}\left(\frac{dt}{dx}\right)\frac{dx^2}{2!} + \cdots$$

∴ Equn (1) becomes

$$-kA_{cs}\left.\frac{dt}{dx}\right|_x = -KA_{cs}\left.\frac{dt}{dx}\right|_x - KA_{cs}\left(\frac{d^2t}{dx^2}\right)_x dx - kA_{cs}\left(\frac{d^3t}{dx^3}\right)\frac{dx^2}{2!} + \cdots$$

$$+ h(P\,dx)(t - t_a)$$

Neglecting higher terms of as dx so we have

$$-KA_{cs}\left.\frac{dt}{dx}\right|_{x} = -KA_{cs}\left.\frac{dt}{dx}\right|_{x} - KA_{cs}\left[\frac{d^2t}{dx^2}\right]dx + h(Pdx)(t-t_a)$$

$$\Rightarrow KA_{cs}\left(\frac{d^2t}{dx^2}\right)dx - h(Pdx)(t-t_a) = 0$$

$$\Rightarrow \boxed{\frac{d^2t}{dx^2} - \frac{hP}{KA_{cs}}(t-t_a) = 0} \quad \text{—— (2)}$$

Let $\theta_x = t_x - t_a$

$$\Rightarrow \frac{d\theta}{dx} = \frac{dt}{dx} \quad \& \quad \frac{d^2\theta}{dx^2} = \frac{d^2t}{dx^2}$$

$$\Rightarrow \boxed{\frac{d^2\theta}{dx^2} - m^2\theta = 0} \quad \text{—— (3)} \quad \text{where } m = \sqrt{\frac{hP}{KA_{cs}}}$$

For a given fin m is constant.

Equ^n (2) & (3) are general energy equ^n for 1-D heat dissipation from fin.

The general sol^n of this equ^n is

$$\boxed{\theta = C_1 e^{mx} + C_2 e^{-mx}} \quad \text{—— (4)}$$

$$\Rightarrow \boxed{t - t_a = C_1 e^{mx} + C_2 e^{-mx}}$$

where C_1 & C_2 are the constants which can be determined from boundary cond^{n}s.

Different cases are –

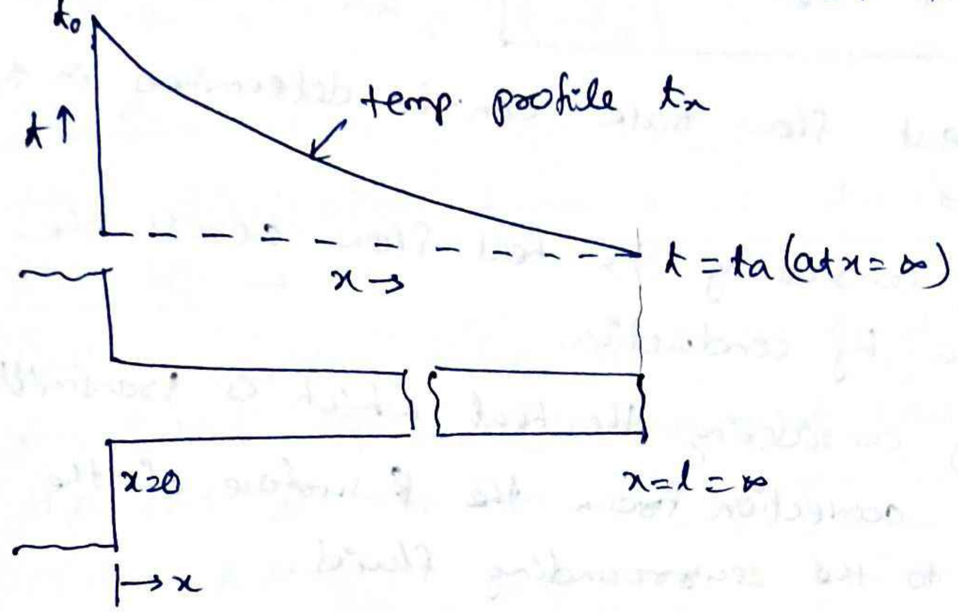
<u>Case 1</u> → The fin is infinitely long & the temp at the end of the fin is essentially that of the ambient

<u>Case 2</u> → The end of the fin is insulated

<u>Case 3</u> → The fin is of finite length & loses heat by convection.

Case - 1 Infinitely long fin ($l \to \infty$) 3mm dia & 30mm long

temp at the end of fin = ambient temp



The boundary cond^n's are →

i) At $x = 0$, $t = t_0 \Rightarrow t - t_a = t_0 - t_a$

Or At $x = 0$, $\theta = \theta_0$

ii) At $x = \infty$, $t = t_a \to t - t_a = t_a - t_a = 0$

Or At $x = \infty$, $\theta = 0$

∴ Equ^n (4) becomes

At $x = 0$, $\theta = \theta_0$

$$\therefore \theta_0 = C_1 e^0 + C_2 e^{-0} = C_1 + C_2$$

At $x = \infty$, $\theta = 0$

$$\Rightarrow 0 = C_1 e^{\infty} + C_2 e^{-\infty}$$

$$\Rightarrow \boxed{C_1 = 0}$$

$$\because C_1 + C_2 = \theta_0 \Rightarrow \boxed{C_2 = \theta_0}$$

∴ The temp. distribution →

$$\theta = \theta_0 e^{-mx}$$

$$\Rightarrow t - t_a = (t_0 - t_a) e^{-mx}$$

$$\Rightarrow \boxed{\frac{t - t_a}{t_0 - t_a} = e^{-mx}} \quad \text{——— (5)}$$

The heat flow rate can be determined in two ways →

1) By considering the heat flow across the base by conduction.

2) By considering the heat which is transmitted by convection from the surface of the fin to the surrounding fluid.

1) The rate of heat flow across the base of the fin is

$$Q_{fin} = -k A_{cs} \left.\frac{dt}{dx}\right|_{x=0}$$

$$\left.\frac{dt}{dx}\right|_0 = \frac{d}{dx}\left(t_a + (t_0 - t_a)e^{-mx}\right)\Big|_{x=0}$$

$$= -m(t_0 - t_a)\, e^{-mx}\Big|_{x=0}$$

$$= -m(t_0 - t_a)$$

$$\therefore Q_{fin} = -kA_{cs}\left[-m(t_0 - t_a)\right] = kA_{cs}m(t_0 - t_a)$$

$$\Rightarrow Q_{fin} = kA \times \sqrt{\frac{hP}{kA_{cs}}}\,(t_0 - t_a)$$

$$\boxed{Q_{fin} = \sqrt{hPkA_{cs}}\,(t_0 - t_a)}$$

<u>Or</u>

$$2)\ Q_{fin} = \int_0^{\infty} h(P \cdot dx)(t - t_a)$$

$$= \int_0^{\infty} hP(t_0 - t_a)\, e^{-mx}\, dx$$

$$= hP(t_0 - t_a)\int_0^{\infty} e^{-mx}\, dx$$

$$= hP(t_0 - t_a)\frac{1}{m} = hP(t_0 - t_a)\sqrt{\frac{kA_{cs}}{hP}}$$

$$\Rightarrow Q_{fin} = \sqrt{hPkA_{cs}}\,(t_0 - t_a)$$

Case-2

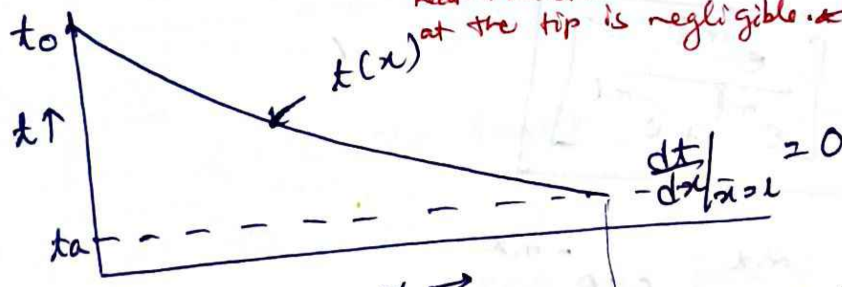
Heat dissipation from a fin insulated at the tip →

Let's consider a fin of finite length with insulated ~~tip~~ end (i.e. no heat loss from the end of the fin)

$$\Rightarrow -kA\left.\frac{dt}{dx}\right|_{x=l} = 0$$

$$\Rightarrow \left.\frac{dt}{dx}\right|_{x=l} = 0$$

It may seem undesirable to use a fin with an insulated tip since such insulation will decrease heat transfer from the fin. However, in practice the heat transfer area at the fin tip is generally small in comparison with the lateral area of the fin for heat transfer. Under these cond's the flow of heat at the tip is negligible. ~~at this cond~~



The boundary conditions are ⇒

1) At $x = 0$, $t = t_0$

$\theta = t_0 - t_a = \theta_0$

$\Rightarrow$ At $x = 0$,

2) At $x = l$, $\left.\frac{dt}{dx}\right|_{x=l} = 0$

Applying the boundary condition in the general sol^n

$$\underset{\theta}{t - t_a} = C_1 e^{mx} + C_2 e^{-mx}$$

$\theta_0 = C_1 + C_2$

Again $\frac{dt}{dx} = mC_1 e^{mx} - mC_2 e^{-mx}$

$\Rightarrow \left.\frac{dt}{dx}\right|_{x=l} = mC_1 e^{ml} - mC_2 e^{-ml} = 0$

$\therefore C_1 e^{ml} - C_2 e^{-ml} = 0$

$\Rightarrow C_1 e^{ml} - (\theta_0 - C_1) e^{-ml} = 0$

$\Rightarrow C_1 (e^{ml} + e^{-ml}) = \theta_0 e^{-ml}$

$$\Rightarrow \boxed{C_1 = \theta_0 \left[\frac{e^{-ml}}{e^{ml} + e^{-ml}}\right]}$$

$$\& \quad C_2 = \theta_0 - \theta_0 \left[\frac{e^{-ml}}{e^{ml} + e^{-ml}}\right]$$

$$\boxed{C_2 = \theta_0 \left[\frac{e^{ml}}{e^{ml} + e^{-ml}}\right]}$$

$$\therefore \theta = C_1 e^{mx} + C_2 e^{-mx}$$

$$= \theta_0 \left[\frac{e^{-ml}}{e^{ml} + e^{-ml}}\right] e^{mx} + \theta_0 \left[\frac{e^{ml}}{e^{ml} + e^{-ml}}\right] e^{-mx}$$

$$\Rightarrow \frac{\theta}{\theta_0} = \frac{e^{m(l-x)} + e^{-m(l-x)}}{e^{ml} + e^{-ml}}$$

$$\Rightarrow \boxed{\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{\cosh[m(l-x)]}{\cosh(ml)}}$$

$\left(\because \cosh x = \frac{e^{x} + e^{-x}}{2}\right)$

→ Expression for temp. distribution

Rate of heat flow from the fin

$$Q_{fin} = -KA_{cs}\left[\frac{dt}{dx}\right]_{x=0}$$

$$\left.\frac{dt}{dx}\right|_{x=0} = (t_0 - t_a)\left[\frac{\sinh\{m(l-x)\}}{\cosh(ml)}\right](-m)$$

$$\Rightarrow \left.\frac{dt}{dx}\right|_{x=0} = -m(t_0 - t_a)\tanh(ml)$$

$$\therefore Q_{fin} = -KA_{cs}\left[-m(t_0 - t_a)\tanh(ml)\right]$$

$$\Rightarrow Q_{fin} = KA_{cs}m(t_0 - t_a)\tanh(ml)$$

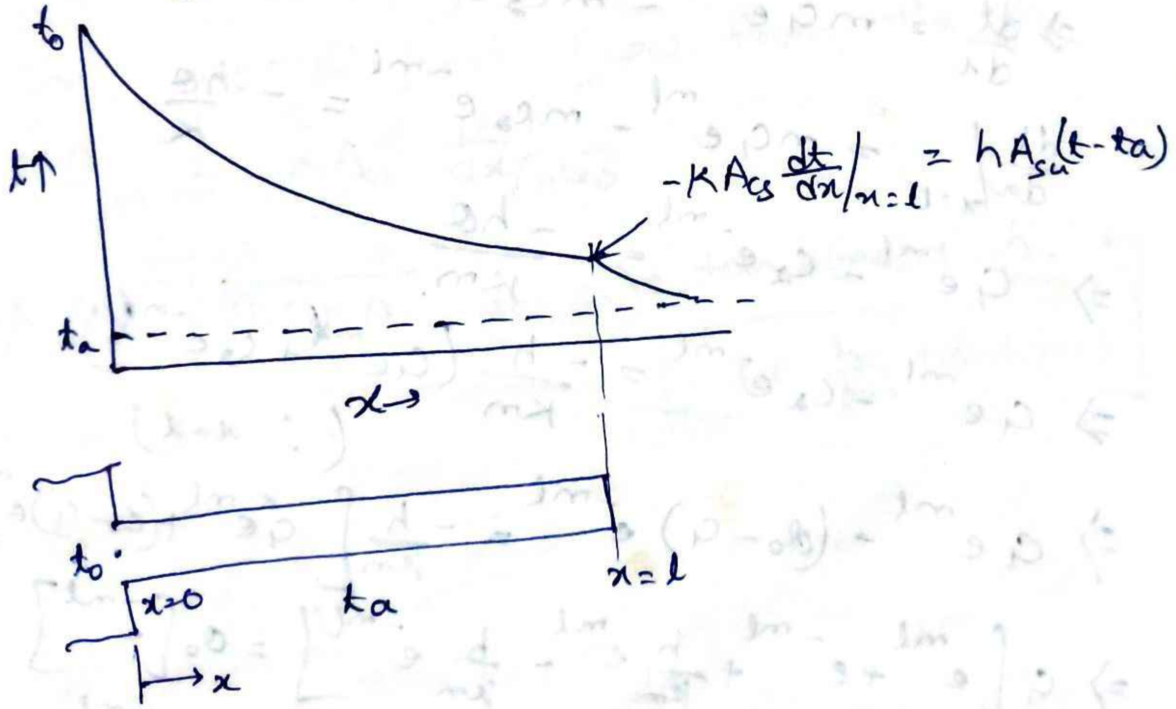
$$= KA_{cs}\sqrt{\frac{hP}{KA_{cs}}}(t_0 - t_a)\tanh(ml)$$

$$\boxed{Q_{fin} = \sqrt{hPKA_{cs}}(t_0 - t_a)\tanh(ml)}$$

Case-3

Fin of finite length & loses heat by convection.

Let's consider a fin of finite length losing heat at the tip



The boundary conditions are →

i) At $x = 0$, $\theta = \theta_0$

ii) Heat conducted to the fin at $x = l$

= Heat convected from the end to the surroundings

$$\Rightarrow -KA_{cs} \left.\frac{dt}{dx}\right|_{x=l} = h A_{su} (t - t_a)$$

A_{cs} = cross-sectional area for heat conduction

A_{su} = surface area from which convective heat transfer takes place

At tip of the fin $A_{cs} = A_{su}$

$$\Rightarrow -K \left.\frac{dt}{dx}\right|_{x=l} = h(t - t_a)$$

$$\Rightarrow \frac{dt}{dx} = -\frac{h\theta}{K} \text{ at } x = l$$

$\therefore$ $C_1 + C_2 = \theta_0$

& $t - t_a = C_1 e^{mx} + C_2 e^{-mx}$

$$\Rightarrow \frac{dt}{dx} = mC_1 e^{mx} - mC_2 e^{-mx}$$

$$\left.\frac{dt}{dx}\right|_{x=l} = mC_1 e^{ml} - mC_2 e^{-ml} = -\frac{h\theta}{K}$$

$$\Rightarrow C_1 e^{ml} - C_2 e^{-ml} = \frac{-h\theta}{Km}$$

$$\Rightarrow C_1 e^{ml} - C_2 e^{-ml} = -\frac{h}{Km}(C_1 e^{ml} + C_2 e^{-ml})$$

$(\because x = l)$

$$\Rightarrow C_1 e^{ml} - (\theta_0 - C_1) e^{-ml} = -\frac{h}{Km}\left[C_1 e^{ml} + (\theta_0 - C_1) e^{-ml}\right]$$

$$\Rightarrow C_1 \left[e^{ml} + e^{-ml} + \frac{h}{Km} e^{ml} - \frac{h}{Km} e^{-ml}\right] = \theta_0 \left[e^{-ml}\right] - \theta_0 \frac{h}{Km} e^{-ml}$$

$$\Rightarrow C_1 = \frac{\theta_0 \left[1 - \frac{h}{km}\right] e^{-ml}}{(e^{ml} + e^{-ml}) + \frac{h}{km}(e^{ml} - e^{-ml})}$$

$$C_2 = \theta_0 - \left[\frac{\theta_0 \left(1 - \frac{K}{hm}\right) e^{-ml}}{e^{ml} + e^{-ml} + \frac{h}{km}(e^{ml} - e^{-ml})}\right]$$

$$= \frac{\theta_0 \left[1 + \frac{h}{km}\right] e^{ml}}{e^{ml} + e^{-ml} + \frac{h}{km}(e^{ml} - e^{-ml})}$$

we know $\theta = C_1 e^{mx} + C_2 e^{-mx}$

$$\therefore \frac{\theta}{\theta_0} = \frac{\left[e^{m(l-x)} + e^{-m(l-x)}\right] + \frac{h}{km}\left[e^{m(l-x)} - e^{-m(l-x)}\right]}{e^{ml} + e^{-ml} + \frac{h}{km}(e^{ml} - e^{-ml})}$$

$$\Rightarrow \boxed{\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{\cosh\{m(l-x)\} + \frac{h}{km}\sinh\{m(l-x)\}}{\cosh(ml) + \frac{h}{km}\sinh(ml)}}$$

The rate of heat flow from the fin

$$Q_{fin} = -KA_{cs} \left.\frac{dt}{dx}\right|_{x=0}$$

$$\Rightarrow Q_{fin} = \sqrt{hPKA_{cs}}\,(t_0 - t_a)\left[\frac{\tanh(ml) + \frac{h}{km}}{1 + \frac{h}{km}\tanh(ml)}\right]$$

Efficiency of fin (η_{fin})

$$\eta_{fin} = \frac{\text{Actual heat transferred by the fin } (Q_{fin})}{\text{Max}^{m} \text{ heat that would be transferred if whole surface of the fin is maintained at the base temp.}}$$

$$\eta_{fin} = \frac{Q_{fin}}{Q_{max}}$$

$$\Rightarrow \eta_{fin} = \frac{\sqrt{hPKA_{cs}}\,(t_0 - t_a)}{hPL\,(t_0 - t_a)} = \sqrt{\frac{KA_{cs}}{hPl^2}}$$

$$= \frac{1}{ml} \rightarrow \text{For fin which is infinitely long}$$

For a fin which is insulated at the tip

$$\eta_{fin} = \frac{\sqrt{hPKA_{cs}}\,(t_0 - t_a)\tanh(ml)}{hPl\,(t_0 - t_a)}$$

$$= \frac{\tanh(ml)}{ml}$$

where $ml = \sqrt{\frac{hP}{KA_{cs}}} \cdot l = \sqrt{\frac{h\,2(b+y)}{kby}} \cdot l$

if b is large compared to y then

$$ml = \sqrt{\frac{2hb}{kby}} \cdot l = \sqrt{\frac{2h}{kyl}} \cdot l^{3/2} = \sqrt{\frac{2h}{KA_p}} \cdot l^{3/2}$$

Fin will always be used where the heat transfer coefficient is less suppose if there is fluid at one side & gas at the other side, it is preferred to use fins in gas side. This is the reason for not using fins in steam condenser tubes.

Effectiveness of fin (ε_{fin})

$$\varepsilon_{fin} = \frac{\text{Heat transfer with fin}}{\text{Heat transfer without fin}} = \frac{Q_{with\,fin}}{Q_{without\,fin}}$$

$$= \frac{\sqrt{hPkA_{cs}}\,(t_o - t_a)}{h A_{cs}\,(t_o - t_a)}$$

$$= \sqrt{\frac{PK}{hA_{cs}}} \quad \text{for infinitely long fin}$$

For a straight rectangular fin

$$\frac{P}{A_{cs}} = \frac{2(b+y)}{b \cdot y} \simeq \frac{2}{y}$$

$$\therefore \varepsilon_{fin} = \sqrt{\frac{2K}{hy}}$$

Notes ⇒

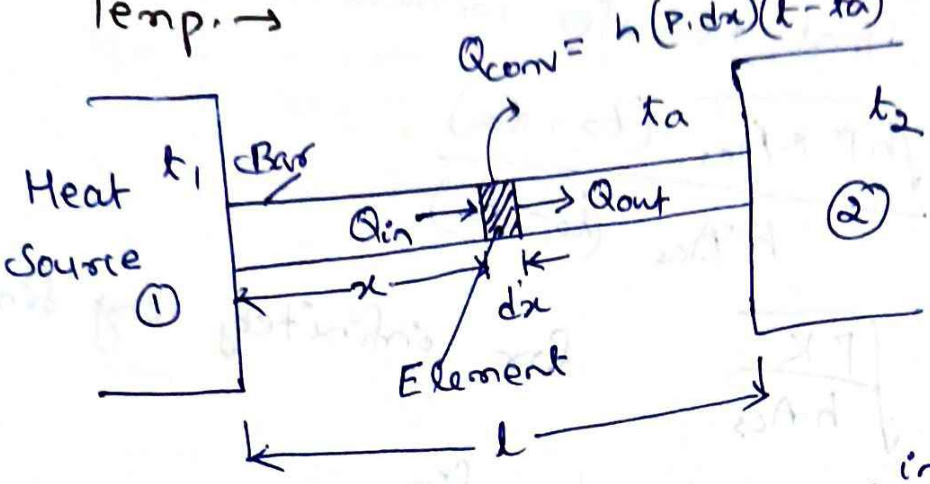
1. Fin effectiveness $\sqrt{\frac{PK}{hA_{cs}}}$ should be greater than 1. The use of fins on surfaces is justified only if $\frac{PK}{hA_{cs}} > 5$

2. If $\frac{P}{A_{cs}} \uparrow \Rightarrow \varepsilon_{fin} \uparrow$ so thin & closely spaced fins are preferred. The lower limit on the distance betn two adjacent fins is governed by the thickness of boundary layer that develops on the surface of the fin.

3. Uses of fin is justified where h is small

4. Use of fins will be more effective with materials of large thermal conductivities. Generally aluminium is used

Heat Transfer from a Bar Connected to the two Heat Sources at different Temp. →



Let l = length of the bar connecting two heat sources.

A_{cs} = cross-sectional area of the bar

P = Perimeter of the bar

t_1 = temp of heat source - 1

t_2 = temp of " " - 2

t_a = temp of air surrounding the bar

h = heat transfer coefficient

K = thermal conductivity of the bar

$\theta_1 = t_1 - t_a$

$\theta_2 = t_2 - t_a$

Consider an infinitely small element of the bar of thickness dx located at a distance x from heat source - 1

$$Q_{in} = -K A_{cs} \frac{dt}{dx}$$

$$Q_{out} = -K A_{cs}\left(t + \frac{dt}{dx}\,\delta x\right)$$

$$Q_{conv} = h\,P.\,dx\,(t - t_a)$$

Q_n

$Q_{n+d} = Q_n + \frac{d}{dx}(Q_n)\,dx$

$+ hp$ -

$= -KA\frac{d}{dx}\left(\frac{dT}{dx}\right)dx$

$+ hp\,(dx)(T - T_a) = 0$

Applying energy balance on the element

$$Q_{in} = Q_{out} + Q_{conv.}$$

$\Rightarrow KA\frac{d^2T}{dx^2} - hp\,(T - T_a) = 0$

$$-KA_{cs}\frac{dt}{dx} = -KA_{cs}\frac{d}{dx}\left(t + \frac{dt}{dx}\delta x\right) + hP.\delta x\,(t - t_a)$$

Upon simplification & rearrangement we have

$$\frac{d^2t}{dx^2} - \frac{hP}{KA_{cs}}(t - t_a) = 0$$

Replacing $(t - t_a)$ by θ

$$\frac{d^2\theta}{dx^2} - \frac{hP}{KA_{cs}}\theta = 0$$

The solⁿ to this differential equⁿ is

$$\theta = C_1 e^{mx} + C_2 e^{-mx} \quad \text{where } m = \sqrt{\frac{hP}{KA_{cs}}}$$

C_1 & C_2 are evaluated from boundary conditions →

i) At $x = 0$, $\theta = \theta_1$

ii) At $x = l$, $\theta = \theta_2$

$$\therefore \theta_1 = C_1 + C_2 \quad \text{— (a)}$$

$$\theta_2 = C_1 e^{ml} + C_2 e^{-ml} \quad \text{— (b)}$$

solving equⁿs (a) & (b) we have

$$C_1 = \frac{\theta_2 - \theta_1 e^{-ml}}{e^{ml} - e^{-ml}}$$

$$C_2 = \frac{\theta_1 e^{ml} - \theta_2}{e^{ml} - e^{-ml}}$$

Substituting the values of C_1 & C_2 we have

$$\theta = \left[\frac{\theta_2 - \theta_1 e^{-ml}}{e^{ml} - e^{-ml}}\right] e^{mx} + \left[\frac{\theta_1 e^{ml} - \theta_2}{e^{ml} - e^{-ml}}\right] e^{-mx}$$

$$= \frac{\theta_1\left[e^{ml} \cdot e^{-mx} - e^{-ml} \cdot e^{mx}\right]}{e^{ml} - e^{-ml}} + \frac{\theta_2\left[e^{mx} - e^{-mx}\right]}{e^{ml} - e^{-ml}}$$

$$= \frac{\theta_1 \sinh\{m(l-x)\} + \theta_2 \sinh(mx)}{\sinh(ml)}$$

The rate of heat loss

$$Q = \int_0^l h\, P\, dx\, (t - t_a) = \int_0^l h\, P\, dx \cdot \theta$$

$$= hP \int_0^l \frac{\theta_1 \sinh(ml - x) + \theta_2 \sinh(mx)}{\sinh(ml)}\, dx$$

$$= \frac{hP}{m \sinh(ml)} (\theta_1 + \theta_2)\{\cosh(ml) - 1\}$$

$$\Rightarrow Q = \sqrt{hPKA_{cs}}\, (\theta_1 + \theta_2) \left[\frac{\cosh(ml) - 1}{\sinh(ml)}\right]$$

$$= \frac{1}{h_i A_i} + \frac{R_{fi}}{A_i} + \frac{\ln(D_o/D_i)}{2\pi k L} + \frac{R_{fo}}{A_o} + \frac{1}{h_o A_o}$$

Analysis of HXs

HX should be selected in such a way that it will achieve a specified temp. change in a fluid stream of known mass flow rate, or to predict the outlet temps of the hot & cold fluid streams in a specified HX.

- Two methods are used in the analysis of HX
  - Log Mean Temp. Difference (LMTD) method
  - Effectiveness - NTU Method

Some general considerations →

- HXs usually operate for long periods of time with no change in their operating cond. Therefore, they can be modeled as steady-flow devices.
- The mass flow rate of each fluid remains constant, & the fluid properties such as tem & velocity at any inlet or outlet remain the same.

- The fluids streams experience little or no change in their velocities & elevations, & thus the KE & PE changes are negligible.

- The specific heat of a fluid, changes with temp. But in a specified temp range, it can be treated as a constant at some avg. value with little loss in accuracy.

- Axial heat conduction along the tube is usually insignificant & can be considered negligible.

- The outer surface of the HX is assumed to be perfectly insulated, so that there is no heat loss to the surrounding medium, & any heat transfer occurs betn the two fluids only.

Under these assumptions, the 1st law of thermodynamics requires that the rate of heat transfer from the hot fluid be equal to the rate of heat transfer to the cold one.

ie. $\dot{Q} = \dot{m}_c C_{pc} (T_{c,o} - T_{c,i})$

& $\dot{Q} = \dot{m}_h C_{ph} (T_{h,i} - T_{h,o})$

c → cold fluid, h → hot fluid

$\dot{m}_c, \dot{m}_h$ - mass flow rate

C_{pc}, C_{ph} - Specific heats

$T_{c,o}, T_{h,o}$ - Outlet temp

$T_{c,i}, T_{h,i}$ - Inlet temp

Heat capacity rate = Mass flow rate x Specific heat of a fluid
(C)

$\Rightarrow C_h = \dot{m}_h c_{ph}$ & $C_c = \dot{m}_c c_{pc}$

The heat capacity rate of a fluid stream represents the rate of heat transfer needed to change the temp of the fluid stream by 1°c as it flows through a HX.

- The fluid with a large heat capacity rate will experience a small temp change & the fluid with a small heat capacity rate will experience a large temp change

⇒ Doubling the mass flow rate of a fluid while leaving everything else unchanged will halve the temp of change of that fluid.

$\therefore \dot{Q} = C_c (T_{co} - T_{ci})$

& $\dot{Q} = C_h (T_{hi} - T_{ho})$

When $C_c = C_h \Rightarrow T_{co} - T_{ci} = T_{hi} - T_{ho}$

- One of the fluids in a condenser or a boiler undergoes a phase-change process.

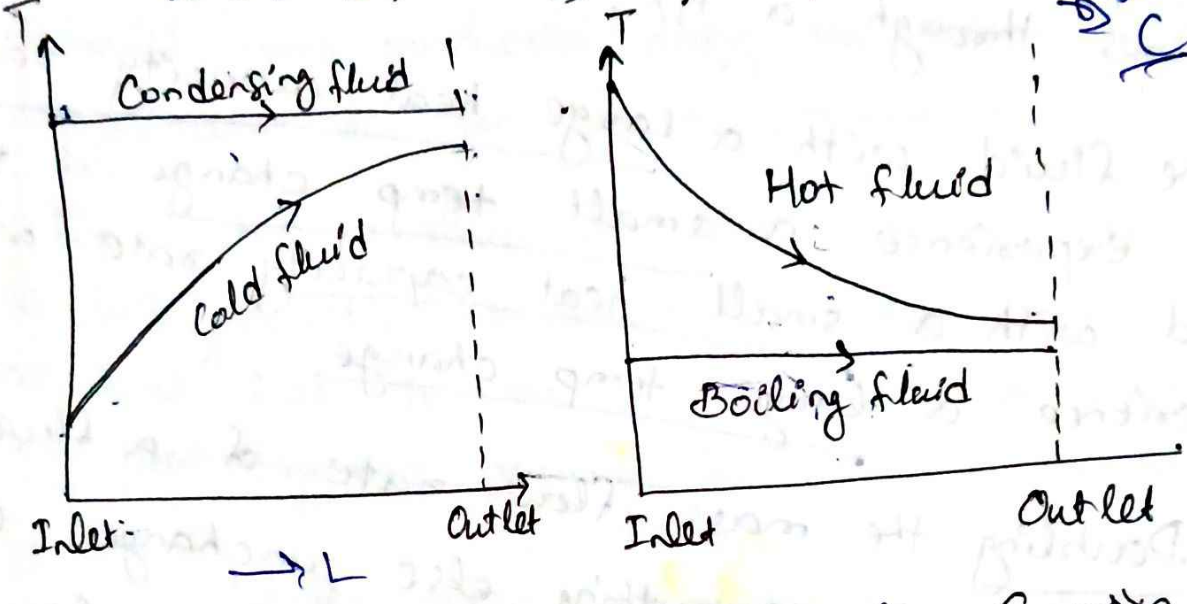
$$\dot{Q} = \dot{m} h_{fg}$$

where $\dot{m}$ = rate of evaporation or condensation of the fluid.

- During a phase-change process the fluid absorbs or releases a large amount of heat at constant temp.
- The heat capacity rate of a fluid during a phase-change-process $\approx \infty$ because the temp change is practically zero. $Q = mc_p \Delta T$

When $\Delta T \to 0 \Rightarrow C = \dot{m}c_p \to \infty$

$\Rightarrow C = \frac{Q}{\Delta T}$



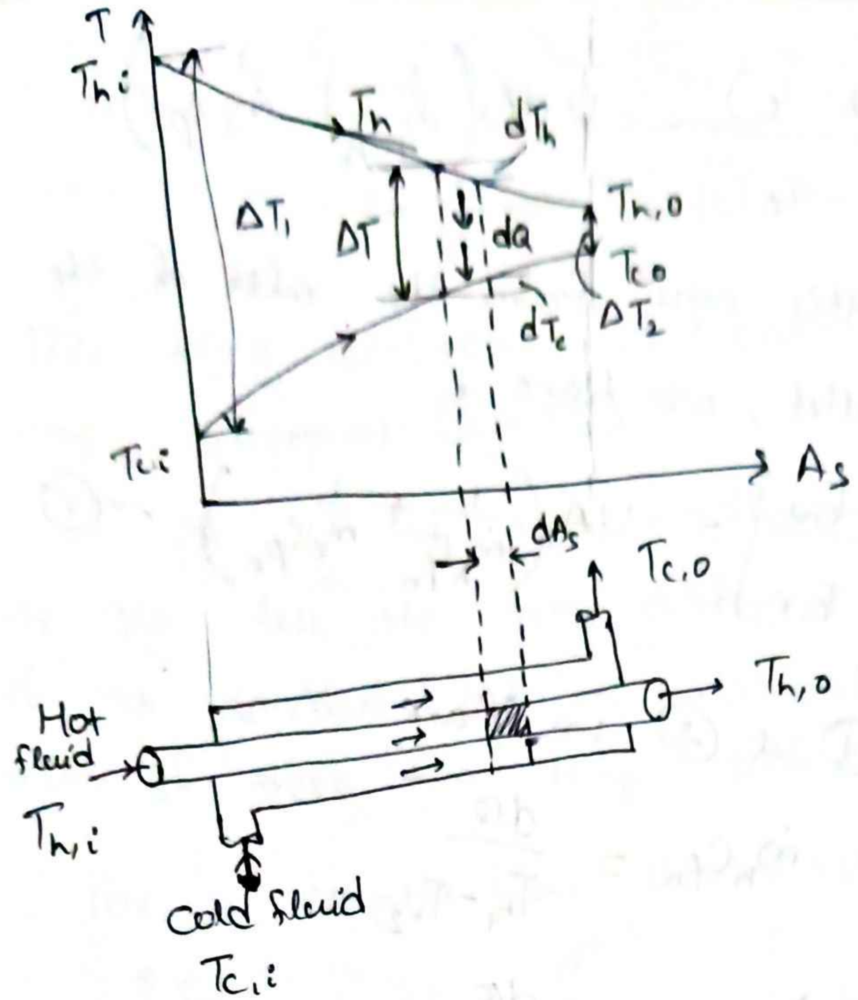
Condenser $C_h \to \infty$ Boiler $C_c \to \infty$

Log Mean Temp. Difference Method

Let's consider the parallel-flow double-pipe HX

The temp difference ΔT betⁿ the hot & cold fluids is large at the inlet of the HX but decreases exponentially toward the outlet.

Assuming the outsurface of the HX to be well insulated so that heat transfer occurs betⁿ the two fluids only



$\Delta T_1 = T_{hi} - T_{ci}$

$\Delta T_2 = T_{h,o} - T_{co}$

$d\dot{Q} = U(T_h - T_c)\, dA_s$

Energy balance on each fluid in a small element of HX

$$d\dot{Q} = -\dot{m}_h C_{p_h} dT_h \quad (\because \text{Temp change is } -ve) \quad —(1)$$

$$\& \; d\dot{Q} = \dot{m}_c C_{p_c} dT_c \quad —(2)$$

$$\Rightarrow dT_h = \frac{-d\dot{Q}}{\dot{m}_h C_{p_h}}$$

$$\& \; dT_c = \frac{d\dot{Q}}{\dot{m}_c C_{p_c}}$$

$$\therefore dT_h - dT_c = d(T_h - T_c) = -d\dot{Q}\left(\frac{1}{\dot{m}_h C_{p_h}} + \frac{1}{\dot{m}_c C_{p_c}}\right)$$

Again $d\dot{Q} = U(T_h - T_c)\, dA_s$

$$\Rightarrow d(T_h - T_c) = -U(T_h - T_c)\left(\frac{1}{\dot{m}_h C_{p_h}} + \frac{1}{\dot{m}_c C_{p_c}}\right) dA_s$$

$$\Rightarrow \frac{d(T_h - T_c)}{T_h - T_c} = -U\, dA_s\left(\frac{1}{\dot{m}_h C_{p_h}} + \frac{1}{\dot{m}_c C_{p_c}}\right)$$

Integrating this equ^n from the inlet of the HX to its outlet, we have

$$\ln\left[\frac{T_{h,o} - T_{c,o}}{T_{h,i} - T_{c,i}}\right] = -UA_s\left(\frac{1}{\dot{m}_h C_{p_h}} + \frac{1}{\dot{m}_c C_{p_c}}\right) \quad \text{—— (3)}$$

From equ^n (1) & (2) we have

$$\dot{m}_h C_{p_h} = \frac{dQ}{T_{h_1} - T_{h_2}}$$

$$\& \quad \dot{m}_c C_{p_c} = \frac{dQ}{T_{c_2} - T_{c_1}}$$

Substituting this in equ^n (3) we have

$$\ln\left[\frac{T_{h,o} - T_{co}}{T_{hi} - T_{ci}}\right] = -UA_s\left[\frac{T_{hi} - T_{ho}}{dQ} + \frac{T_{co} - T_{ci}}{dQ}\right]$$

$$= \frac{UA_s}{dQ}\left(T_{ho} - T_{hi} - T_{co} + T_{ci}\right)$$

$$\Rightarrow dQ = UA_s \frac{(T_{ho} - T_{co}) - (T_{hi} - T_{ci})}{\ln\left[(T_{h,o} - T_{co})/(T_{hi} - T_{ci})\right]} \quad \text{—— (4)}$$

We know $dQ = UA\,\Delta T_m$ —— (5)

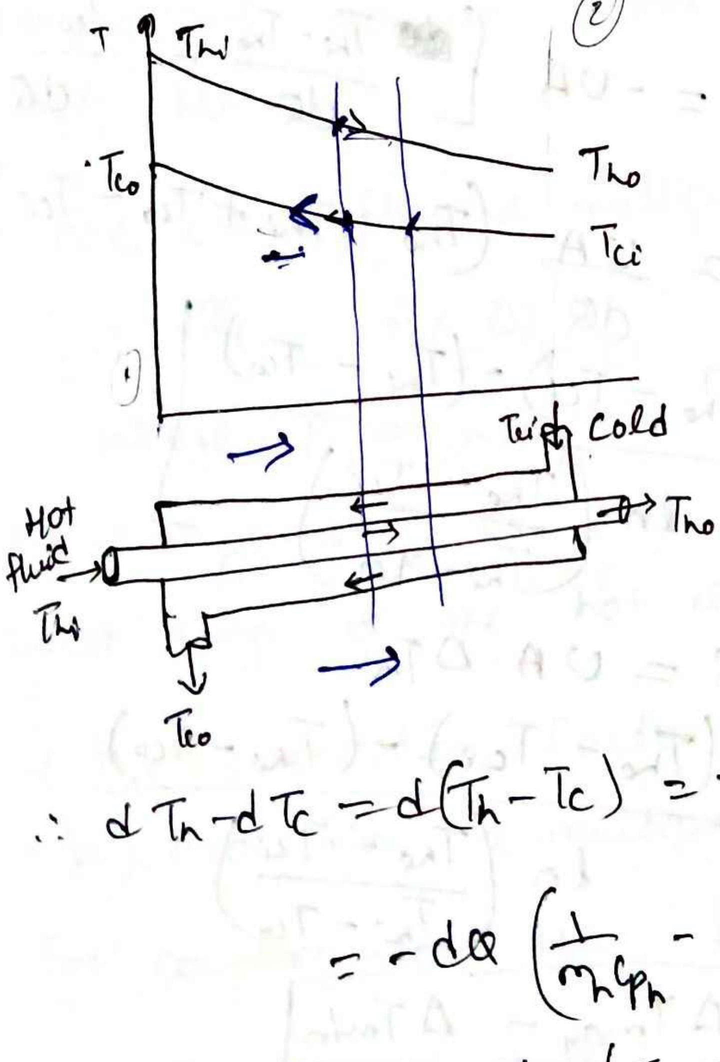
Comparing equ^n (4) & (5) we have

$$\Delta T_{lm} = \frac{(T_{ho} - T_{co}) - (T_{hi} - T_{ci})}{\ln\left[(T_{ho} - T_{co}) / (T_{hi} - T_{ci})\right]}$$

This temp difference is called the log mean temp. difference (LMTD)

- It is the temp difference at one end of the Hx less the temp difference at the other end of Hx divided by the natural logarithm of the ratio of these two temp. differences.

For a counter-flow double-pipe Hx



We know

$$dQ = -m_h c_{p_h} \, dT_h \quad -(1)$$

$$\& \; dQ = -m_c c_{p_c} \, dT_c \quad -(2)$$

$$\Rightarrow dT_h = -\frac{dQ}{m_h c_{p_h}}$$

$$\& \; dT_c = -\frac{dQ}{m_c c_{p_c}}$$

$$\therefore dT_h - dT_c = d(T_h - T_c) = -\frac{dQ}{m_h c_{p_h}} + \frac{dQ}{m_c c_{p_c}}$$

$$= -dQ\left(\frac{1}{m_h c_{p_h}} - \frac{1}{m_c c_{p_c}}\right)$$

Again $dQ = U \, dA_s (T_h - T_c)$

$$\Rightarrow d(T_h - T_c) = -U \, dA_s (T_h - T_c)\left(\frac{1}{m_h c_{p_h}} - \frac{1}{m_c c_{p_c}}\right)$$

$$\Rightarrow \frac{d(T_h - T_c)}{T_h - T_c} = -U\,dA_s\left(\frac{1}{m_h C_{ph}} - \frac{1}{m_c C_{pc}}\right)$$

Integrating this equ^n from inlet to outlet, we have

$$\ln\left(\frac{T_{ho} - T_{ci}}{T_{hi} - T_{co}}\right) = -UA\left(\frac{1}{m_h C_{ph}} - \frac{1}{m_c C_{pc}}\right)$$

From equ^n ① & ② we have

$$m_h C_{ph} = \frac{dQ}{T_{hi} - T_{ho}}$$

$$\& \quad m_c C_{pc} = \frac{dQ}{T_{co} - T_{ci}}$$

$$\Rightarrow \ln\left(\frac{T_{ho} - T_{ci}}{T_{hi} - T_{co}}\right) = -UA\left[\frac{T_{hi} - T_{ho}}{dQ} + \frac{T_{co} - T_{ci}}{dQ}\right]$$

$$= \frac{UA}{dQ}\left(T_{ho} - T_{hi} + T_{co} - T_{ci}\right)$$

$$\Rightarrow dQ = UA\left[\frac{(T_{ho} - T_{ci}) - (T_{hi} - T_{co})}{\ln\left(\frac{T_{ho} - T_{ci}}{T_{hi} - T_{co}}\right)}\right]$$

We know $dQ = UA\,\Delta T_m$

$$\therefore \Delta T_m = \frac{(T_{ho} - T_{ci}) - (T_{hi} - T_{co})}{\ln\left(\frac{T_{ho} - T_{ci}}{T_{hi} - T_{co}}\right)}$$

$$\therefore \boxed{\Delta T_m = \frac{\Delta T_{max} - \Delta T_{min}}{\ln \frac{\Delta T_{max}}{\Delta T_{min}}}}$$

For specified inlet & outlet temp., the log mean temp. difference for a counter-flow HX is always greater than that for a parallel-flow HX

$\Rightarrow \Delta T_{lm,CF} > \Delta T_{lm,PF}$

Thus a smaller surface area & thus a smaller HX is needed to achieve a specified heat transfer rate in a counter-flow HX.

So counter-flow HX is commonly used.

- A condenser or a boiler can be considered to be either a parallel or counter flow HX.

- For cross-flow & multipass shell-&-tube HX

$$\Delta T_{lm} = F \times \Delta T_{lm,CF}$$

Where $F \rightarrow$ correction factor which depends on the geometry of the HX & the inlet & outlet temp of the hot & cold fluid streams.

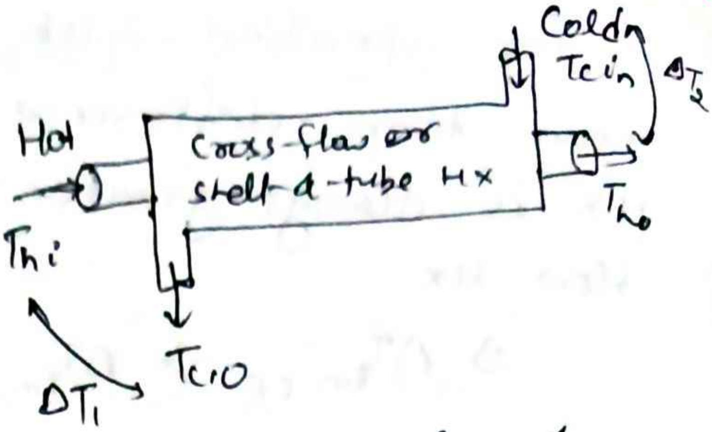
- The correction factor is less than unity for a cross-flow & multipass shell-&-tube HX

- For counter-flow HX, $F = 1$

F for cross-flow & shell-&-tube HX can be found by the graphs plotted betn P & R where

$$P = \frac{t_2 - t_1}{T_1 - t_1}$$

$$\& \quad R = \frac{T_1 - T_2}{t_2 - t_1}$$



Where 1 & 2 represents the inlet & outlet

For a shell-&-tube-HX

T → Shell side temp

t → Tube " "

- The correction factor for a condenser or boiler is $F = 1$

LMTD Method → Procedure

1. Select the type of HX suitable for the application
2. Determine any unknown inlet or outlet temp & heat transfer rate using an energy balance
3. Calculate log mean temp. difference ΔT_{lm} & correction factor F, if necessary.
4. Find the overall heat transfer coefficient U.
5. Calculate heat transfer surface area As

HE calcu^n → 10 qu^s → m_h, c_{ph}, m_c, c_{pc}, T_{h1}, T_{h2}, T_{c1}, T_{c2}, U, Q

For Desig HE → m_h, c_{ph}, m_c, c_{pc}, T_{h1}, T_{c1}, T_2 (or T_{c2}) & U_0 are given

From the equ^n $Q = m_h c_{ph} (T_{h1} - T_{h2}) = m_c c_{pc} (T_{c2} - T_{c1}) = U A \theta_m$

T_{c2} (or T_{h2}), θ_m & A_0 → calculated

T_{c2} & T_{h2} → calcu Trial &

Heat exchangers are devices that facilitate the exchange of heat bet^n two fluids that are at different temps while keeping them from mixing with each other.

- Heat exchangers are used in a wide range of applications, from heating & air-conditioning systems in a household, to chemical processing & power production in large plants.

- Heat transfer in a HX usually involves convection in each fluid & conduction through the wall separating the two fluids

- The rate of heat transfer bet^n the two fluids at a location in a HX depends on the magnitude of the temp. difference at that location, which varies along the heat exchanger.

- The weight & size of HX used in space or aeronautical applications are very important parameters

Classification of Heat Exchangers

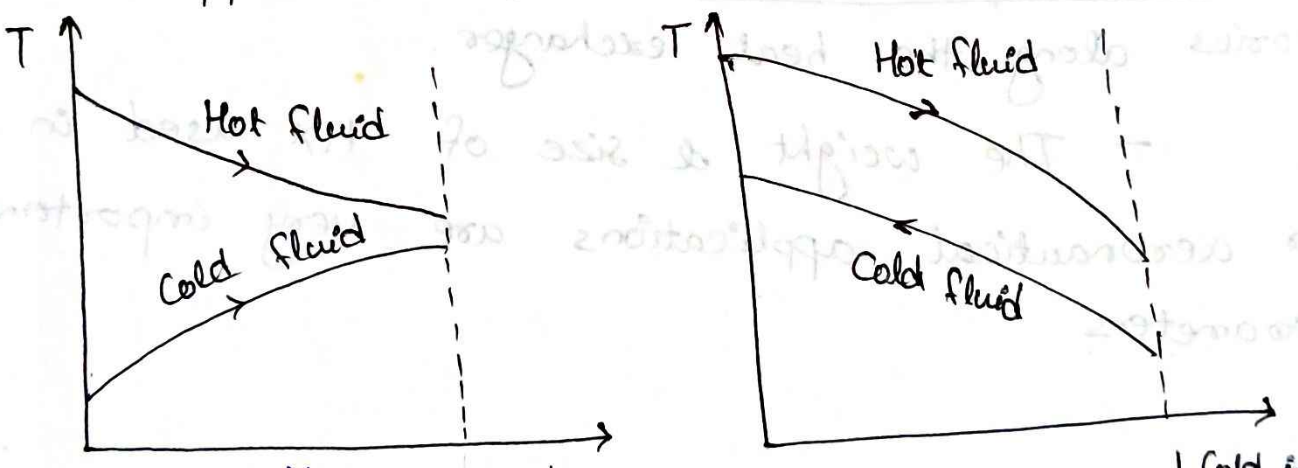
- The simplest type of heat exchanger consists of two concentric pipes of different diameters, called the <u>double-pipe</u> heat exchanger.

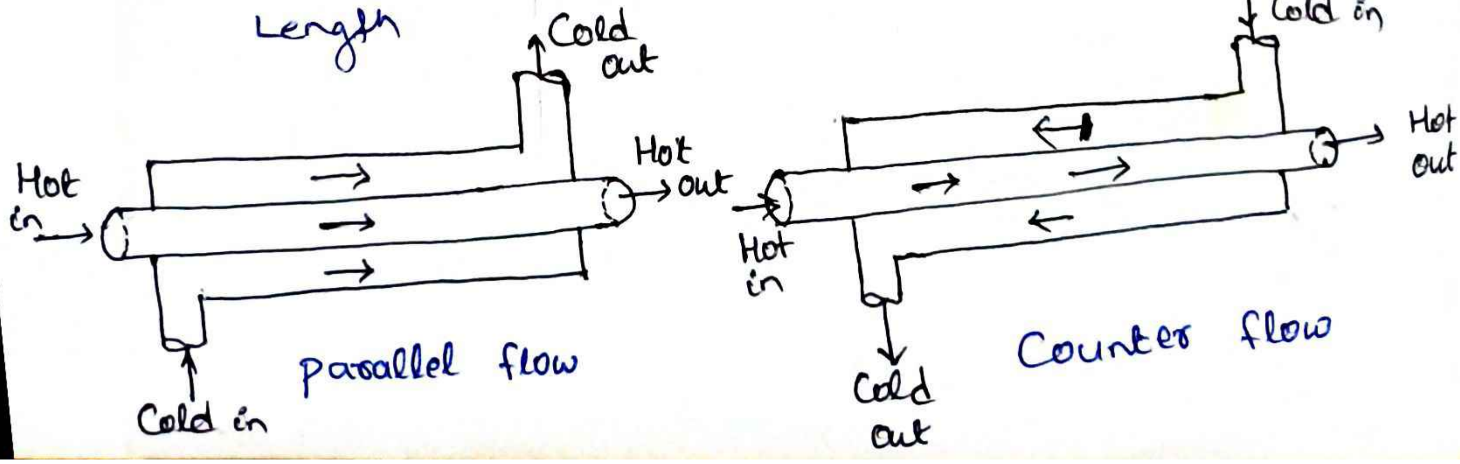
- One fluid in a double-pipe HX flows through the smaller pipe while the other fluid flows through the annular space betⁿ the two pipes. Nature of heat exchange process → direct contact, Indirect contact

- Two types of flow arrangement are possible in a double-pipe HX →

  - In <u>parallel flow</u>, both the hot & cold fluids enter the heat exchanger at the same end and move in the same direction.

  - In <u>counter flow</u>, the hot & cold fluids enter the HX at opposite ends and flow in opposite directions.



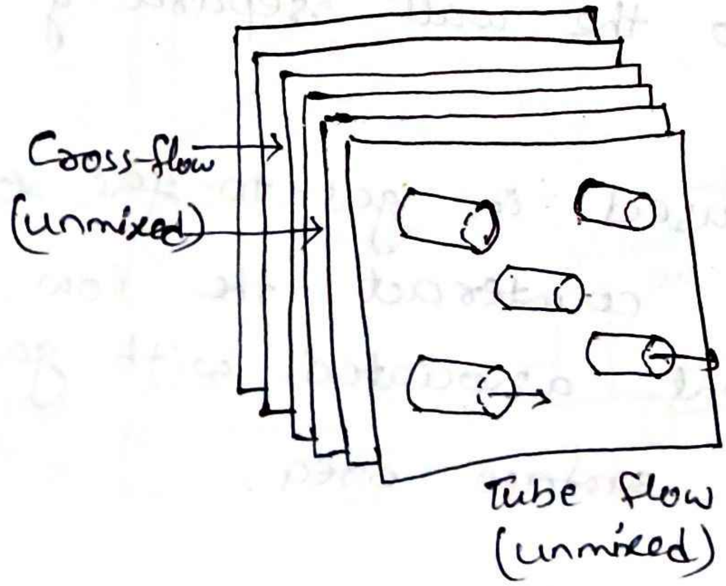


- The type of HX, which is specifically designed to realize a large heat transfer surface area per unit volume, is the compact HX.
- The ratio of heat transfer surface area of a heat exchanger to its volume is called the area density (β)
- A HX with $\beta > 700\ m^2/m^3$ is called compact

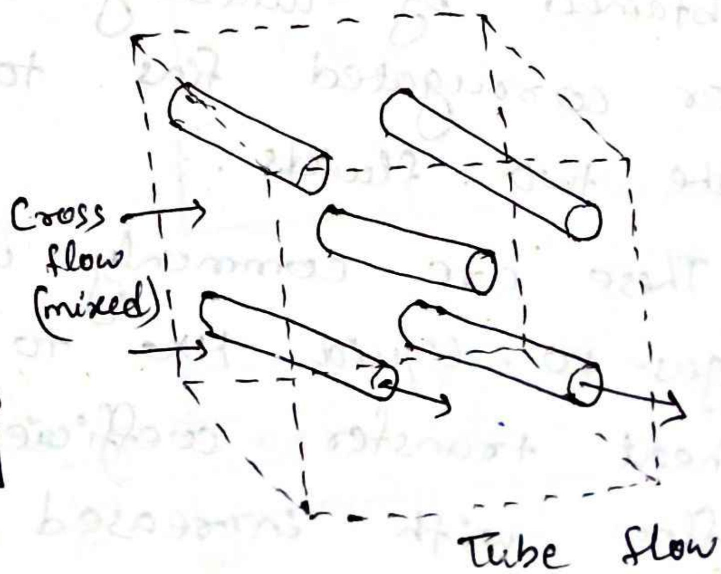
Examples → Car Radiators $\beta \approx 1000\ m^2/m^3$
Glass ceramic gas turbine HX
Regenerator of a Stirling engine
Human lung $\beta \approx 20{,}000\ m^2/m^3$

- Compact HXs enable us to achieve high heat transfer rates betⁿ two fluids in a small volume
- They are commonly used in applications with strict limitations on the weight & volume of HX.
- They have low overall heat-transfer coefficients
- The large surface area in compact HXs is obtained by attaching closely spaced thin plate or corrugated fins to the wall separating the two fluids.
- These are commonly used in gas-to-gas & gas-to-liquid HXs to counteract the low heat transfer coefficient associated with gas flow with increased surface area.

- Car radiator → Water-to-air compact HX
  - Fins are attached to the air side of the tube surface.
- In Compact HX, the two fluids usually move ⊥r to each other & such flow is called **cross-flow**.
- Cross-flow is further classified as
  - Unmixed
  - & mixed flow
- In fig-1 the cross-flow is said to be unmixed since the plate fins force the fluid to flow through a particular interfin spacing & prevent it from moving in the transverse dirn.
- In fig-2 the flow is said to be mixed since the fluid now is free to move in the transverse dirn.
- Car radiator → Both fluids are unmixed



Both fluids unmixed



One fluid mixed, one fluid unmixed

- If a fluid is unmixed, there can be a temp. gradient both parallel & normal to the flow direction, whereas when the fluid is mixed, there will be a tendency for the fluid temp to equalize in the dirⁿ normal to the flow.

- The fluid is mixed or unmixed influences the overall heat transfer in the exchanger because this heat transfer is dependent on the temp difference betⁿ the hot & cold fluids.

Shell-&-tube HX contain a large nos. of tubes packed in a shell with their axes parallel to that of the shell.

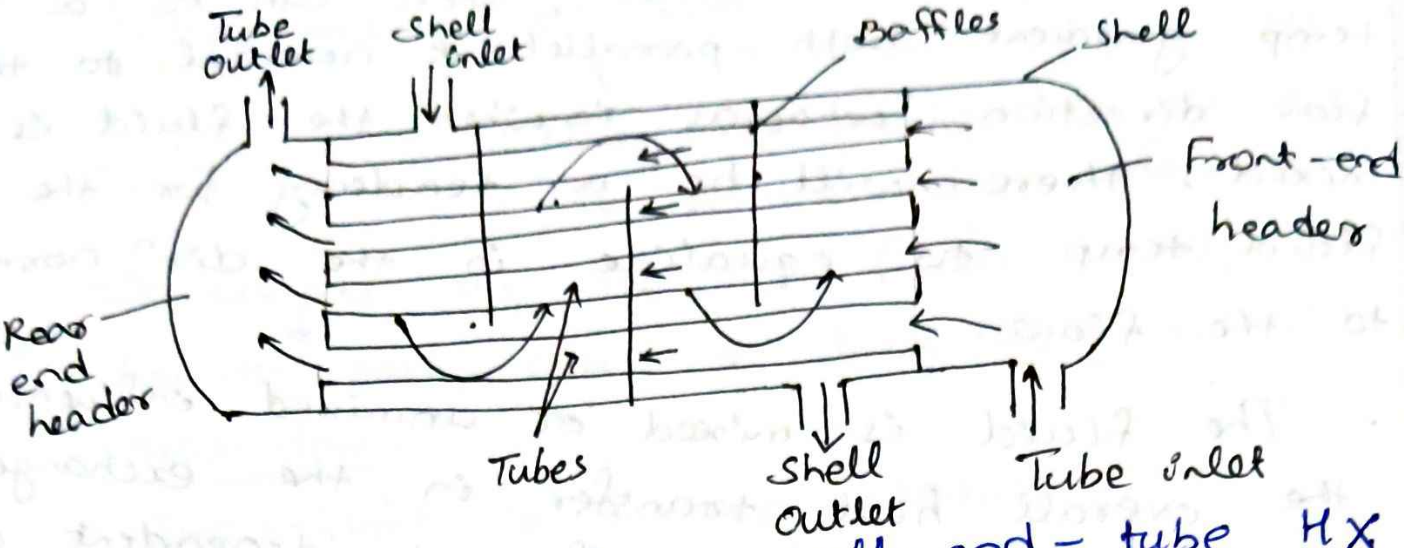
- Heat transfer takes place as one fluid flows inside the tubes while the other fluid flows outside the tubes through the shell.

- Baffles are commonly placed in the shell to force the shell-side fluid to flow across the shell to enhance heat transfer & to maintain uniform spacing betⁿ the tubes.

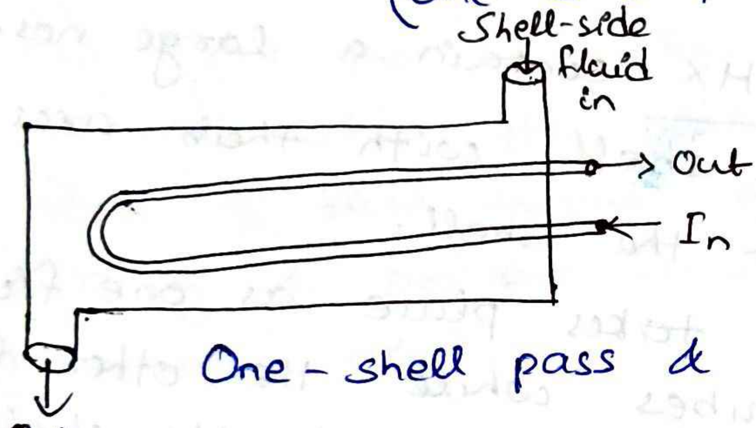
- The tubes in a shell-&-tube HX open to some large flow areas called headers at both ends of the shell.

- These are widely used in the chemical-process industries.

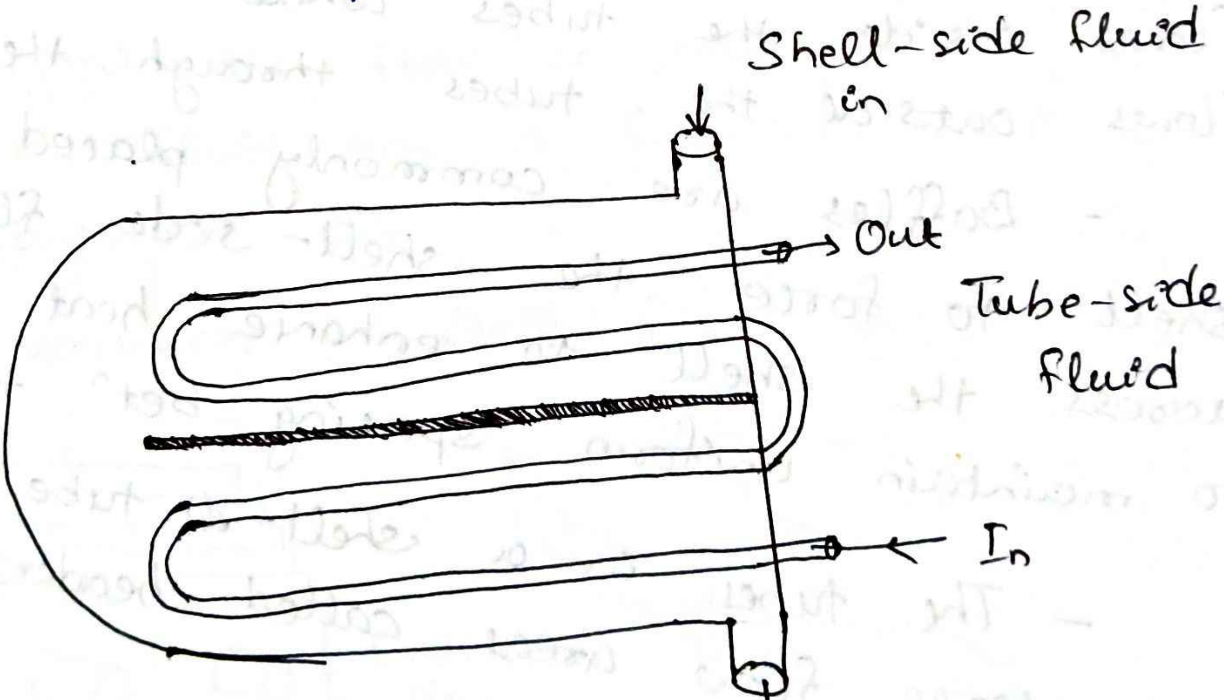
- These are not suitable for use in automobile



Schematic of a shell-and-tube HX (One-shell pass & one-tube pass)



One-shell pass & two-tube passes



Two-shell passes & 4-tube passes

- Shell-&-tube HX are further classified according to the no. of shell & tube passes involved.
- HX in which all the tubes make one U-turn in the shell, are called one-shell-pass & two-tube-passes HX.
- A HX that involves two passes in the shell & 4 passes in the tubes is called a two-shell-passes & 4-tube-passes HX.

Plate & frame or just Plate HX → consists of a series of plates with corrugated flat flow passes.
- The hot & cold fluids flow in alternate passages, & thus each cold fluid stream is surrounded by two hot fluid streams, resulting in very effective heat transfer.
- These are suitable for liquid-to-liquid HX provided that the hot & cold fluid streams are at about the same pressure.

Regenerative HX → It involves the alternate passage of the hot & cold fluid streams through the same flow area.
- The static-type regenerative HX is basically a porous mass that has a large heat storage capacity
- Hot & cold fluids flow through this porous mass alternatively.

- Heat is transferred from the hot fluid to the matrix of the regenerator during the flow of the hot fluid, & from the matrix to the cold fluid during the flow of the cold fluid.
- The matrix serves as a temporary heat storage medium.

- The <u>dynamic-type regenerator</u> involves a rotating drum & continuous flow of the hot & cold fluid through different portions of the drum so that any portion of the drum passes periodically through the hot stream, storing heat, & then through the cold stream, rejecting this stored heat.
- Drum serves as the medium to transport the heat from the hot to the cold fluid stream.

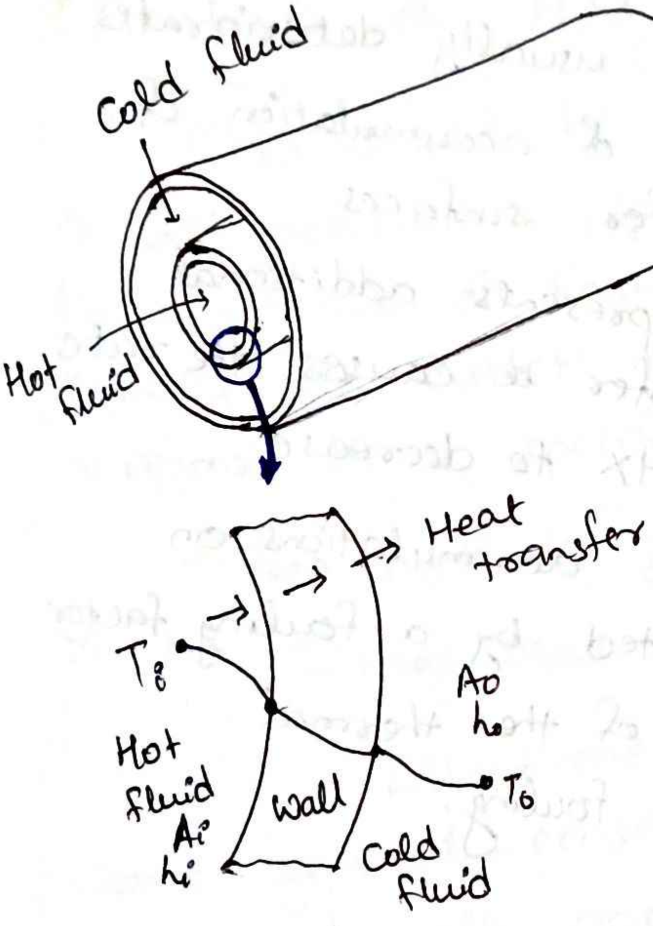
✓ <u>Condenser</u> → It is a HX in which one of the fluids is cooled & condenses as it flows through the HX.

✓ <u>Boiler</u> → It is a HX in which one of the fluids absorbs heat & vaporizes.

<u>Space radiator</u> → It is a HX that transfers heat from the hot fluid to the surrounding space by radiation.

Overall Heat Transfer Coefficient

- A HX involves two flowing fluids separated by a solid wall.
- Heat is first transfered from the hot fluid to the wall by convection, through the wall by conduction, & from the wall to the cold fluid by convection.



$i \rightarrow$ inner surface
$o \rightarrow$ outer " of the inner tube

$A_i = \pi D_i L$

$A_o = \pi D_o L$

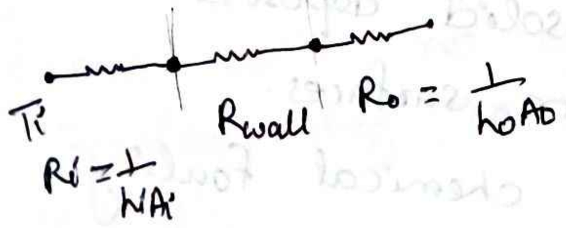
Thermal resistance of the tube wall

$$R_{wall} = \frac{\ln(D_o/D_i)}{2\pi K L}$$

$L \rightarrow$ Length of the tube

$$\therefore R_{total} = R_i + R_{wall} + R_o$$

$$= \frac{1}{h_i A_i} + \frac{\ln(D_o/D_i)}{2\pi K L} + \frac{1}{h_o A_o}$$



Rate of heat transfer

$$\dot{Q} = \frac{\Delta T}{R} = UA\Delta T = U_i A_i \Delta T = U_o A_o \Delta T$$

Where $U \rightarrow$ Overall heat transfer coefficient
Unit $\rightarrow$ $W/m^2\,^\circ C$

$$\Rightarrow \frac{1}{UA} = \frac{1}{U_i A_i} = \frac{1}{U_o A_o} = R_{total} = \frac{1}{h_i A_i} + R_{wall} + \frac{1}{h_o A_o}$$

$$\Rightarrow U_i = \frac{1}{\frac{1}{h_i} + \frac{A_i \ln(r_o/r_i)}{2\pi k L} + \frac{A_i}{A_o h_o}}$$

R_{wall} may often be neglected, since a thin wall of large thermal conductivity is generally used.

$$\& \quad U_o = \frac{1}{\frac{A_o}{A_i h_i} + \frac{A_o \ln(r_o/r_i)}{2\pi k L} + \frac{1}{h_o}}$$

Fouling Factor

- The performance of HXs usually deteriorates with time as a result of accumulation of deposits on heat transfer surfaces.
- A layer of deposits represents additional resistance to heat transfer & causes the rate of heat transfer in a HX to decrease.
- The net effect of these accumulations on heat transfer is represented by a fouling factor R_f, which is a measure of the thermal resistance introduced by fouling.

→ Types of fouling ⇒

- the precipitation of solid deposits in a fluid on the heat transfer surfaces.
- corrosion & other chemical fouling

Experimentally $R_f = \frac{1}{U_{dirty}} - \frac{1}{U_{clean}}$

- The fouling factor is zero for a new HX & increases with time as the solid deposits build up on the HX surface.
- The fouling factor depends on the operating temp & the velocity of the fluids, as well as the length of service.
- Fouling increases with increasing temp & decreasing velocity.

$$\frac{1}{UA_s} = \frac{1}{U_iA_i} = \frac{1}{U_oA_o} = R$$

$$= \frac{1}{h_iA_i} + \frac{R_{fi}}{A_i} + \frac{\ln(D_o/D_i)}{2\pi kL} + \frac{R_{fo}}{A_o} + \frac{1}{h_oA_o}$$

Radiation

- Thermal radiation or radiation heat transfer refers to the heat energy emitted by the bodies because of their temperatures
- All bodies at a temp. above absolute zero temp. emit energy by a process of electro-magnetic radiation.
- The intensity of radⁿ depends upon the temp and nature of the surface.
- The energy transfer by radiation does not require any medium betⁿ hot & cold surfaces.
- The energy transfer by radiation is the fastest (at the speed of light because the rate of heat transfer by conduction & convection varies as the temp difference to the 1st power. whereas the radiant heat exchange betⁿ two bodies depends on the difference betⁿ their temp to the 4th power)
- The heat transfer through an evacuated space can occur only by radiation.

Example → When a person sits infront of a fire, he gets most of the heat energy by radiation.

- Radiation heat transfer can also occur betⁿ two bodies separated by a medium that is colder than the both bodies.

Example → The energy emitted by sun reaches the earth surface after travelling through space & extremely cold air layers at high altitudes.

- Two concepts are used in study of thermal radiation → - Maxwell theory
  - Max Plank's theory

Maxwell's Theory

According to Maxwell electromagnetic theory, the energy is transferred from a hot body to cold body in the form of electromagnetic waves.

The electromagnetic waves are characterised by their frequency ν and wavelength λ, in a medium as

$$\lambda = \frac{c}{\nu} \qquad c = \text{speed of light}$$

In vacuum $c = c_0 = 2.998 \times 10^8$ m/s

This concept is useful in studies for the prediction of the rad" properties of the surfaces & materials.

Max Planck's Theory

According to Max Planck's concept, the propagation of thermal radiation takes place in form of discrete quanta called photons, each quantum having an energy of

$$E = h\nu = \frac{hc}{\lambda}$$

where $h \rightarrow$ Planck's constant

$$= 6.6256 \times 10^{-34} \text{ Js}$$

$$\therefore E \propto \frac{1}{\lambda}$$

$\Rightarrow$ Shorter wavelength radiation possesses the larger photon energy.

- This theory is used to predict the magnitude of emitted energy by a body at a given temp under ideal conditions.

Black body Radiation

A blackbody is defined as a body which is a perfect emitter and absorber of radiation.

The properties of a black body →

1. A blackbody absorbs all incident radⁿ from all directions at all wavelengths

2. At a specified temp & wavelength, no body can emit energy more than a blackbody.

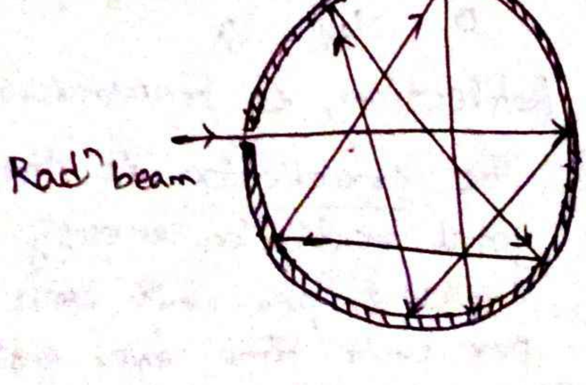
3. The radⁿ emitted by a blackbody depends upon wavelength & temp, but it is independent of direction.

4. A blackbody neither reflects nor transmits any amount of incident radⁿ.

In actual, no surface is a ~~pre~~ perfect black body.

The ice & snow are almost black to radⁿ at all wavelength.

Consider a hollow enclosure with a very small hole ~~P~~ for the passage of incident radⁿ. It experiences many reflections within the enclosure, & almost entire beam is absorbed by the cavity & the blackbody behaviour is experienced.



Spectral & Total Emissive Power

- All surfaces at a temp above absolute zero temp emit energy in all directions over a wide range of wavelength.
- At a given temp, the total amount of heat energy emitted by a surface in all direction over entire wavelength per unit area, per unit time is called the total emissive power
- The emissive power $E = f(\varepsilon, \lambda, T)$

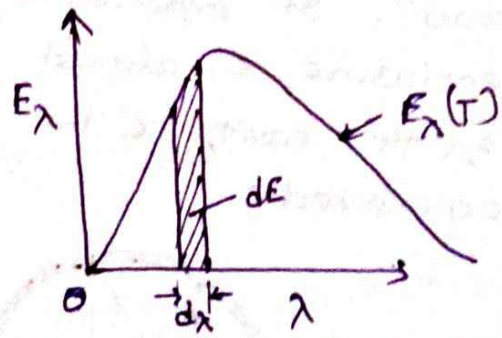
  where $\varepsilon \rightarrow$ emissivity, a surface characteristic

  λ - wavelength of radiation

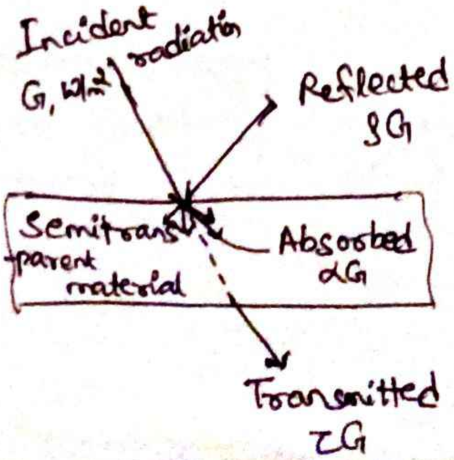
  T - absolute temp of the surface
- The spectral or monochromatic emissive power (E_λ) is defined as amount of radiation energy emitted by a surface at an absolute temp T, per unit time, per unit surface area and per unit wavelength $d\lambda$ about the wavelength λ.

The total emissive power

$$E = \int_0^\infty E_\lambda \, d\lambda$$



Surface Absorption, Reflection, & Transmission



The irradiation is the total radiation energy incident per unit area per unit time over entire wave length from all direction.

Absorptivity :- (α) is defined as fraction of radiation energy incident on the surface from all directions, over entire wavelength, that is absorbed by the surface.

$$\alpha = \frac{G_a}{G_i}$$

$G_a \rightarrow$ energy absorbed by the surface W/m²
G_i = irradiation W/m²

~~Reflectivity (ρ)~~ :-

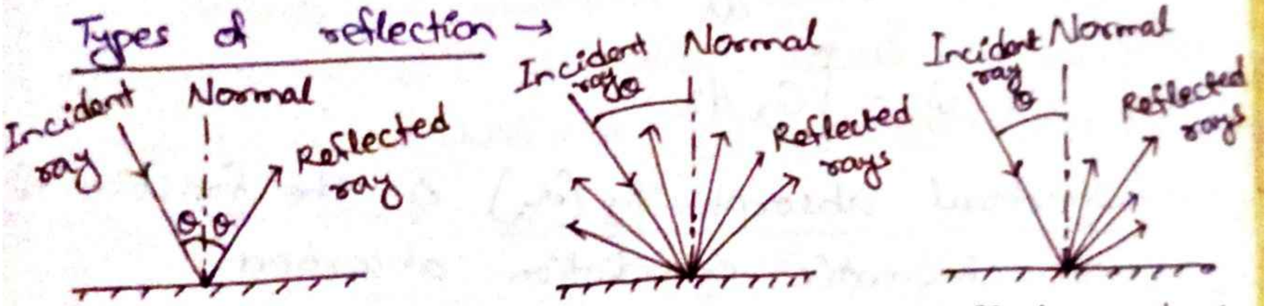
- A blackbody absorbs all incident radⁿ
∴ α for blackbody = 1

Reflectivity (ρ) :- is defined as the fraction of radiation energy incident on a surface from all directions over entire wavelength, that is reflected.

$$\rho = \frac{G_\rho}{G}$$

G_ρ = energy reflected by the surface.

Types of reflection →



Specular or mirror like reflection | Diffuse reflection | Reflection which is betⁿ diffuse and specular (a real surface)

If the surface is perfectly smooth and the angle θ of incident and reflected rays is equal then the reflection is called the specular reflection.

If the surface has some roughness, the incident radⁿ is scattered in all directions after reflection, such reflection is called the diffuse reflection.

The reflection from real surfaces is neither specular nor diffuse but combination of D & S behaviour.

Transmissivity (τ) is the fraction of incident energy transmitted through the surface.

$$\tau = \frac{G_\tau}{G} \qquad G_\tau \rightarrow \text{energy transmitted by the surface}$$

$$G_\alpha + G_\rho + G_\tau = G$$

$$\therefore \alpha + \rho + \tau = 1$$

Spectral or monochromatic irradiation (G_λ)

For a specific wavelength or direction, the irradiation is called as monochromatic irradiation.

It is defined as the radiant heat flux incident on a surface per unit wavelength about a wavelength λ from all directions.

- Unit $\rightarrow$ W/m^2

$$G_\lambda = \frac{dG}{d\lambda}$$

$$\Rightarrow G = \int_0^\infty G_\lambda \, d\lambda$$

Spectral absorptivity (α_λ) is the fraction of monochromatic irradiation absorbed.

Spectral reflectivity (ρ_λ) is the fraction of monochromatic irradiation reflected.

Spectral transmissivity (τ_λ) is the fraction of monochromatic irradiation transmitted.

$$\alpha_\lambda = \frac{G_{\lambda,a}}{G_\lambda}, \quad \rho_\lambda = \frac{G_{\lambda,\rho}}{G_\lambda}, \quad \tau_\lambda = \frac{G_{\lambda,\tau}}{G_\lambda}$$

$$\alpha_\lambda + \rho_\lambda + \tau_\lambda = 1$$

$$\therefore \alpha = \frac{\int_0^\infty \alpha_\lambda G_\lambda \, d\lambda}{\int_0^\infty G_\lambda \, d\lambda}$$

Opaque body → For an opaque surface, there is no transmission. $\tau = 0$ Most solids do not transmit any rad^n & are opaque.

$\therefore \alpha + \rho = 1$ & $\alpha_\lambda + \rho_\lambda = 1$

White body → A body which reflects all almost all radiation incident upon it & does not absorb or transmit any part of it.

For white body $\alpha = 0$, $\tau = 0$, $\rho = 1$

Black body → A black body neither reflects nor transmits any part of the incident radiation but it absorbs all of it.

$\rho = 0$, $\tau = 0$ & $\alpha = 1$

Most gases have high value of τ & low values of α & ρ. Air at atm pr & temp is transparent to thermal rad^n $\tau = 1$, $\rho = \alpha = 0$

Blackbody Radiation Laws

① Blackbody Spectral Emissive Power

According to Max Planck's distribution law based on quantum theory, the spectral or monochromatic emissive power for a black surface is highest at every wavelength at any given temp

$$E_{b\lambda}(T) = \frac{C_1}{\lambda^5\{\exp(C_2/\lambda T) - 1\}}$$

Where C_1 & C_2 are constant

$C_1 = 2\pi h C_0^2 = 3.742 \times 10^8\ W.\mu m^4/m^2$

$C_2 = hC_0/K_B = 1.438 \times 10^4\ \mu m.K$

K_B - Boltzmann constant $= 1.3805 \times 10^{-23}$

C_0 = Velocity of light in vacuum

$= 3 \times 10^8$ m/s

h = Plank's constant $= 6.625 \times 10^{-34}$ Js

T = Absolute temp, K

λ = wavelength, μm

$E_{b\lambda}(T)$ = spectral blackbody emissive power at T, W/m^2

$\therefore E_b = \frac{C_1}{C_2^4} T^4 \frac{\pi^4}{90}$

$\Rightarrow E_b = \sigma T^4$

where $\sigma = \frac{C_1}{C_2^4} \times \frac{\pi^4}{15} = \frac{3.742 \times 10^8 \times \pi^4}{(1.438 \times 10^4)^4 \times 15}$

$= 5.672 \times 10^{-8} \ W/m^2 . K^4$

= Stefan Boltzmann constant.

Emissivity (ϵ)

The radn emitted by a real body at temp T is always less than that of black body.

Emissivity is defined as the ratio of the radiation energy emitted by a surface to that emitted by a blackbody at the same temp.

It varies betn 0 & 1

The emissivity of real surfaces varies with temp of surface, wavelength & direction of emission. Therefore, different emissivities may be defined for a surface, depending upon the effect considered.

Hemispherical & Total Emissivity

The emissivity of a surface that is averaged over all directions is called the hemispherical emissivity.

The emissivity averaged over all wavelength is called the total emissivity.

The total hemispherical emissivity $\epsilon(T)$ of a surface is defined as ratio of the radiation heat flux emitted over all wavelength in a hemispherical space (all directions) to that

The directional emissivity ϵ_θ of a surface in normal direction $(\theta = 0)$ represents the hemispherical emissivity of the surface.

Kirchhoff's Law

It states that at thermal equilibrium, the ratio of the total emissive power to the total absorptivity is constant for all bodies.

$$\frac{E_1}{\alpha_1} = \frac{E_2}{\alpha_2} = E_b$$

- At a specific wavelength the spectral emissivity is equal to spectral absorptivity at thermal equilibrium for all bodies.
- This law is applicable when the radiation properties are independent of wavelength i.e for graybodies or when incident & emitted radⁿ have same spectral distribution.

Gray & Diffuse Surfaces

A surface is said to be diffuse surface, if its radiation properties such as absorptivity emissivity etc are independent of direction.

A surface is said to be gray surface if its properties are independent of wavelength.

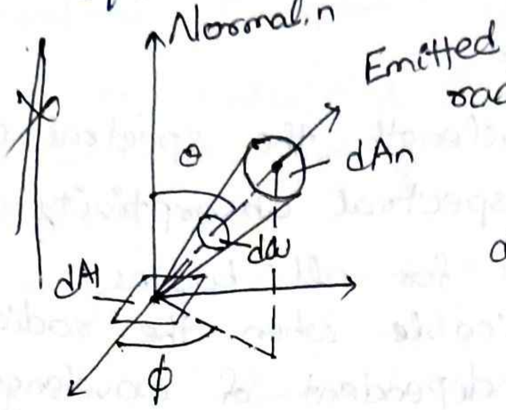
Radiation Intensity

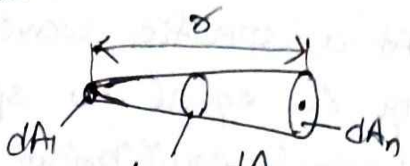
The radⁿ emitted from a surface propagates in all possible direction. Similarly, the radⁿ incident on a surface may come from different directions. In both cases, effect of radⁿ on the surface depends on the directional distribution. Such directional effect may be treated by introducing the concept of radiation intensity.

Solid Angle

It is defined as the space enclosed inside a sphere enclosed by a conical surface with the vertex of the cone at the centre of the sphere.

- It is measured by the ratio of the spherical surface enclosed by the cone to the square of the radius of the sphere.





$d\omega = \frac{dA_n}{r^2}$

Let's consider the emission from the differential area dA_1 towards the normal area dA_n. The differential area dA_n, through which the emission passes, subtends an angle called the solid angle when viewed from dA_1. Mathematically

$$d\omega = \frac{dA_n}{r^2}$$

where A_n = Normal area
r = radius of sphere

- The solid angle is measured in steradians (sr)
- The solid angle subtended by hemisphere from its centre is

$$d\omega = \frac{dA_n}{r^2} = \frac{2\pi r^2}{r^2} = 2\pi$$

- By full sphere $d\omega = \frac{4\pi r^2}{r^2} = 4\pi$

* A solid angle is defined as the ratio of the spherical surface enclosed by a cone with its vertex at the centre of the sphere, to the square of the radius of the sphere.

Spectral Intensity of Radiation ($I_{b\lambda}$)

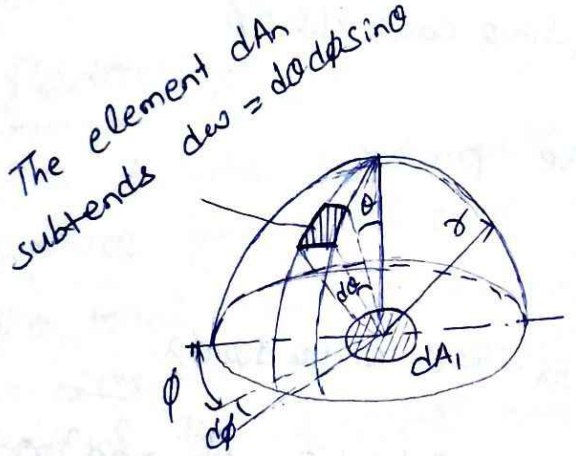
It is the radiant energy emitted by a blackbody at a temp T, streaming through a unit area normal to direction of propagation per unit wavelength about a wavelength per unit solid angle about the direction of propagation of beam

$$I_{b\lambda} = \frac{\text{Energy emitted}}{\text{Projected area} \times \text{Wavelength} \times \text{Solid angle}} \quad \frac{W}{m^2 . \mu m . sr}$$

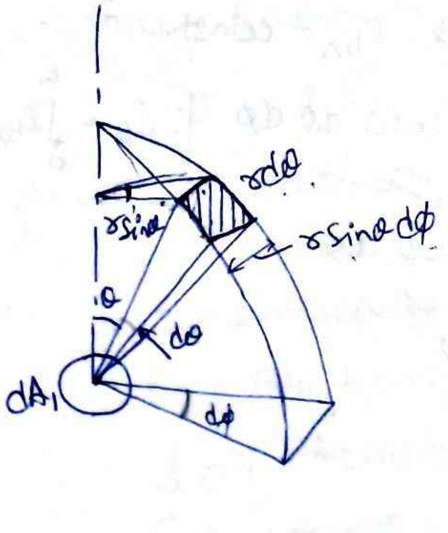
Radiation Intensity (I_b)

It is the radiation emitted by a black body at temp T over all wavelength per unit projected area per unit solid angle.

Mathematically $I_b = \int_0^\infty I_{b\lambda} \cdot d\lambda \quad (W/m^2 . sr)$



Let's consider the emission from an elemental area dA_1 at the centre of the hemisphere. Let this emission is absorbed by elemental area dA_n, a portion of the hemi-sphere.



The solid angle subtended by dA_n

$$d\omega = \frac{dA_n}{r^2}$$

Now $dA_n = r\,d\theta \times r\sin\theta\,d\phi$

$= r^2 \sin\theta\,d\theta\,d\phi$

$$\therefore d\omega = \frac{r^2 \sin\theta\,d\theta\,d\phi}{r^2}$$

$= \sin\theta\,d\theta\,d\phi$.

The projected area on a plane normal to the line joining dA_1

$$dA_n = dA_1 \cos\theta$$

We know spectral intensity of rad^n

$$I_{b\lambda} = \frac{\text{Energy emitted}}{\text{Projected area} \times \text{Wave length} \times \text{Solid angle}}$$

$$= \frac{dQ/d\lambda}{dA_1 \cos\theta \times \frac{dA_n}{r^2}}$$

Where $\frac{dQ}{d\lambda} = dQ_\lambda$

$$\Rightarrow dQ_\lambda = I_{b\lambda}\, dA_1 \cos\theta \times \frac{r^2 \sin\theta\, d\theta\, d\phi}{r^2}$$

$\therefore$ Spectral emissive power

$$dE_\lambda = \frac{dQ_\lambda}{dA_1} = I_{b\lambda} \sin\theta \cos\theta\, d\theta\, d\phi$$

$$\Rightarrow E_{b\lambda} = \int dE_\lambda$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{b\lambda} \sin\theta \cos\theta\, d\theta\, d\phi$$

Now total emissive power

$$E_b = \int_0^\infty E_{b\lambda}\, d\lambda$$

$$= \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} I_{b\lambda} \sin\theta \cos\theta\, d\theta\, d\phi\, d\lambda$$

For a diffuse surface, intensity of rad^n is independent of direction & $I_{b\lambda} = \text{const}$

$$E_b = I_b \times \int_0^{2\pi} \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta\, d\phi \quad \left[\because I_b = \int_0^\lambda I_{b\lambda}\, d\lambda\right]$$

$$= I_b \times 2\pi \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta$$

$$= \pi I_b \int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$\boxed{E_b = \pi I_b}$$ ✓

Thus the total emissive power of a black body is π times the intensity of radiation

Lambert Cosine Law

It states that the total emission from a surface in any direction is directly proportional to the cosine of the angle of the emission.

- The angle of emission θ is the angle subtended by the normal to the radiating surface and the direction vector of emission of the receiving surface.

If E_n is total emissive power in normal direction, then the emissive power

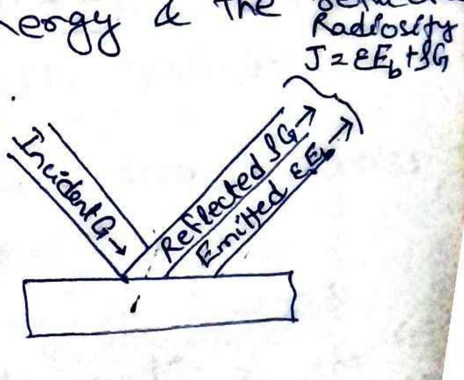
$$E = E_n \cos\theta$$

- It is true only for diffuse radiating surface.
- For the surfaces, obey Lambert cosine law, the intensity of radⁿ in any direction is same. i.e. $I_n = I = \text{Const.}$

Radiosity (J)

It is the total radiant energy leaving a surface per unit area per unit time.

- The total radiant energy leaving a surface consists of the emitted energy & the reflected part of the incident energy.



Thus Radiosity

$$J = \varepsilon E_b + \rho G$$

where J = radiosity of the surface (W/m²)

ε = Emissivity of the surface

E_b = Black body emissive power

ρ = Reflectivity of the surface

G = Incident radiant flux (W/m²)

For a gray, diffuse opaque surface ($\tau = 0$)

$$\alpha + \rho = 1$$

$$\Rightarrow \rho = 1 - \alpha$$

& at thermal equilibrium $\varepsilon = \alpha$

$$\therefore J = \varepsilon E_b + \rho G$$

$$= \varepsilon E_b + (1-\varepsilon) G$$

$$\Rightarrow G = \frac{J - \varepsilon E_b}{1-\varepsilon}$$

The total rate of energy leaving the surface $= AJ$

" " " " energy incident on the surface $= AG$

Thus the net radiant energy leaving the surface

$$Q = AJ - AG = A(J - G)$$

$$= A\left[J - \frac{J - \varepsilon E_b}{1-\varepsilon}\right] = \frac{A\varepsilon (E_b - J)}{1-\varepsilon}$$

This equn is not valid for black surface

For black surface $\varepsilon = \alpha = 1$ & $\rho = 0$

$$\boxed{J = E_b}$$

& $$\boxed{Q = A(E_b - G)}$$